Math 597b, Spring 2015, Problems 3

Due Tuesday 3rd February

1. (i) Prove that if Q is a real number such that $Q \geq 1$, then $\displaystyle\sum_{n\leq Q} \frac{1}{n} \geq \log Q$.

(ii) A natural number q is squarefree when it has no repeated prime factors, i.e. $\mu(q)^2 = 1$. Let $s(n)$ denote the squarefree kernel of n, $s(n) = \prod_{p|n} p$. Prove that if q is squarefree, then $\displaystyle\frac{1}{\phi(q)} = \sum_{\substack{n=1 \\ s(n)=q}}^{\infty} \frac{1}{n}$.

(iii) Prove that $\displaystyle\sum_{n\leq Q} \frac{\mu(n)^2}{\phi(q)} = \sum_{\substack{n=1 \\ s(n)\leq Q}}^{\infty} \frac{1}{n}$. Hence deduce that $\displaystyle\sum_{n\leq Q} \frac{\mu(n)^2}{\phi(q)} \geq \log Q$.

(iv) Prove that $\displaystyle\sum_{q\leq Q} \frac{\mu(q)^2}{\phi(q)} = \sum_{r|k} \sum_{\substack{m\leq Q/r \\ (m,k/r)=1}} \frac{\mu(mr)^2}{\phi(mr)}$.

(v) Note that if $\mu(mr) \neq 0$, then $(m,r) = 1$. Show that $\displaystyle\sum_{r|k} \sum_{\substack{m\leq Q/r \\ (m,k/r)=1}} \frac{\mu(mr)^2}{\phi(mr)} = \sum_{r|k} \frac{\mu(r)^2}{\phi(r)} \sum_{\substack{m\leq Q/r \\ (m,k)=1}} \frac{\mu(m)^2}{\phi(m)} \leq \frac{k}{\phi(k)} \sum_{\substack{q\leq Q \\ (q,k)=1}} \frac{\mu(q)^2}{\phi(q)}$ and $\displaystyle\sum_{\substack{q\leq Q \\ (q,k)=1}} \frac{\mu(q)^2}{\phi(q)} \geq \frac{\phi(k)}{k} \log Q$.

2. Suppose that M and N are integers, $N \geq 1$, $T(x)$ is as given in (19.8), $\delta > 0$ and the points x_r are well spaced in the sense of (19.9). Suppose also that there are constants A, B and a real valued function $f(N,\delta)$ such that $N^{-1} \sup_{\delta} f(N,\delta) \to 0$ as $N \to \infty$ and for any choice of the above we have

$$\sum_{r=1}^{R} |T(x_r)|^2 \leq (AN + B\delta^{-1} + f(N,\delta)) \sum_{n=M+1}^{M+N} |c_n|^2. \qquad (*)$$

Let $H \in \mathbb{N}$ and define $x_{rh} = \dfrac{x_r + h}{H} \quad 1 \leq r \leq R,\ 0 \leq h < H$,

$$b_n = \begin{cases} c_{n/H} & \text{when } H|n, \\ 0 & \text{when } H\nmid n, \end{cases} \qquad T^*(x) = \sum_{n=HM+H}^{HM+HN} b_n e(nx).$$

(i) Prove that $\min \|x_{rh} - x_{sj}\| \geq \delta/H$ where the min is over pairs r,h; s,j with $r,h \neq s,j$ and deduce that

$$\sum_{r=1}^{R} \sum_{h=0}^{H-1} |T^*(x_{rh})|^2 \leq (A(HN - H + 1) + BH\delta^{-1} + f(HN - H + 1, \delta/H)) \sum_{n=M+1}^{M+N} |c_n|^2.$$

(ii) Prove that $\displaystyle\sum_{r=1}^{R} \sum_{h=0}^{H-1} |T^*(x_{rh})|^2 = H \sum_{r=1}^{R} |T(x_r)|^2$.

(iii) Assuming only (*) deduce that $\displaystyle\sum_{r=1}^{R} |T(x_r)|^2 \leq (A(N-1) + B\delta^{-1}) \sum_{n=M+1}^{M+N} |c_n|^2$.