MATH 597B, SPRING 2015, PROBLEMS 4

Due Tuesday 10th February

1. (i) Prove that for any $s \in \mathbb{C}$ for which $\frac{1}{\zeta(s)}$ and $G(s)$ exist we have

$$\frac{1}{\zeta(s)} = 2G(s) - G(s)^2\zeta(s) - \left(\frac{1}{\zeta(s)} - G(s)\right)(\zeta(s)G(s) - 1).$$

(ii) Let u be a real parameter with $u \ge 1$ and define $G(s) = \sum_{n \le u} \mu(n)n^{-s}$. Prove that when $\Re s > 1$

$$G(s)^2\zeta(s) = \sum_{l \le u} \mu(l) \sum_{m \le u} \mu(m) \sum_{n=1}^{\infty} (lmn)^{-s}.$$

(iii) Prove that when $\Re s > 1$ we have $\zeta(s)G(s) = \sum_{m=1}^{\infty} \tau_u(m)m^{-s}$ where $\tau_u(m) = \sum_{\substack{k \le u \\ k|m}} \mu(m)$.

(iv) Prove that $\tau_u(1) = 1$ and $\tau_u(m) = 0$ when $1 < m \le u$. Deduce that when $\Re s > 1$

$$\left(\frac{1}{\zeta(s)} - G(s)\right)(\zeta(s)G(s) - 1) = \sum_{l > u} \sum_{m > u} \mu(l)\tau_u(m)(lm)^{-s}.$$

(v) Prove that if $n \in \mathbb{N}$ we have $\mu(n) = c_1(n) - c_2(n) - c_3(n)$ where

$$c_1(n) = \begin{cases} 2\mu(n) & (n \le u) \\ 0 & (n > u), \end{cases} \quad c_2(n) = \sum_{\substack{k \le u, l \le u, m \\ klm = n}} \mu(k)\mu(l), \quad c_3(n) = \sum_{\substack{l > u, m > u \\ lm = n}} \mu(l)\tau_u(m).$$

(vi) Prove that for an arbitrary arithmetical function f and for any real $x \ge u$ we have $\sum_{n \le x} \mu(n)f(n) = 2S_1 - S_2 - S_3$ where $S_1 = \sum_{n \le u} \mu(n)f(n)$,

$$S_2 = \sum_{\substack{l,m \\ l \le u, m \le u}} \mu(l)\mu(m) \sum_{n \le x/(lm)} f(lmn), \quad S_3 = \sum_{\substack{l > u, m > u \\ lm \le x}} \mu(l)\tau_u(m)f(lm).$$

(vii) Explain how this can be used to show that if $\alpha \in \mathbb{R}$, $q \in \mathbb{N}$, $a \in \mathbb{Z}$, $(q,a) = 1$ and $|\alpha - a/q| \le q^{-2}$, then

$$\sum_{n \le x} \mu(n)e(\alpha n) \ll x(\log x)^3 \big(q^{-1/2} + x^{-1/5} + (q/x)^{1/2}\big).$$