MATH 597B BOUNDED GAPS IN THE PRIMES, SPRING 2015, PROBLEMS 1

Due 20th January

1. (The 'Larger Sieve' of Gallagher [1971]) Let $\mathcal{N}$ be a subset of the integers in an interval $[M+1, M+N]$ and suppose $\mathrm{card}(\mathcal{N}) = Z$. Let $Z(q,h)$ denote the number of $n \in \mathcal{N}$ such that $n \equiv h \pmod q$. For any primepower q let $r(q)$ denote the number of $h \pmod q$ for which $Z(q,h) > 0$.

(a) Explain why $Z^2 = \left(\sum_{h=1}^{q} Z(q,h) \right)^2 \le r(q) \sum_{h=1}^{q} Z(q,h)^2$.

(b) Let $\mathcal{Q}$ be a finite set of primepowers q with $r(q) \neq 0$. Deduce that

$$Z^2 \sum_{q \in \mathcal{Q}} \frac{\Lambda(q)}{r(q)} \le \sum_{q \in \mathcal{Q}} \Lambda(q) \sum_{\substack{n_1, n_2 \in \mathcal{N} \\ n_1 \equiv n_2 \ (q)}} 1.$$

(c) Group pairs n_1, n_2 of members of $\mathcal{N}$ according to their common difference, and hence show that the right hand side above is

$$= \sum_{d=-N+1}^{N-1} \sum_{\substack{n_1, n_2 \in \mathcal{N} \\ n_1 - n_2 = d}} \sum_{\substack{q \in \mathcal{Q} \\ q \mid d}} \Lambda(q).$$

(d) Show that

$$\sum_{d \neq 0} \sum_{\substack{n_1, n_2 \in \mathcal{N} \\ n_1 - n_2 = d}} 1 = Z^2 - Z.$$

(e) Deduce that the expression in (c) is $\le Z \sum_{q \in \mathcal{Q}} \Lambda(q) + (Z^2 - Z) \log N$.

(f) Conclude that

$$Z \le \frac{\sum_{q \in \mathcal{Q}} \Lambda(q) - \log N}{\sum_{q \in \mathcal{Q}} \Lambda(q)/r(q) - \log N}$$

provided that the denominator is positive.

(g) Suppose that $r(p) = (p+1)/2$ for each odd prime p. By choosing $\mathcal{Q}$ suitably deduce that $Z \ll N^{1/2}$. One can see that this is essentially best possible by taking $\mathcal{N}$ to be a set of perfect squares.