Math 597b Bounded Gaps in the Primes, Spring 2015, Problems 2

Due Tuesday 27th January 2015

1. Let K, $R \in \mathbb{N}$, $M \in \mathbb{Z}$, $N = KR+1$, $x_r = r/R$ $(1 \le r \le R)$, $a_n = 1$ when $n \equiv M+1 \pmod R$ and $a_n = 0$ otherwise. Show that

$$\sum_{r-1}^{R} \left| \sum_{n=M+1}^{M+N} a_n e(nx_r) \right|^2 = (N - 1 + 1/\delta) \sum_{n=M+1}^{M+N} |a_n|^2$$

where $\delta = \min_{r \ne s} \|x_r - x_s\|$.

2. When $M+1 \le n \le M+N$ let $a_n \in \mathbb{C}$ and put $T(\alpha) = \sum_{n=M+1}^{M+N} a_n e(n\alpha)$. Given $q \in \mathbb{N}$, $h \in \mathbb{Z}$ define $Z(q,h) = \sum_{n=M+1, n\equiv h \pmod q}^{M+N} a_n$ and for each prime p define $\mathcal{D}(p)$ to be the residue classes $h \pmod p$ such that $Z(p,h) = 0$. Let $\delta(p) = \operatorname{card} \mathcal{D}(p)$ and let $\mathcal{R}(p)$ denote the residue classes not in $\mathcal{D}(p)$ so that $\operatorname{card} \mathcal{R}(p) = p - \delta(p)$. For general q let $\mathcal{R}(q)$ denote the residue classes r modulo q so that $r \in \mathcal{R}(p)$ for every $p|q$ and define $r(q) = \operatorname{card} \mathcal{R}(q)$. Throughout suppose that q is squarefree and $\delta(p) < p$.

(i) Prove that $r(q) = \prod_{p|q}(p - \delta(p))$.

(ii) Prove that if $k \in \mathcal{R}(q)$, then $\displaystyle\sum_{h \in \mathcal{R}(q)} c_q(h-k) = \prod_{p|q} \delta(p)$ where $c_q(n)$ denotes Ramanujan's sum.

(iii) Prove that $\displaystyle\sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left| \sum_{r \in \mathcal{R}(q)} e(ar/q)(Z(q,r) - Z/r(q)) \right|^2 = T_1 - 2\Re T_2 + T_3$ where

$$T_1 = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left| \sum_{r \in \mathcal{R}(q)} e(ar/q) Z(q,r) \right|^2 \text{ and } T_2 = |Z|^2 \prod_{p|q} \tfrac{\delta(p)}{p-\delta(p)} = T_3.$$

(iv) Prove that $\displaystyle\sum_{\substack{a=1 \\ (a,q)=1}}^{q} |T(a/q)|^2 \ge |Z|^2 \prod_{p|q} \frac{\delta(p)}{p - \delta(p)}$.

(v) Suppose further that $a_n = 0$ or 1. Prove that $Z \ll \dfrac{N+Q^2}{L}$ where

$$L = \sum_{q \le Q} \mu(q)^2 \prod_{p|q} \frac{\delta(p)}{p - \delta(p)}.$$