Multiplicative Number Theory III

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Contents

Notation

Some symbols are used in more than one way. The intended interpretation should be clear from the context in which it arises.

Symbol	Meaning
$\mathbb{C}$	The set of complex numbers. See page 1.
$\mathbb{Q}$	The set of rational numbers.
$\mathbb{R}$	The set of real numbers.
$\mathbb{T}$	$\mathbb{R}/\mathbb{Z}$, i.e., the real numbers modulo 1. See page xx.
$\mathbb{Z}$	The set of rational integers.
A^*	The adjoint of the matrix A. See page **??**.
$\lVert\boldsymbol{x}\rVert$	Norm of the vector $\boldsymbol{x}$.
$\lVert A\rVert$	The operator norm of the matrix A. See page **??**.
$\lVert\alpha\rVert$	The distance from α to the nearest integer. See page **??**.
$E_0(\chi)$	$= 1$ if $\chi = \chi_0$, $= 0$ otherwise. See page **??**
$s(x)$	The sawtooth function. See page **??**.
$\mathrm{si}(x)$	The sine integral. See page **??**.
$\deg P$	The degree of the polynomial P.
$\Delta(s)$	$\zeta(s) = \Delta(s)\zeta(1-s)$. See page 81.
$\sum_\chi^\star$	A sum over primitive characters modulo q. See page **??**.

23
Probabilistic number theory

C:ProbNoThy

We say that an arithmetic function f is *additive* if

$$f(mn) = f(m) + f(n) \tag{23.1}$$

E:defaddfcn

whenever $(m,n) = 1$. The values of an additive function are determined by its values on prime-powers, since

$$f(n) = \sum_{p^k \| n} f\big(p^k\big)\,. \tag{23.2}$$

E:formaddfcn

If the identity (23.1) holds for all pairs m, n, then we say that f is *totally* (or *completely*) *additive*. If f is additive and $f\big(p^k\big) = f(p)$ for all p and all $k \ge 1$, then we say that f is *strongly additive*. For example, $\log n$ and $\Omega(n)$ are totally additive functions, $\log(n/\varphi(n))$ and $\omega(n)$ are strongly additive, while $\Omega(n) - \omega(n)$ and $\log d(n)$ are additive but neither totally additive nor strongly additive.

In our study of sieves in Chapter 3 we saw that things do not always work out as one would expect on probabilistic grounds. However, we find that the distribution of the values of an additive function follow the natural probabilistic model very closely. Suppose that f is an additive function. The asymptotic density of integers n for which $p^k \| n$ is $p^{-k}(1 - 1/p)$. It is with this 'probability' that the term $f\big(p^k\big)$ is one of the terms in the sum. Accordingly, for each prime number p we define a random variable X_p that has the distribution

$$\begin{aligned} \mathsf{P}\big(X_p = f\big(p^k\big)\big) &= \frac{1 - 1/p}{p^k} \qquad (k = 1, 2, \ldots), \\ \mathsf{P}\big(X_p = 0\big) &= 1 - 1/p\,. \end{aligned} \tag{23.3}$$

E:DistrvX_p

If p and q are distinct primes, then by the Chinese remainder theorem we see that the asymptotic density of the integers n for which both

$p^k \| n$ and $q^\ell \| n$ is $p^{-k}(1-1/p)q^{-\ell}(1-1/q)$. Hence the two events $p^k \| n$ and $q^\ell \| n$ are asymptotically independent. Thus it is natural to take the random variables X_p to be independent, and we set

$$X = \sum_p X_p\,. \tag{23.4}$$

E:defrvX

This sum either converges almost always or almost nowhere. We shall find that when it converges almost always, the values of f have a limiting distribution that is the same as the distribution of X, and that when it converges almost nowhere, f does not have a limiting distribution.

We have already established a scattering of results concerning a few additive functions. In §2.3 we estimated the mean of $\omega(n)$, and also its variance about its mean. In §2.4 we determined the distribution of the additive function $\Omega(n) - \omega(n)$ by calculating the mean value of the multiplicative function $z^{\Omega(n)-\omega(n)}$. In §7.4 we put

$$\alpha_n = \frac{\Omega(n) - \log\log n}{\sqrt{\log\log n}},$$

and found that the distribution of α_n is asymptotically normal with mean 0 and variance 1. In this chapter we are more concerned with developing a general theory than with special examples.

23.1 The Turán–Kubilius inequality

S:TKIneq

PT34 Turán (1934) showed (cf Theorem 2.12) that

$$\sum_{n\le x} (\omega(n) - \log\log x)^2 \ll x \log\log x,$$

and JK56 Kubilius (1956) generalized this to arbitrary additive functions.

Suppose that f is an additive function. Then

$$\begin{aligned}\sum_{n\le x} f(n) &= \sum_{n\le x}\sum_{p^k\| n} f\big(p^k\big)\\ &= \sum_{p^k\le x} f\big(p^k\big)\sum_{\substack{n\le x\\ p^k\| n}} 1\\ &= \sum_{p^k\le x} f\big(p^k\big)\Big(\Big[\frac{x}{p^k}\Big]-\Big[\frac{x}{p^{k+1}}\Big]\Big)\\ &= x\sum_{p^k\le x}\frac{f\big(p^k\big)}{p^k}\Big(1-\frac{1}{p}\Big)+O\Big(\sum_{p^k\le x}\big|f\big(p^k\big)\big|\Big). \qquad (23.5)\end{aligned}$$

E:meanaddfcn

For ease of reference, we set

$$A(x)=A(f,x)=\sum_{p^k\le x}\frac{f\big(p^k\big)}{p^k}\Big(1-\frac{1}{p}\Big). \qquad (23.6)$$

E:defA

We anticipate that the variance of f about its mean should not be much more than

$$B(x)=B(f,x)=\sum_{p^k\le x}\frac{\big|f\big(p^k\big)\big|^2}{p^k}. \qquad (23.7)$$

E:defB

By Cauchy's inequality,

$$\sum_{p^k\le x}\big|f\big(p^k\big)\big| \;\le\; B(x)^{1/2}\Big(\sum_{p^k\le x}p^k\Big)^{1/2} \;\ll\; B(x)^{1/2}\frac{x}{\sqrt{\log x}}.$$

Thus from (23.5) we see that

$$\sum_{n\le x} f(n)=A(x)x+O\big(B(x)^{1/2}x(\log x)^{-1/2}\big). \qquad (23.8)$$

E:meanfest1

As concerns the potential size of $A(x)$ relative to $B(x)$, we note by Cauchy's inequality that

$$|A(x)|^2\le\Big(\sum_{p^k\le x}\frac{\big|f\big(p^k\big)\big|^2}{p^k}\Big)\Big(\sum_{p^k\le x}\frac{1}{p^k}\Big)\ll B(x)\log\log x. \qquad (23.9)$$

E:ArelBest

We now show that $B(x)$ is within a constant factor of being an upper bound for the variance of the values of f about its mean.

T:TKIneq **Theorem 23.1** (The Turán–Kubilius inequality) *Let f be an additive function, with $A(x)$ and $B(x)$ defined as in* (23.6) *and* (23.7). *Then*

$$\sum_{n\le x}\big(f(n)-A(x)\big)^2 \;\ll\; xB(x). \qquad (23.10)$$

E:TKIneq

The implicit constant in the above is absolute: It is independent of both f and x. From (23.9) we see that

$$\sum_{n\le x} |A(x)|^2 \ll xB(x)\log\log x,$$

and by (23.10) it follows that also

$$\sum_{n\le x} |f(n)|^2 \ll xB(x)\log\log x\,.$$

Thus we see that the estimate (23.10) is never more than a factor of $\log\log x$ from being trivial. Despite this lack of quantitative depth, the Turán–Kubilius inequality turns out to be a quite useful result.

Proof We expand the sum on the left hand side, and obtain three terms. The simplest is

$$\begin{aligned} T_0 = \sum_{n\le x} |A(x)|^2 &= |A(x)|^2[x] = |A(x)|^2 x + O\big(|A(x)|^2\big) \\ &= x|A(x)|^2 + O(B(x)\log\log x) \end{aligned} \tag{23.11}$$

E:T_0Est

by (23.9). The intermediate term is $T_1 = -2\,\mathrm{Re}\,\overline{A(x)}\sum_{n\le x} f(n)$. Thus by (23.8),

$$T_1 = -2|A(x)|^2x + O\big(|A(x)|B(x)^{1/2}x(\log x)^{-1/2}\big),$$

and by (23.9) this is

$$= -2|A(x)|^2x + O\big(B(x)x(\log x)^{-1/2}(\log\log x)^{1/2}\big)\,. \tag{23.12}$$

E:T_1Est

Finally,

$$\begin{aligned} T_2 = \sum_{n\le x} |f(n)|^2 &= \sum_{n\le x}\sum_{p^k\|n} f\big(p^k\big)\sum_{q^\ell\|n}\overline{f}\big(q^\ell\big) \\ &= \sum_{p^k\le x}\sum_{q^\ell\le x} f\big(p^k\big)\overline{f}\big(q^\ell\big)\sum_{\substack{n\le x\\ p^k\|n\\ q^\ell\|n}} 1 \end{aligned}$$

where q denotes a prime number. The contribution of those terms for which $p = q$ is

$$T_2' = \sum_{p^k\le x} \big|f\big(p^k\big)\big|^2\Big(\Big[\frac{x}{p^k}\Big] - \Big[\frac{x}{p^{k+1}}\Big]\Big) \le xB(x)\,. \tag{23.13}$$

E:T2diag

The remaining terms contribute

$$T_2'' = \sum_{\substack{p^kq^\ell\le x\\ p\ne q}} f(p^k)\overline{f}(q^\ell)\Big(\Big[\frac{x}{p^kq^\ell}\Big]-\Big[\frac{x}{p^{k+1}q^\ell}\Big]-\Big[\frac{x}{p^kq^{\ell+1}}\Big]+\Big[\frac{x}{p^{k+1}q^{\ell+1}}\Big]\Big)$$

$$= x\sum_{p^kq^\ell\le x}\frac{f(p^k)}{p^k}\,\frac{\overline{f}(q^\ell)}{q^\ell}\Big(1-\frac{1}{p}\Big)\Big(1-\frac{1}{q}\Big) \qquad (23.14) \quad \boxed{\texttt{E:T2nondiag1}}$$

$$+\,O\Big(\sum_{p^kq^\ell\le x}\big|f(p^k)\overline{f}(q^\ell)\big|\Big)+O\Big(x\sum_{p^{k+\ell}\le x}\frac{|f(p^k)\overline{f}(p^\ell)|}{p^{k+\ell}}\Big). \qquad (23.15) \quad \boxed{\texttt{E:T2nondiag2}}$$

By Cauchy's inequality the first error term is

$$\ll\Big(\sum_{p^kq^\ell\le x}\frac{\big|f(p^k)\big|^2\big|f(q^\ell)\big|^2}{p^kq^\ell}\Big)^{1/2}\Big(\sum_{p^kq^\ell\le x}p^kq^\ell\Big)^{1/2}.$$

Here the first sum is $\le B(x)^2$, and the second sum is

$$\le x\sum_{\substack{n\le x\\ \omega(n)\le 2}}1\ll x^2(\log x)^{-1}\log\log x$$

by (7.54). By Cauchy's inequality the second error term in (23.15) is

$$\ll x\Big(\sum_{p^{k+\ell}\le x}\frac{\big|f(p^k)\big|^2}{p^{k+\ell}}\Big)^{1/2}\Big(\sum_{p^{k+\ell}\le x}\frac{\big|f(p^\ell)\big|^2}{p^{k+\ell}}\Big)^{1/2}$$

$$= x\sum_{p^k\le x}\frac{\big|f(p^k)\big|^2}{p^k}\sum_{\substack{\ell\\ p^\ell\le x/p^k}}\frac{1}{p^\ell}\ll xB(x)\,.$$

The expression (23.14) is

$$= x|A(x)|^2 - x\sum_{\substack{p^k\le x\\ q^\ell\le x\\ p^kq^\ell>x}}\frac{f(p^k)}{p^k}\,\frac{\overline{f}(q^\ell)}{q^\ell}\Big(1-\frac{1}{p}\Big)\Big(1-\frac{1}{q}\Big)\,.$$

By Cauchy's inequality the sum on the right has absolute value

$$\le\Big(\sum_{\substack{p^k\le x\\ q^\ell\le x\\ p^kq^\ell>x}}\frac{\big|f(p^k)\big|^2}{p^k}\,\frac{\big|f(q^\ell\big|^2)}{q^\ell}\Big)^{1/2}\Big(\sum_{\substack{p^k\le x\\ q^\ell\le x\\ p^kq^\ell>x}}\frac{1}{p^kq^\ell}\Big)^{1/2}.$$

Here the first sum is $\le B(x)^2$, and the second sum is

$$= 2 \sum_{p^k \le x^{1/2}} \frac{1}{p^k} \sum_{x/p^k < q^\ell \le x} \frac{1}{q^\ell} + \Big(\sum_{x^{1/2} < p^k \le x} \frac{1}{p^k} \Big)^2$$
$$\ll \sum_{p^k \le x^{1/2}} \frac{k \log p}{p^k \log x} + 1 \ll 1 \, .$$

On assembling our estimates we deduce that

$$T_2'' = x|A(x)|^2 + O(x(B(x))) \, .$$

The stated result now follows by combining this with (23.11)–(23.13). □

23.1.1 Exercises

1. Show that almost all integers n have $(1/2 + o(1)) \log\log n$ prime factors $\equiv 1 \pmod 4$.
2. Let k be a fixed positive integer. Show that $d_k(n) = (\log n)^{(1+o(1)) \log k}$ for almost all integers.
3. Show that

$$\sum_{n \le x} \Omega(n)\Omega(n+k) = x(\log\log x)^2 + cx \log\log x + O(x)$$

where

$$c =$$

4. Show that $\sum_{n \le x} \big(\omega\big(n^2+1\big) - \log\log n\big)^2 \ll x \log\log x$.
5. Show that $\sum_{p \le x} \big(\omega(p+1) - \log\log p\big)^2 \ll x \log\log x$.
6. Suppose that f is an additive function, and let $A(x)$ and $B(x)$ be defined as in (23.6) and (23.7).
   (a) Show that if $n \le x$, then

$$|A(n) - A(x)|^2 \le B(x) \sum_{n < p^k \le x} \frac{1}{p^k} \, .$$

   (b) Show that

$$\sum_{n \le x} |A(n) - A(x)|^2 \ll B(x)x/\log x \, .$$

(c) Conclude that

$$\sum_{n\le x} |f(n) - A(n)|^2 \ll xB(x)\,.$$

7. Suppose that f is an additive function, that $A(x)$ and $B(x)$ are defined as in (23.6) and (23.7) respectively, and that $\alpha \in \mathbb{R}$. Let

$$S(\alpha) = \sum_{n\le x} f(n)e(\alpha n).$$

(a) Prove that

$$S(\alpha) = A(x)\sum_{n\le x} e(\alpha n) + O\big(xB(x)^{1/2}\big).$$

(b) Prove that if $\eta(x) \to 0+$ as $x \to \infty$, $B(x) \ll \eta(x)^2A(x)^2$ and $\|\alpha\|^{-1} \ll \eta(x)x$, then for all sufficiently large x we have

$$S(\alpha) \ll \eta(x)|S(0)|$$

8. Suppose that f is an additive function and that $B(x)$ is defined as in (23.7).

(a) Show that

$$\Big|\sum_{p^k\le x}\frac{f(p^k)}{p^{k+1}}\Big|^2 \ll B(x)\,.$$

(b) Show that

$$\Big|\sum_{\substack{p^k\le x\\ k>1}}\frac{f(p^k)}{p^k}\Big|^2 \ll B(x)\,.$$

(c) Put $A'(x) = \sum_{p\le x} f(p)/p$. Show that if f is an additive function, then

$$\sum_{n\le x} |f(n) - A'(x)|^2 \ll xB(x)\,.$$

9. The *Kubilius class* $\mathcal{H}$ consists of those additive functions f with the two properties

(i) $B(f,x) \to \infty$ as $x \to \infty$;

(ii) $$\sum_{x^{1/2}<p^k\le x}\frac{\big|f(p^k)\big|^2}{p^k} = o(B(x)) \qquad (x\to\infty)\,.$$

Show that if $f \in \mathcal{H}$, then

$$\sum_{n\le x} |f(n) - A(x)|^2 = (1 + o(1))xB(x)$$

as $x \to \infty$.

10. Let $f(n) = \log n$.
    (a) Show that $A(x) = \log x + O(1)$.
    (b) Show that $B(x) = \frac{1}{2}(\log x)^2 + O(\log x)$.
    (c) Deduce that $f \notin \mathcal{H}$.
    (d) Show that $\sum_{n\le x} |f(n) - A(x)|^2 \ll x$.
11. Suppose that f is strongly additive, so that $f(n) = \sum_{p|n} f(p)$ for all n. Consider the bilinear form inequality

$$\sum_{n\le x} \Big| \sum_{p|n} f(p) - \sum_{p\le x} \frac{f(p)}{p} \Big|^2 \le \Delta \sum_{p\le x} \frac{|f(p)|^2}{p} \qquad (23.16) \quad \boxed{\texttt{E:biform1}}$$

in the variables $f(p)$.

(a) By the change of variables $g(p) = f(p)/\sqrt{p}$, show that the above is equivalent to the bilinear form inequality

$$\sum_{n\le x} \Big| \sum_{p|n} g(p)p^{1/2} - \sum_{p\le x} \frac{g(p)}{p^{1/2}} \Big|^2 \le \Delta \sum_{p\le x} |g(p)|^2 . \qquad (23.17) \quad \boxed{\texttt{E:biform2}}$$

(b) Use Theorem **??** to show that the above is equivalent to the bilinear form inequality

$$\sum_{p\le x} p \Big| \sum_{\substack{n\le x\\ p|n}} h(n) - \frac{1}{p} \sum_{n\le x} h(n) \Big|^2 \le \Delta \sum_{n\le x} |h(n)|^2 \qquad (23.18) \quad \boxed{\texttt{E:biform3}}$$

in the variables $h(n)$.

(c) Apply the large sieve, as discussed in §xx.x to show that

$$\sum_{p\le x^{1/2}} p \Big| \sum_{\substack{n\le x\\ p|n}} h(n) - \frac{1}{p} \sum_{n\le x} h(n) \Big|^2 \ll x \sum_{n\le x} |h(n)|^2 .$$

(d) Show that if $x^{1/2} < p \le x$, then

$$\Big| \sum_{\substack{n\le x\\ p|n}} h(n) \Big|^2 \ll \frac{x}{p} \sum_{\substack{n\le x\\ p|n}} |h(n)|^2 .$$

(e) Show that

$$\sum_{x^{1/2}<p\le x} p\Big|\sum_{\substack{n\le x\\ p|n}} h(n)\Big|^2 \ll x\sum_{n\le x}|h(n)|^2\,.$$

(f) Show that

$$\sum_{x^{1/2}<p\le x} \frac{1}{p}\Big|\sum_{n\le x} h(n)\Big|^2 \ll x\sum_{n\le x}|h(n)|^2\,.$$

(g) Deduce that (23.18) and hence also (23.16) hold with $\Delta \ll x$.

23.2 Mean values of multiplicative functions

S:MVMF

Suppose that$f$ is a multiplicative function, which is to say that $f(mn) = f(m)f(n)$ whenever $(m,n)=1$. Hence $f(n) = \prod_{p^k\|n} f\big(p^k\big)$. We let $\mathcal{M}_0$ denote the class of those multiplicative functions f for which $|f(n)|\le 1$ for all n. Our object is to characterize those members of $\mathcal{M}_0$ that have an asymptotic mean value. If f is a real-valued additive function, then $e(tf(n))\in\mathcal{M}_0$, so an ability to compute mean values of multiplicative functions will help us to determine the Fourier transform of the distribution of additive functions. We begin with several simple results concerning (not necessarily multiplicative) arithmetic functions.

T:MVf=g*1 **Theorem 23.2** *If* $f(n)=\sum_{d|n} g(d)$, *if the series* $\sum_{d=1}^\infty g(d)/d$ *converges, say to* a, *and if* $\sum_{d\le x}|g(d)| = o(x)$ *as* $x\to\infty$, *then*

$$S(x) = \sum_{n\le x} f(n) = ax + o(x)\,. \tag{23.19}$$

E:MVf

Proof Clearly

$$S(x) = \sum_{n\le x}\sum_{d|n} g(d) = \sum_{d\le x} g(d)[x/d] = x\sum_{d\le x} g(d)/d + O\Big(\sum_{d\le x}|g(d)|\Big).$$

Thus we have the stated result. □

Co:Wintner **Corollary 23.3** (Wintner) *If* $f(n)=\sum_{d|n} g(d)$ *and* $\sum_{d=1}^\infty |g(d)|/d < \infty$, *then* (23.19) *holds with* $a=\sum_{d=1}^\infty g(d)/d$.

Proof From the hypothesis that $\sum_{d=1}^\infty |g(d)|/d<\infty$, it follows by partial summation that $\sum_{d\le x}|g(d)| = o(x)$. □

Co:MVmultfcn1 **Corollary 23.4** *If f is multiplicative, if*

$$\sum_p \frac{|1-f(p)|}{p} < \infty, \tag{23.20}$$ E:sum|1-f(p)|/p<infty

and if

$$\sum_{\substack{p^k\\ k>1}} \frac{\left|f\left(p^k\right)\right|}{p^k} < \infty,$$

then (23.19) *holds with*

$$a = \prod_p \Big(1-\frac{1}{p}\Big)\Big(1+\frac{f(p)}{p}+\frac{f\left(p^2\right)}{p^2}+\cdots\Big). \tag{23.21}$$ E:MVaform

Proof Let g be the multiplicative function for which $g\left(p^k\right) = f\left(p^k\right) - f\left(p^{k-1}\right)$. Then $f(n) = \sum_{d|n} g(d)$, and

$$\sum_{d=1}^{\infty} \frac{|g(d)|}{d} = \prod_p \Big(1+\frac{|f(p)-1|}{p}+\frac{|f(p^2)-f(p)|}{p^2}+\cdots\Big) < \infty,$$

and thus

$$\sum_{d=1}^{\infty} \frac{g(d)}{d} = \prod_p \Big(1+\frac{f(p)-1}{p}+\frac{f(p^2)-f(p)}{p^2}+\cdots\Big) = a$$

where a is defined by (23.21). □

In the same vein we have

T:fnearg **Theorem 23.5** *If $f(n) = \sum_{d|n} g(d)h(n/d)$, if $\sum_{m=1}^{\infty} |h(m)|/m < \infty$, and if $\sum_{d\le x} g(d) = bx + o(x)$, then we have* (23.19) *with $a = b\sum_{m=1}^{\infty} h(m)/m$.*

Here we see that a mean value for g yields one for f, provided that f is near g in the sense that $\sum |h(m)|/m < \infty$. If $h(1) = 1$ and $h(m) = 0$ for all $m > 1$, then $f = g$.

Proof Put

$$r(x) = \sum_{d\le x} g(d) \; - \; bx\,.$$

Then

$$S(x) = \sum_{n\le x}\sum_{d|n} g(d)h(n/d) = \sum_{m\le x} h(m) \sum_{d\le x/m} g(d)$$
$$= bx\sum_{m\le x}\frac{h(m)}{m} + \sum_{m\le x} h(m)r(x/m)\,.$$

There is a constant C (depending on g) such that $|r(x)| \le Cx$ for all $x \ge 1$, and for every $\varepsilon > 0$ there is a δ such that $|r(x)| \le \varepsilon x$ for all $x \ge 1/\delta$. Thus the second sum above has absolute value not exceeding

$$\varepsilon x \sum_{m\le \delta x} |h(m)|/m \;+\; Cx \sum_{\delta x< m\le x} |h(m)|/m\,.$$

Here the first term is $\ll \varepsilon x$, and the second sum is small since it is part of the tail of a convergent series. Thus we have the stated result. □

In Theorem 23.2 we found a connection between the mean value of f and the convergence of the series $\sum g(d)/d$, but we find it more productive to pursue the line suggested by Corollary 23.4, which we now sharpen.

T:Delange **Theorem 23.6** (Delange) *Suppose that $f \in \mathcal{M}_0$, and that the series*

$$\sum_p \frac{1-f(p)}{p} \tag{23.22}$$ E:sum1-f(p)/pconv

converges. Then (23.19) *holds with a given by* (23.21).

Since $\mathrm{Re}\, f(p) \le |f(p)| \le 1$, we see that the convergence of the series (23.22) implies that the sum of the real parts is absolutely convergent, just as it was in Corollary 23.4. Thus Theorem 23.6 is stronger by virtue of the fact that we are no longer assuming that the sum of the imaginary parts is absolutely convergent. Given the convergence of the product (23.21), we see that $a \ne 0$ unless one of the individual factors vanishes. This happens only in the single case that

$$f\big(2^k\big) = -1 \qquad (k = 1, 2, 3, \ldots)\,. \tag{23.23}$$ E:f(2^k)=-1

Proof We suppose first that in addition to the stated hypotheses, f has the further properties that

$$f\big(p^k\big) = f(p)^k \qquad (k = 1, 2, \ldots), \tag{23.24}$$ E:fTotMult

and that

$$\mathrm{Re}\, f(p) \ge 1/2 \tag{23.25}$$ E:Reflarge

for all p. Once we have established the theorem for such f, we extend the result to general f by an appeal to Theorem 23.5. Let P be a large parameter, let $\mathcal{P}_1$ denote the set of primes not exceeding P, and let $\mathcal{P}_2$ denote the primes larger than P. Let f_i be multiplicative, $f_i(p^k) = f(p)^k$ for $p \in \mathcal{P}_i$, and $f_i(p^k) = 1$ for $p \notin \mathcal{P}_i$. Thus $f = f_1 f_2$, and by Corollary 23.4,

$$\sum_{n \le x} f_1(n) = a(P)x + o(x)$$

with

$$a(P) = \prod_{p \le P} \Big(1 - \frac{1}{p}\Big)\Big(1 - \frac{f(p)}{p}\Big)^{-1}.$$

Since $|f(p^k)| \le 1$, we see by (23.24) and (23.25) that we may write $f_2(n) = e^{g(n)}$ where g is an additive function such that $-\log 2 \le \operatorname{Re} g(p) \le 0$ and $|\operatorname{Im} g(p)| \le \pi/3$. If $\operatorname{Re} z \le 0$, then

$$\big|e^z - 1\big| = \Big|\int_0^z e^w \, dw\Big| \le |z|,$$

so we see that

$$\Big|\sum_{n \le x} f(n) - f_1(n)\Big| \le \sum_{n \le x} \big|f_2(n) - 1\big| \le \sum_{n \le x} |g(n)|.$$

Let $A(x) = A(g, x)$ be defined as in (23.6). Then by Cauchy's inequality the above is

$$\le x|A(x)| + \sum_{n \le x} |g(n) - A(x)| \le x|A(x)| + x^{1/2}\Big(\sum_{n \le x} |g(n) - A(x)|^2\Big)^{1/2},$$

and by the Turán–Kubilius inequality (Theorem 23.1), this is

$$\ll x|A(x)| + xB(x)^{1/2}.$$

We now relate $A(x)$ and $B(x)$ to $\sum_p (1 - f(p))/p$. While the imaginary part of this sum is not necessarily absolutely convergent, the real part of each term is nonnegative, and so the sum of the real parts is absolutely convergent. Also, $|1 - f(p)| = 1 - 2\operatorname{Re} f(p) + |f(p)|^2 \le 2 - 2\operatorname{Re} f(p)$, $g(p^k) \ll k$, and $|g(p)| \asymp |1 - f(p)|$, so that

$$B(x) = \sum_{p^k \le x} \frac{|g(p^k)|}{p^k} \ll \sum_{p > P} \frac{1}{p^2} + \sum_{P < p \le x} \frac{|1 - f(p)|^2}{p}$$
$$\ll \frac{1}{P} + \operatorname{Re} \sum_{P < p \le x} \frac{1 - f(p)}{p}.$$

We also observe that $g(p) = f(p) - 1 + O\big(|1-f(p)|^2\big)$, so that

$$A(x) = \sum_{p^k \le x} \frac{g(p^k)}{p^k}\Big(1 - \frac{1}{p}\Big) = \sum_{P<p\le x} \frac{f(p)-1}{p}$$
$$+ O\Big(\sum_{P<p\le x} \frac{|1-f(p)|^2}{p}\Big) + O\Big(\sum_{P<p\le x} \frac{1}{p^2}\Big)$$
$$\ll \frac{1}{P} + \Big|\sum_{P<p\le x} \frac{1-f(p)}{p}\Big| .$$

On assembling our estimates, we find that

$$S(x) = a(P)x + o(x) + O\big(x/P^{1/2}\big) + O\bigg(x\Big|\sum_{P<p\le x} \frac{1-f(p)}{p}\Big|^{1/2}\bigg).$$

Since P can be arbitrarily large, this gives the desired result, subject to (23.24) and (23.25).

To complete the proof we now suppose only that $f \in \mathcal{M}_0$ and that $\sum_p (1-f(p))/p$ converges. Let $\mathcal{P}$ denote the set of primes p for which $\operatorname{Re} f(p) \le 1/2$. We note that

$$\sum_{p\in\mathcal{P}} \frac{1}{p} \;\le\; 2\operatorname{Re}\sum_p \frac{1-f(p)}{p} \;<\; \infty\,.$$

We define multiplicative functions g and h by the Euler products

$$G(s) \;=\; \sum_{n=1}^\infty \frac{g(n)}{n^s} \;=\; \prod_{p\in\mathcal{P}}\Big(1-\frac{1}{p^s}\Big)^{-1} \prod_{p\notin\mathcal{P}}\Big(1-\frac{f(p)}{p^s}\Big)^{-1},$$
$$H(s) \;=\; \sum_{n=1}^\infty \frac{h(n)}{n^s} \;=\; \prod_{p\in\mathcal{P}}\Big(1-\frac{1}{p^s}\Big)\Big(1+\frac{f(p)}{p^s}+\frac{f(p^2)}{p^{2s}}+\cdots\Big)$$
$$\times \prod_{p\notin\mathcal{P}}\Big(1-\frac{f(p)}{p^s}\Big)\Big(1+\frac{f(p)}{p^s}+\frac{f(p^2)}{p^{2s}}+\cdots\Big).$$

Thus

$$F(s) = \sum_{n=1}^\infty \frac{f(n)}{n^s} = \prod_p \Big(1+\frac{f(p)}{p^s}+\frac{f(p^2)}{p^{2s}}+\cdots\Big) = G(s)H(s)\,.$$

We observe that $g \in \mathcal{M}_0$, $\sum_p (1-g(p))/p$ converges, and that g satisfies (23.24) and (23.25). Hence

$$\sum_{n\le x} g(n) = x\prod_{p\notin\mathcal{P}}\Big(1-\frac{1}{p}\Big)\Big(1-\frac{f(p)}{p}\Big)^{-1} + o(x),$$

and we obtain the desired result from Theorem 23.5, since

$$\sum_{m=1}^{\infty} \frac{|h(m)|}{m} \le \prod_{p\in\mathcal{P}} \Big(1+\frac{2}{p}+\frac{2}{p^2}+\cdots\Big) \prod_{p\notin\mathcal{P}} \Big(1+\frac{2}{p^2}+\frac{2}{p^3}+\cdots\Big)$$
$$\ll \exp\Big(2\sum_{p\in\mathcal{P}}\frac{1}{p}\Big) < \infty\,.$$

□

In Delange's Theorem (Theorem 23.6), the mean value is nonzero unless (23.23) holds. In §23.5 we shall characterize those $f \in \mathcal{M}_0$ with vanishing mean value, in terms of the behaviour of the generating Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \qquad (\sigma > 1)\,. \tag{23.26}$$ E:DefF(s)

To prepare for the proof of our next result we establish a variant of the Hardy–Littlewood tauberian theorem (Theorem 5.7).

L:HLTaub4primes **Lemma 23.7** *Suppose that the numbers $c(p)$ are bounded, and that*

$$\lim_{\sigma\to 1^+} \sum_p \frac{c(p)}{p^\sigma}$$

exists and has the (finite) value c. Then $\sum_p c(p)/p$ converges, and has the value c.

Proof Put

$$a(u) = \sum_{e^{u-1}<p\le e^u} \frac{c(p)}{p}\,.$$

Then $a(u) \ll 1/u$ for $u \ge 1$, and

$$I(u) = \int_0^\infty a(u)e^{-\delta u}\,du = \frac{1-e^{-\delta}}{\delta}\sum_p \frac{c(p)}{p^{1+\delta}}$$

tends to c as $\delta \to 0^+$. Thus by the Hardy–Littlewood tauberian theorem (Theorem 5.7 with $\beta = 0$) it follows that $\int_0^U a(u)\,du$ tends to c as $U \to \infty$. But

$$\int_0^U a(u)\,du = \sum_{p\le e^U} \frac{c(p)}{p} - \sum_{e^{U-1}<p\le e^U} \frac{c(p)}{p}(U-\log p),$$

and the second sum is $\ll 1/U$, so $\sum_p c(p)/p$ converges to c. □

For members of $\mathcal{M}_0$ with nonzero mean value, we have the following comprehensive result.

T:Delange2 **Theorem 23.8** (Delange) *Suppose that $f \in \mathcal{M}_0$, and let $S(x)$, the number a, and the function $F(s)$ be defined as in* (23.19), (23.21) *and* (23.26), *respectively. Then the following assertions are equivalent*:

(a) $S(x) \sim ax$ *and* $a \neq 0$;

(b) $\displaystyle\sum_{n\le x} \frac{f(n)}{n} \sim a \log x$ *and* $a \neq 0$;

(c) $F(\sigma) \sim \dfrac{a}{\sigma - 1}$ *as* $\sigma \to 1^+$ *and* $a \neq 0$;

(d) $\displaystyle\lim_{\sigma\to 1^+} \sum_p \frac{1 - f(p)}{p^\sigma}$ *exists and* (23.23) *fails*;

(e) $\displaystyle\sum_p \frac{1 - f(p)}{p}$ *converges and* (23.23) *fails.*

Proof We deduce (b) from (a) by partial summation, and similarly deduce (c) from (b). But (c) asserts that $\lim_{\sigma\to 1^+} F(\sigma)/\zeta(\sigma) = a$, which is to say that

$$\lim_{\sigma\to 1^+} \prod_p \Big(1 - \frac{1}{p^\sigma}\Big)\Big(1 + \frac{f(p)}{p^\sigma} + \frac{f(p^2)}{p^{2\sigma}} + \cdots\Big) = a \neq 0.$$

Each factor of the product has modulus not exceeding 1, so if (23.23) were to hold, then the limit would be 0. Thus (23.23) fails and the product is comparable to

$$\exp\Big(\sum_p \frac{1 - f(p)}{p^\sigma}\Big).$$

Hence we have (d). That (d) implies (e) is immediate from Lemma 23.7, and that (e) implies (a) follows from Theorem 23.6. □

23.2.1 Exercise

1. Suppose that $\sum_{d=1}^\infty g(d)/d$ converges, say to a.
   (a) Show that $\sum_{d\le x} g(d) = o(x)$.
   (b) Suppose also that $\sum_{d\le x} |g(d)| \ll x$. Use Axer's Theorem (Theorem 8.1) to show that $\sum_{d\le x} g(d)\{x/d\} = o(x)$.

(c) Put $f(n) = \sum_{d|n} g(d)$. Under the above hypotheses, show that $\sum_{n\le x} f(n) = ax + o(x)$. (Note that this improves upon Theorem 23.2.)

23.3 The distribution of additive functions

S:DistAddFcns

We now employ our understanding of the mean values of multiplicative functions to establish

T:E-WThm **Theorem 23.9** (Erdős–Wintner) *Let f be a real-valued additive function. The following are equivalent*:

(a) *Each of the following series is convergent*:

$$\sum_{\substack{p \\ |f(p)|\le 1}} \frac{f(p)}{p}, \qquad \sum_{\substack{p \\ |f(p)|\le 1}} \frac{|f(p)|^2}{p}, \qquad \sum_{\substack{p \\ |f(p)|>1}} \frac{1}{p}. \tag{23.27}$$

E:3series

(b) *There is an increasing function $F(u)$ such that* $\lim_{u\to-\infty} F(u) = 0$, $\lim_{n\to+\infty} = 1$, *and such that*

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{card}\{n \le N : f(n) \le u\} = F(u), \tag{23.28}$$

E:DistFcnf

whenever u is not a point of discontinuity of F.

Since F is increasing, the set of its discontinuities is is at most countable. Later we shall see that if we define F to be right-continuous, so that $F(u) = F(u^+)$, then (23.28) holds for all values of u. When (b) holds we may say that F is the asymptotic distribution of f. Given F with the above properties, there is a unique probability measure μ such that $F(u) = \int_{-\infty}^u 1\,d\mu$. Moreover, we can construct a probability measure μ_N that attaches weight $1/N$ to each of the points $f(n)$ for $1 \le n \le N$. Then (23.28) asserts that

$$\lim_{N\to\infty} \int_{-\infty}^u 1\,d\mu = \int_{-\infty}^u 1\,d\mu,$$

which is to say that the measures μ_N tend weakly to μ. Our proof of the Erdős–Wintner Theorem depends on our discussion in §**??** concerning the weak convergence of measures.

Proof Suppose that (a) holds. By virtue of Theorem I.7, in order to show that (b) holds it suffices to show that

$$\widehat{\mu}_N(t) = \int_{\mathbb{R}} e(-tu)\, d\mu_N(u) = \frac{1}{N}\sum_{n=1}^{N} e(-tf(n)) \qquad (23.29)$$

E:muhatN

has a limit $r(t)$ as $N \to \infty$, and that r is continuous at $t = 0$. Let $g(n) = g_t(n) = e(-tf(n))$. Then $g \in \mathcal{M}_0$, so by Theorem 23.6 the above tends to

$$r(t) = \prod_p \Big(1 - \frac{1}{p}\Big)\Big(1 + \frac{e(-tf(p))}{p} + \frac{e(-tf(p^2))}{p^2} + \cdots\Big), \qquad (23.30)$$

E:rform

provided that the sum

$$\sum_p \frac{1 - e(-tf(p))}{p} \qquad (23.31)$$

E:sumpconv1

converges. The above is

$$\begin{aligned} &= 2\pi it \sum_{|f(p)|\le 1} \frac{f(p)}{p} + \sum_{|f(p)|\le 1} \frac{1 - 2\pi itf(p) - e(-tf(p))}{p} \\ &\quad + \sum_{|f(p)|>1} \frac{1 - e(-tf(p))}{p}. \end{aligned} \qquad (23.32)$$

E:sumpexpanded

By the hypotheses (a) we see that the first sum is a constant, and that the third sum is absolutely and uniformly convergent. Since $e(\theta) = 1 + 2\pi i\theta + O(\theta^2)$, the second sum is absolutely convergent, and uniformly so for t in a bounded set. Thus the sum (23.31) converges. Moreover, the expression (23.32) tends to 0 as $t \to 0$, so $r(t)$ tends to 1, and hence we have (b).

We now show that (b) implies (a). If the μ_N tend weakly to μ, then by Theorem I.6 it follows that $\widehat{\mu}_N(t) \to \widehat{\mu}(t)$. Since $\widehat{\mu}(t)$ is continuous and $\widehat{\mu}(0) = 1$, it follows from (23.29) that the multiplicative function g_t has a non-zero mean value for all t near 0, and that this mean value tends to 1 as $t \to 0$. Hence by Theorem 23.8 we deduce that the sum (23.31) converges for all small t, and tends to 0 as $t \to 0$. Let $s(t)$ denote the real part of the series (23.31). Since each term has non-negative real part, the sum of the real parts is absolutely convergent, and uniformly bounded for $|t| \le \delta$. But then

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} s(t)\, dt = \sum_p \frac{1}{p}\Big(1 - \frac{\sin 2\pi\delta f(p)}{2\pi\delta f(p)}\Big),$$

and hence this latter sum is finite. But

$$1 - \frac{\sin\theta}{\theta} \gg \min(1, \theta^2),$$

so the second and third sums in (23.27) are convergent. Hence the second and third sums in (23.32) are convergent, and since the sum (23.31) is convergent, it follows that the first sum in (23.32) is convergent. □

In the next section we shall find that much can be said about the distribution function F of a real-valued additive function f. At this point we content ourselves with the following simple result.

T:DVC **Theorem 23.10** *Let f be a real-valued additive function with limiting distribution function F. If the series*

$$\sum_{f(p)\neq 0} \frac{1}{p} \tag{23.33}$$ E:sumpconv2

converges, then each value assumed by f is attained on a set of positive density, so that F has jump discontinuities but is otherwise constant (i.e., the associated measure μ is discrete). If the series diverges, then the distribution function F is continuous, and hence any value of f is assumed only on a set of density 0.

Proof Let $\mathcal{P}$ denote the set of primes p for which $f(p) \neq 0$, let $\mathcal{N}_1$ denote the set of positive integers composed entirely of primes $p \in \mathcal{P}$, let $\mathcal{N}_2$ denote the set of integers composed entirely of primes $p \notin \mathcal{P}$ with each prime occurring with multiplicity > 1, and finally let $\mathcal{N}_3$ denote the set of squarefree integers composed entirely of primes $p \notin \mathcal{P}$. Each n can be written uniquely in the form $n = n_1 n_2 n_3$ with $n_i \in \mathcal{N}_i$ and $(n_2, n_3) = 1$, and $f(n)$ depends only on n_1 and n_2. The number of $n \le x$ with prescribed n_1 and n_2 is the number of squarefree $n_3 \le x/(n_1 n_2)$ such that $n_3 \in \mathcal{N}_3$ and $(n_2, n_3) = 1$. By Corollary 23.4, this is

$$\begin{aligned} &\sim \frac{x}{n_1 n_2} \prod_{p\in\mathcal{P}} \Big(1 - \frac{1}{p}\Big) \prod_{p \mid n_2} \Big(1 - \frac{1}{p}\Big) \prod_{\substack{p\notin\mathcal{P} \\ p \nmid n_2}} \Big(1 - \frac{1}{p^2}\Big) \\ &= \frac{x}{n_1 n_2} \prod_{p\in\mathcal{P}} \Big(1 - \frac{1}{p}\Big) \prod_{p \mid n_2} \Big(1 + \frac{1}{p}\Big)^{-1} \prod_{p\notin\mathcal{P}} \Big(1 - \frac{1}{p^2}\Big). \end{aligned}$$

Moreover, these densities sum to 1 as n_1 and n_2 range over $\mathcal{N}_1$ and $\mathcal{N}_2$.

Now suppose that the series (23.33) diverges. By Theorem 23.33 it suffices to show that

$$\int_{-T}^{T} |\widehat{\mu}(t)|^2 \, dt = o(T) \qquad (23.34) \quad \boxed{\texttt{E:avemusmall}}$$

as $T \to \infty$. In the case at hand we know that

$$\widehat{\mu}(t) = \prod_p \Big(1 - \frac{1}{p}\Big)\Big(1 + \frac{e(-tf(p))}{p} + \frac{e(-tf(p^2))}{p^2} + \cdots\Big). \qquad (23.35) \quad \boxed{\texttt{E:muhatform}}$$

Let $\mathcal{P}$ be a finite set of primes for which $f(p) \ne 0$, and put $s = \sum_{p\in\mathcal{P}} 1/p$. In the above product, each prime contributes a factor whose absolute value is ≤ 1. Thus

$$\begin{aligned} |\widehat{\mu}(t)| &\le \prod_{p\in\mathcal{P}} \Big|\Big(1 - \frac{1}{p}\Big)\Big(1 + \frac{e(-tf(p))}{p} + \cdots\Big)\Big| \\ &\ll \prod_{p\in\mathcal{P}} \Big|1 - \frac{1}{p} + \frac{e(-tf(p))}{p}\Big| \ll \exp\Big(-2\sum_{p\in\mathcal{P}} \frac{\sin^2 \pi t f(p)}{p}\Big). \end{aligned}$$

Hence by Hölder's inequality

$$\begin{aligned} \frac{1}{2T}\int_{-T}^{T} |\widehat{\mu}(t)|^2 \, dt &\ll \frac{1}{2T}\int_{-T}^{T} \exp\Big(-4\sum_{p\in\mathcal{P}} \frac{\sin^2 \pi t f(p)}{p}\Big) \, dt \\ &\le \prod_{p\in\mathcal{P}} \Big(\frac{1}{2T}\int_{-T}^{T} \exp(-4s \sin^2 \pi t f(p)) \, dt\Big)^{1/(sp)}. \end{aligned}$$

Suppose that $f(p) > 0$. The integrand has period $1/f(p)$, and

$$\begin{aligned} f(p)\int_0^{1/f(p)} \exp(-4s \sin^2 \pi t f(p)) \, dt &= \int_0^1 \exp(-4s \sin^2 \pi t) \, dt \\ \le \int_{-1/2}^{1/2} \exp(-16st^2) \, dt &\le \int_{-\infty}^{\infty} \exp(-16st^2) \, dt = \frac{\sqrt{\pi}}{4\sqrt{s}}. \end{aligned}$$

Hence

$$\frac{1}{2T}\int_{-T}^{T} \exp(-4s \sin^2 \pi t f(p)) \, dt \le \frac{1}{\sqrt{s}}$$

for all sufficiently large T, and so

$$\limsup_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |\widehat{\mu}(t)|^2 \, dt \le \frac{1}{\sqrt{s}}.$$

By choosing $\mathcal{P}$ suitably, we may make s as large as we please. Thus we have (23.34), and the proof is complete. □

23.3.1 Exercises

1. (a) Show that $\log \sigma(n)/n$ has a limiting distribution.
   (b) Show that this limiting distribution is continuous.
   (c) Deduce that the set of perfect numbers (i.e., those for which $\sigma(n) = 2n$ is a set of density 0.
2. Show that an integer-valued additive function f has a limiting distribution if and only if
$$\sum_{f(p)\neq 0} \frac{1}{p} < \infty\,.$$
3. Let f be a multiplicative function that takes only positive real values. Show that f has a limiting distribution if and only if each of the following four series converges:
$$\sum_{1/2\le|f(p)|\le 2} \frac{1-f(p)}{p}, \quad \sum_{1/2\le|f(p)|\le 2} \frac{|1-f(p)|^2}{p}, \quad \sum_{|f(p)|>2} \frac{1}{p}, \quad \sum_{|f(p)|<1/2} \frac{1}{p}\,.$$
4. Let $f_1, \ldots, f_k$ be real-valued additive functions, and put $\boldsymbol{f}(n) = (f_1(n), \ldots, f_k(n))$. Give necessary and sufficient conditions that $\boldsymbol{f}$ should have a limiting distribution in $\mathbb{R}^k$. Deduce a variant of the Erdős–Wintner Theorem (Theorem 23.9) for complex-valued additive functions.
5. Let f be a real-valued additive function with limiting distribution F, and let μ denote the associated limiting measure. Show that either $\widehat{\mu}(t)$ is never 0, or that its zeros form an arithmetic progression of the form $c(2k + 1)$ for $k \in \mathbb{Z}$.

23.4 Applications of probability theory

S:AppProbThy

Let f be a real-valued additive function, and for each prime p let X_p denote the random variable defined in (23.3). We take the X_p to be independent, and ask whether the random variable X defined in (23.4) exists. In this connection we quote without proof

Theorem 23.11 (Kolmogorov's Three Series Theorem) *Let Y_n be independent random variables. If each of the three series*
$$\sum_n \int_{|Y_n|\le 1} Y_n, \qquad \sum_n \int_{|Y_n|\le 1} \int |Y_n|^2, \qquad \sum_n \int_{|Y_n|>1} 1$$

converges, then the sum $Y = \sum_n Y_n$ *converges almost everywhere. If any one of these series diverges, then the sum* $\sum_n Y_n$ *diverges almost everywhere.*

For our sum (23.4), the conditions of Kolmogorov's theorem are precisely the conditions of part (a) of the Erdős–Wintner Theorem (Theorem 23.9). Hence the random variable X exists precisely when f has a limiting distribution F. In the context of Kolmogorov's Three Series Theorem, when Y exists its Fourier transform is

$$\widehat{Y}(t) = \int e(-tY) = \int \prod_n e(-tY_n) = \prod_n \int e(-tY_n) = \prod_n \widehat{Y}_n(t)$$

by the independence of the Y_n. In the case of the variable X, we find that

$$\begin{aligned}\widehat{X}_p(t) = \int e(-tX_p) &= 1 - \frac{1}{p} + e(-tf(p))\frac{1}{p}\Big(1 - \frac{1}{p}\Big) + \cdots \\ &= \Big(1 - \frac{1}{p}\Big)\Big(1 + \frac{e(-tf(p))}{p} + \frac{e(-tf(p^2))}{p^2} + \cdots\Big),\end{aligned}$$

and hence

$$\widehat{X}(t) = \prod_p \Big(1 - \frac{1}{p}\Big)\Big(1 + \frac{e(-tf(p))}{p} + \frac{e(-tf(p^2))}{p^2} + \cdots\Big).$$

But this is the same as $\widehat{\mu}(t)$ given in (23.35), so by the uniqueness of the Fourier transform (Corollary I.5) it follows that F is the distribution function of X. A great deal is known concerning the distribution function of a sum of random variables, so by appealing to this theory we obtain further information concerning F. In particular, we note the Law of Pure Types:

Theorem 23.12 (Jessen–Wintner) *Let* Y_n *be independent random variables such that* $Y = \sum_n Y_n$ *converges almost everywhere, and suppose that there is a countable set* $\mathcal{C}$ *such that* $P(Y_n \in \mathcal{C}) = 1$ *for all* n*. Then the distribution of* Y *is of pure type: Either it is discrete, singular, or absolutely continuous.*

Hence we see that the distribution function F of a real-valued additive function is of pure type. In Theorem 23.10 we characterized the situation in which the distribution is discrete; this can also be obtained by applying a general theorem of Lévy (get reference) concerning sums of independent random variables. We have no similar criterion to distinguish between singular and absolutely continuous distributions, although all three types do occur. In particular, the distribution of $\log \varphi(n)n$ is singular, as we now show.

T:DistPhiSing **Theorem 23.13** *Let μ denote the probability measure such that*

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{card}\{n \le N : \varphi(n)/n \le c\} = \mu((-\infty, c])\,.$$

Let α be fixed, $1 < \alpha < e-1$, put $I_k = [\varphi(k)/k - 1/k^\alpha, \varphi(k)/k]$, and set

$$\mathcal{S} = \{x \in [0,1] : x \in I_k \text{ for infinitely many } k\}.$$

Then $m(\mathcal{S}) = 0$, and $\mu(\mathcal{S}) = 1$.

Here $m(\mathcal{S})$ denotes the Lebesgue measure of $\mathcal{S}$.

Proof The first assertion is clear, since

$$\mathcal{S} \subseteq \bigcup_{k>K} I_k$$

for any K, so that

$$m(\mathcal{S}) \le \sum_{k>K} k^{-\alpha}\,.$$

As for the second assertion, we show that for any $\varepsilon > 0$ and any K there is an L such that

$$\mu\Big(\bigcup_{K<k\le L} I_k\Big) \ge 1 - \varepsilon\,. \qquad (23.36)$$ E:muDist

The advantage of this finite form of the assertion is that we can estimate the left hand side by considering densities of set of integers: If $\mathcal{N} = \big\{n : \varphi(n)/n \in \bigcup_{K<k\le L} I_k\big\}$, then $d(\mathcal{N}) = \mu\big(\bigcup_{K<k\le L} I_k\big)$. In establishing (23.36), we may assume that K is large, for if the above holds for one value of K, then it also holds for all smaller values of K. For a given number n, write $n = \prod_{i=1}^{\Omega(n)} p_i$ with $p_1 \le p_2 \le \cdots \le p_{\Omega(n)}$, and set $d_r = \prod_{i\le r} p_i$. We shall show that if L is sufficiently large, then most integers n have a divisor d_r, $K < d_r \le L$, such that

$$\frac{\varphi(n/d_r)}{n/d_r} \ge 1 - d_r^{-\alpha}.$$

In this case $\varphi(n)/n \in I_{d_r}$, since

$$\frac{\varphi(d_r)}{d_r} \ge \frac{\varphi(n)}{n} \ge \frac{\varphi(d_r)}{d_r}\,\frac{\varphi(n/d_r)}{n/d_r} \ge \frac{\varphi(d_r)}{d_r}\big(1 - d_r^{-\alpha}\big) \ge \frac{\varphi(d_r)}{d_r} - d_r^{-\alpha}\,.$$

Let $\mathcal{N}_0$ denote the complementary set of numbers, i.e., the n for which $\varphi(n/d_r)/(n/d_r) < 1 - d_r^{-\alpha}$ for all $d_r \in (K, L]$. To estimate the size of $\mathcal{N}_0$ we consider various possibilities. Let $\mathcal{N}_1$ be the set of n such that the interval $(K, \log L]$ contains none of the special divisors d_r. Let $\beta = \alpha/2 + (e-1)/2$, so that $1 < \alpha < \beta < e-1$, and let $\mathcal{N}_2$ be the set of numbers $n$such that $p_{r+1} < d_r^\beta$ whenever $d_r \in (K, L]$. Finally, let $\mathcal{N}_3$ be the set of those n such that there is a $d_r \in (K, L]$ for which $p_{r+1} > d_r^\beta$ and $\varphi(n/d_r)/(n/d_r) < 1 - d_r^{-\alpha}$. The sets $\mathcal{N}_i$ possess asymptotic densities, but for our present purpose it suffices to bound their upper asymptotic densities where the upper asymptotic density of a set $\mathcal{A}$ is

$$\overline{d}(\mathcal{A}) = \limsup_{x\to\infty} \frac{1}{x} \operatorname{card}\{n \le x : n \in \mathcal{A}\}.$$

The main estimates to be established are that

$$\overline{d}(\mathcal{N}_1) \ll \frac{\log K}{\log\log L}, \tag{23.37}$$ E:DensityN1

$$\overline{d}(\mathcal{N}_1^c\mathcal{N}_2) \ll \frac{1}{\log\log L}, \tag{23.38}$$ E:DensityN2

$$\overline{d}(\mathcal{N}_3) \ll K^{\alpha-e+1}. \tag{23.39}$$ E:DensityN3

Once these estimates are in place, we argue that $\mathcal{N}_2 = \mathcal{N}_1\mathcal{N}_2 \cup \mathcal{N}_1\mathcal{N}_2^c \subseteq \mathcal{N}_1 \cup \mathcal{N}_1^c\mathcal{N}_2$. Since $\overline{d}(\mathcal{A}\cup\mathcal{B}) \le \overline{d}(\mathcal{A}) + \overline{d}(\mathcal{B})$, it follows from (23.37) and (23.38) that

$$\overline{d}(\mathcal{N}_2) \ll \frac{\log K}{\log\log L}. \tag{23.40}$$ E:DensityN2Est

We also observe that $\mathcal{N}_0\mathcal{N}_2^c \subseteq \mathcal{N}_3$. Thus $\mathcal{N}_0 = \mathcal{N}_0\mathcal{N}_2 \cup \mathcal{N}_0\mathcal{N}_2^c \subseteq \mathcal{N}_2 \cup \mathcal{N}_3$, so from (23.39) and (23.40) we deduce that

$$\overline{d}(\mathcal{N}_0) \le \overline{d}(\mathcal{N}_2) + \overline{d}(\mathcal{N}_3) \ll K^{\alpha-e+1} + \frac{\log K}{\log\log L}.$$

Thus $\overline{d}(\mathcal{N}_0) < \varepsilon$ if K is sufficiently large and if L is sufficiently large compared with K.

To prove (23.37), we suppose, as we may, that $L > \exp(K^2)$. For $n \in \mathcal{N}_1$, choose r so that $d_r < K$ and $d_{r+1} > \log L$. Thus $n = d_r m$ with m composed entirely of primes $> (\log L)/d_r$. This decomposition is unique, since d_r is composed entirely of primes $< K$, and m is composed entirely of primes $> (\log L)/d_r > K^2/d_r \ge K$. Hence

$$\operatorname{card}\{n \le x : n \in \mathcal{N}_1\} = \sum_{d<K} \operatorname{card}\{m \le x/d : p|m \implies p > (\log L)/d\}.$$

By the theorem of Eratosthenes–Legendre (Theorem 3.1), this is

$$\sim x\sum_{d<K}\frac{1}{d}\prod_{p<\frac{\log L}{d}}\Big(1-\frac{1}{p}\Big)\le x\Big(\sum_{d<K}\frac{1}{d}\Big)\prod_{p<\frac{\log L}{K}}\Big(1-\frac{1}{p}\Big)$$
$$\ll x\frac{\log K}{\log((\log L)/K)},$$

so we have (23.37).

As for (23.38), suppose that $n\in\mathcal{N}_1^c\mathcal{N}_2$ and that $d_r\in(K,L]$. Then $d_{r+1}=d_rp_{r+1}\le d_r^{1+b}$. Thus by induction, if r_0 is the least r for which $d_r>K$, then

$$d_r\le d_{r_0}^{(1+b)^{r-r_0}}<(\log L)^{(1+b)^{r-r_0}}$$

provided that this bound is $\le L$. Set

$$R=\Big[\frac{\log\log L-\log\log\log L}{\log(1+b)}\Big].$$

Then $d_r\le L$ for $r\le r_0+R$. Let

$$\Omega_L(n)=\sum_{\substack{p^k\|n\\ p\le L}}1.$$

Thus $\Omega_L(n)\ge R$ if $n\in\mathcal{N}_2$. By the Turán–Kubilius inequality (Theorem 23.1),

$$\sum_{n\le x}(\Omega_L(n)-\log\log L)^2\ll x\log\log L.$$

Let $c=1/\log(1+\beta)$. Here $c>1$, since $1+\beta<e$, and $R>(c-\varepsilon)\log\log L$ if L is sufficiently large. Thus $\Omega_L(n)-\log\log L\gg\log\log L$ when $n\in\mathcal{N}_2$, and so we have (23.38).

If $n\in\mathcal{N}_3$, then we may write $n=dm$ where $K<d\le L$, $p|m$ implies $p>d^\beta$, and $\varphi(m)/m<1-d^{-\alpha}$. This decomposition may not be unique, but

$$\operatorname{card}\{n\le x : n\in\mathcal{N}_3\}$$
$$\le\sum_{K<d\le L}\operatorname{card}\{m\le x/d : p|m\implies p>d^\beta,\ \varphi(m)/m<1-d^{-\alpha}\}. \tag{23.41}$$

E:DensityN3Est

Let

$$f_y(m)=\sum_{\substack{p|n\\ p>y}}\log(1-1/p)^{-1}.$$

This is an additive function with

$$A(f_y, z) \le (1 + o(1))(y \log y)^{-1}, \qquad B(f_y, z) \ll y^{-2}(\log y)^{-1}\,.$$

Thus if $V \ge 2/(y \log y)$, then by the Turán–Kubilius inequality we see that

$$\operatorname{card}\{m \le z \,:\, f_y(n) > V\} \ll \frac{z}{V^2 y^2 \log y}\,.$$

On taking $z = x/d$, $y = d^\beta$, $V = \log(1 - d^{-\alpha})^{-1} \asymp d^\alpha$, we see that the $m \le x/d$ for which $f_y(m) > V$ includes the m in (23.41), and hence

$$\operatorname{card}\{n \le x \,:\, n \in \mathcal{N}_3\} \ll x \sum_{K<d\le L} d^{-1+2\alpha-2\beta}(\log d)^{-1} \ll \frac{xK^{2\alpha-2\beta}}{\log K}\,.$$

This gives (23.39), in view of the definition of β. Thus the proof is complete. □

23.5 Multiplicative functions with vanishing mean value

S:MFMV0

Suppose that

$$S(x) = \sum_{n\le x} f(n)\,. \tag{23.42}$$

E:DefS(x)

If $S(x) \ll x$, then by Theorem 1.3 the Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} f(n) n^{-s} \tag{23.43}$$

E:DefGenFcnF

converges for $\sigma > 1$, and

$$F(s) = s\int_1^\infty S(x)x^{-s-1}\,dx$$

for $\sigma > 1$. From this formula it is immediate that if $S(x) = ax + o(x)$, then

$$F(s) = \frac{a}{s-1} + o\Big(\frac{\tau}{\sigma - 1}\Big)$$

as $\sigma \to 1^+$. This is a simple abelian theorem. In prior discussions of tauberian converses, such as in §5.2, we imposed a bound on the size of $f(n)$ so that $S(x)$ could not change to quickly. In the present context, the hypothesis that $|f(n)| \le 1$ for all n does not yield a converse (cf Exercise 28.5.1.1), but we find that the hypothesis that $f \in \mathcal{M}_0$ *is* sufficient. The lesson is that for $f \in \mathcal{M}_0$, the quantity $|S(x)|$ changes more slowly on average than it might under the weaker assumption that $|f(n)| \le 1$.

T:MFMV0 **Theorem 23.14** *Suppose that $f \in \mathcal{M}_0$, let $S(x)$ and $F(s)$ be defined by* (23.42) *and* (23.43), *and for $\alpha > 0$ put*

$$M(\alpha) = \Big(\sum_{k=-\infty}^{\infty} \frac{1}{k^2+1} \max_{\substack{\sigma \ge 1+\alpha \\ |t-k| \le 1/2}} |F(s)|^2 \Big)^{1/2} .$$

Then

$$S(x) \ll \frac{x}{\log x} \int_{1/\log x}^{1} \frac{M(\alpha)}{\alpha}\, d\alpha . \tag{23.44}$$ E:MVEst

From the trivial bound $F(s) \ll 1/(\sigma-1)$ it follows that $M(\alpha) \ll 1/\alpha$, and when this is inserted in (23.44) we find that $S(x) \ll x$, which is also trivial. However, if for every T we have $F(s) = o(1/(\sigma-1))$ uniformly for $|t| \le T$, then $M(\alpha) = o(1/\alpha)$, and hence $S(x) = o(x)$.

We show below that

$$M(\alpha) \gg 1 \tag{23.45}$$ E:M>>1

uniformly for $f \in \mathcal{M}_0$. Thus the right hand side of (23.44) is

$$\gg x \frac{\log\log x}{\log x} .$$

That this should be the limit of the method is not surprising, in view of the example considered in Exercise 27.5.1.2, for which $f \in \mathcal{M}_0$, $M(\alpha) \asymp 1$, and yet there is a large x for which $|S(x)| \gg x(\log x)^{-1} \log\log x$.

To establish (23.45), we write

$$F(s) = (1 + D(s))G(s)H(s) \tag{23.46}$$ E:DecompF

where

$$D(s) = \sum_{k=1}^{\infty} \frac{f(2^k)}{2^{ks}}, \ G(s) = \prod_{p>2} \Big(1 - \frac{f(p)}{p^s}\Big)^{-1},$$
$$H(s) = \prod_{p>2} \Big(1 - \frac{f(p)}{p^s}\Big)\Big(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \cdots\Big) .$$

Here

$$\Big(1-\frac{f(p)}{p^s}\Big)\Big(1+\frac{f(p)}{p^s}+\frac{f(p^2)}{p^{2s}}+\cdots\Big)$$
$$=1+\frac{f(p^2)-f(p)^2}{p^{2s}}+\frac{f(p^3)-f(p)f(p^2)}{p^{3s}}+\cdots,$$

which is 1 plus an amount not exceeding $2p^{-2\sigma}(1-p^{-\sigma})^{-1}$ in absolute value. Thus the product $H(s)$ is absolutely and uniformly convergent for $\sigma \ge 2/3$, and so

$$\log H(s) \ll 1 \qquad (\sigma \ge 2/3). \tag{23.47}$$ E:H<<1

Choose a real number t_0 with $|t_0| \le \pi/\log 2$, so that $f(2)/2^{it_0}$ is positive real. Then $\operatorname{Re} f(2)/2^{it} \ge 0$ for $|t-t_0| \le \pi/(2\log 2)$, and so $|1+D(s)| \ge 1/2$ for $\sigma \ge 1$, $|t-t_0| \le \pi/(2\log 2)$. Also,

$$\int_{t_0-1}^{t_0+1} \log G(\sigma+it)\,dt \;=\; \sum_{p>2}\sum_{k=1}^{\infty}\frac{f(p)^k}{kp^{k(1+\alpha)}}\int_{t_0-1}^{t_0+1} p^{-ikt}\,dt$$
$$\ll\; \sum_{p}\sum_{k=1}^{\infty}\frac{1}{k^2p^k\log p} \;\ll\; 1\,.$$

Thus there is an absolute constant $c>0$, and a t_1, $|t_0-t_1| \le 1$, such that $|1+D(1+\alpha+it_1)| \ge 1/2$ and $|G(1+\alpha+it_1)| \ge c$, so we have (23.45).

Proof of Theorem 23.14 We shall establish the two main estimates

$$S(x) \ll \frac{x}{\log x}\int_1^x \frac{|S(u)|}{u^2}\,du + \frac{x\log\log x}{\log x}, \tag{23.48}$$ E:S(x)Est1

$$\int_1^x \frac{|S(u)|\log u}{u^2}\,du \ll M(2/\log x)\log x\,. \tag{23.49}$$ E:S(x)Est2

These suffice to give the stated result, since from (23.49) it is evident that

$$\int_{x^{1/2}}^x \frac{|S(u)|}{u^2}\,du \ll M(2/\log x) \ll \int_{1/\log x}^{2/\log x}\frac{M(\alpha)}{\alpha}\,d\alpha\,.$$

We replace x by $x^{1/2^k}$ and sum over k to show that

$$\int_1^x \frac{|S(u)|}{u^2}\,du \ll \int_{1/\log x}^{1}\frac{M(\alpha)}{\alpha}\,d\alpha\,.$$

We insert this in (23.48) to obtain the stated result. The second term in (23.48) can be neglected, in view of (23.45).

To establish (23.48) we first observe that

$$(\log x)\sum_{n\le x} f(n) - \sum_{n\le x} f(n)\log n = \sum_{n\le x} f(n)\log x/n \ll \sum_{n\le x}\log x/n \ll x\,. \tag{23.50}$$

E:f(n)logx/n

Furthermore,

$$\begin{aligned}\sum_{n\le x} f(n)\log n &= \sum_{n\le x} f(n)\sum_{d|n}\Lambda(d)\\ &= \sum_{d\le x}\Lambda(d)\sum_{m\le x/d} f(md)\\ &= \sum_{p\le x}(\log p)\sum_{m\le x/p} f(mp) + O\Big(x\sum_{\substack{p^k\le x\\ k>1}}\frac{\log p}{p^k}\Big)\\ &= \sum_{p\le x}(\log p) f(p) S(x/p)\\ &\quad + O\Big(\sum_{p\le x}(\log p)\sum_{m\le x/p}|f(mp)-f(m)f(p)|\Big) + O(x)\,.\end{aligned}$$

Since $f(mp) = f(m)f(p)$ unless $p|m$, we see that the sum over $m \le x/p$ is $\ll x/p^2$, and so the first error term above is $\ll x$. On combining this with (23.50), we deduce that

$$S(x)\log x \ll x + \sum_{p\le x}|S(x/p)|\log p\,. \tag{23.51}$$

E:S(x)Est3

Here we have a bound for $|S(x)|$ in terms of S at smaller arguments. The trivial bound for either side is $x\log x$. Thus if it were the case that $S(x)$ were of the order of x, then $S(x/p)$ would have to be of the order of x/p for many primes p. If the primes were exactly uniformly distributed, then the sum over p would be

$$\int_1^x |S(x/v)|\,dv = \int_1^x \frac{|S(u)|}{u^2}\,du\,.$$

Of course the primes are rather irregularly distributed, but as x varies the points x/p also move, so by averaging over x we can pass to a smoother average of $|S|$. Suppose that $X > Y > Z \ge 2$. Since $|S(X) - S(x)| \le |X - x| + 1$, we see that

$$|S(X)|\log X \ll \frac{1}{Y}\int_X^{X+Y} |S(x)|\log x\,dx + Y\log X\,.$$

By (27.50) this is

$$\ll X + Y\log X + \frac{1}{Y}\int_X^{X+Y}\sum_{p\le x}|S(x/p)|\log p\;dx\,.$$

We bound the contribution of the smaller primes trivially:

$$\sum_{p\le X/Z}|S(x/p)|\log p \ll x\sum_{p\le X/Z}\frac{\log p}{p} \ll X\log X/Z\,.$$

As for the contribution of the larger primes, we note that

$$\begin{aligned}\int_X^{X+Y}\sum_{X/Z<p\le 2X}|S(x/p)|\log p\;dx &= \sum_{X/Z<p\le 2X}\int_X^{X+Y}|S(x/p)|\,dx\;\log p\\ &= \sum_{X/Z<p\le 2X}\int_{X/p}^{(X+Y)/p}|S(u)|\,du\;p\;\log p\\ &= \int_1^{2Z}|S(u)|\sum_{\substack{X/Z<p\le 2X\\ X/u<p\le (X+Y)/u}} p\log p\;du\,.\end{aligned}$$

Here we can restrict to $u \le 2Z$ because the two intervals that p must lie in are disjoint if $u > 2Z$. In estimating the above, we now drop the condition $X/Z < p \le 2X$. The remaining condition stipulates that p must lie in an interval whose length is $Y/u \ge Y/(2Z) \ge 2$ if $Z \le Y/4$. Thus by the Brun–Titchmarsh inequality (Corollary 3.4) the number of primes in the interval is bounded by the length of the interval divided by the logarithm of its length. Hence the above sum over primes is

$$\ll \frac{XY\log X/u}{u^2\log Y/u}\,.$$

Here the quotient of logarithms is an increasing function of u, so the above is uniformly

$$\ll \frac{XY\log X/(2Z)}{u^2\log Y/(2Z)}\,.$$

We take $Y = X/\log X$ and $Z = X/(\log X)^2$, and on assembling our estimates discover that

$$S(X)\log X \ll X\int_1^X\frac{|S(u)|}{u^2}\,du + X\log\log X\,.$$

That is, we have (23.48).

Finally we prove (23.49). Let $S_1(x) = \sum_{n\le x} f(n)\log n$. By (23.45) and (23.50) it suffices to show that

$$\int_1^x \frac{|S_1(u)|}{u^2}\,du \ll M(2/\log x)\log x\,. \tag{23.52}$$

E:S1Est1

By the Cauchy–Schwarz inequality,

$$\int_1^x \frac{|S_1(u)|}{u^2}\,du \le \Big(\int_1^x \frac{|S_1(u)|^2}{u^3}\,du\Big)^{1/2}\Big(\int_1^x \frac{1}{u}\,du\Big)^{1/2}.$$

It now suffices to show that

$$\int_1^\infty \frac{|S_1(u)|^2}{u^{3+2\alpha}}\,du \ll \frac{M(\alpha)^2}{\alpha} \tag{23.53}$$

E:S1Est2

for $0 < \alpha \le 1$, since we obtain (23.52) by taking $\alpha = 2/\log x$. By Plancherel's formula as in (5.26), we see that

$$\begin{aligned}\int_1^\infty \frac{|S_1(u)|^2}{u^{3+2\alpha}}\,du &= \frac{1}{2\pi}\int_{-\infty}^\infty \Big|\frac{F'(1+\alpha+it)}{1+\alpha+it}\Big|\,dt \\ &\ll \sum_{k=-\infty}^\infty \frac{1}{k^2+1}\int_{k-1/2}^{k+1/2} |F'(1+\alpha+it)|^2\,dt\,.\end{aligned}$$

We multiply and divide by $|F(1+\alpha+it)|^2$ to see that the above is

$$\le \sum_{k=-\infty}^\infty \frac{\mu(k)}{k^2+1}\int_{k-1/2}^{k+1/2} \Big|\frac{F'}{F}(1+\alpha+it)\Big|^2\,dt$$

where

$$\mu(k) = \max_{|t-k|\le 1/2} |F(1+\alpha+it)|^2\,.$$

Thus to obtain (23.53) it suffices to show that

$$\int_{k-1/2}^{k+1/2} \Big|\frac{F'}{F}(1+\alpha+it)\Big|^2\,dt \ll \frac{1}{\alpha}\,. \tag{23.54}$$

E:Int|F'/F|^2Est

By (23.46) we see that

$$\frac{F'}{F}(s) = \frac{D'(s)}{1+D(s)} + \frac{G'}{G}(s) + \frac{H'}{H}(s)\,.$$

From (23.47) we deduce that $\frac{H'}{H}(s) \ll 1$ uniformly for $\sigma \ge 1$. By Theorem J.1 we see that

$$\int_{k-1/2}^{k+1/2} \Big|\frac{G'}{G}(1+\alpha+it)\Big|^2\,dt \le 3\int_{-1/2}^{1/2} \Big|\frac{\zeta'}{\zeta}(1+\alpha+it)\Big|^2\,dt\,.$$

If $0 < \alpha \le 1$, then by Theorem 6.7 this latter integral is

$$\ll \int_{-1/2}^{1/2} |\alpha + it|^{-2}\, dt \ll \frac{1}{\alpha}\,.$$

Clearly $D'(s) \ll 1$ for $\sigma \ge 1$. For $\sigma > 1$ we write

$$\frac{1}{1 + D(s)} = \sum_{j=0}^{\infty} (-D(s))^j\,.$$

This is a Dirichlet series whose coefficients do not exceed those of

$$1 + \sum_{j=1}^{\infty} \Big(\sum_{k=1}^{\infty} 2^{-ks} \Big)^j = 1 + \sum_{j=1}^{\infty} (2^s - 1)^{-j} = \frac{2^s - 1}{2^s - 2}\,.$$

Hence by Lemma **??**,

$$\int_{k-1/2}^{k+1/2} |1 + D(1 + \alpha + it)|^{-2}\, dt \le 3 \int_{-1/2}^{1/2} \Big| \frac{2^{1+\alpha+it} - 1}{2^{1+\alpha+it} - 2} \Big|^2 dt\,. \quad (23.55)$$ E:|D|^-2Est

Here $|2^{1+\alpha+it} - 1| \asymp 1$ uniformly for $0 \le \alpha \le 1$, and

$$2^{1+\alpha+it} - 2 = 2(\log 2) \int_0^{\alpha+it} 2^s\, ds\,.$$

This integrand has real part $\ge 1/2$ for $\sigma \ge 0$ and $|t| \le \pi/(3\log 2) = 1.5107867\ldots$, so $|2^{1+\alpha+it} - 2| \gg |\alpha + it|$ for $\alpha \ge 0$ and $|t| \le 1/2$. Thus the right hand side of (23.55) is

$$\ll \int_{-1/2}^{1/2} |\alpha + it|^{-2}\, dt \ll \frac{1}{\alpha},$$

so we have (23.54), and the proof is complete. □

We comment that the first part of our proof is reminiscent of the elementary proof of the Prime Number Theorem, as found in §8.2. Moreover, the identity on the left hand side of (23.50) is equivalent to integrating by parts in Perron's formula, as

$$\frac{\log x}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{F(s)x^s}{s}\, ds = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{F(s)x^s}{s^2}\, ds - \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{F'(s)x^s}{s}\, ds\,.$$

This is expected to produce a gain, since we expect that $F(s)/s$ is not generally very rapidly changing, while x^s is spinning fairly rapidly. Indeed, suppose we tried to do something as simple as using Perron's formula to show that $[x] \ll x$. Since

$$\int_{-1}^{1} |\zeta(1+\alpha+it)| x^{1+\alpha}\, dt \asymp x^{1+\alpha} \log \frac{1}{\alpha},$$

we are unable to obtain a bound better than $x \log\log x$. On the other hand, if we were to use Perron's formula to show that $\sum_{n\le x} \log n \ll x\log x$, we fare better, since

$$\int_{-1}^{1} |\zeta'(1+\alpha+it)| x^{1+\alpha}\, dt \asymp \frac{x^{1+\alpha}}{\alpha},$$

and we can take $\alpha = 1/\log x$. In both of these approaches, we would still have the problem that the kernel in Perron's formula decays only like an inverse first power. This could be overcome by smoothing, but in the argument just completed we avoided that problem by averaging over x, which allows us to appeal to Plancherel's identity.

The proof just completed depends only on properties of the zeta function in a neighbourhood of $s = 1$, but if we take $f(n) = \mu(n)$, so that $F(s) = 1/\zeta(s)$, then the further information that $\zeta(1 + it) \ne 0$ allows us to deduce that $M(x) = o(x)$.

We now relate the behaviour of $F(s)$ to the values of $f(p)$.

T:HalaszThm **Theorem 23.15** (Halász) *Suppose that $f \in \mathcal{M}_0$, and let $S(x)$ and $F(s)$ be defined as in* (23.42) *and* (23.43). *Then the following are equivalent:*

(a) $S(x) = o(x)$ *as* $x \to \infty$;
(b) *For each* $T > 0$, $F(s) = o(1/(\sigma - 1))$ *as* $\sigma \to 1^+$ *uniformly for* $|t| \le T$;
(c) *For each fixed* t, $F(\sigma + it) = o(1/(\sigma - 1))$ *as* $\sigma \to 1^+$;
(d) *For each* t, *at least one of the following holds*:

$$\text{(i)}\quad \sum_p \frac{1 - \operatorname{Re}(f(p)p^{-it})}{p} = +\infty, \qquad\qquad (23.56)$$
$$\text{(ii)}\quad f(2^k) = -2^{ikt} \text{ for } k = 1, 2, \ldots .$$

E:Conditions

Moreover, there is at most one real number t for which (23.56)(i) *fails, and at most one real number t for which* (23.56)(ii) *is true.*

Delange's Theorem (Theorem 23.8), when combined with Halász's Theorem (Theorem 23.15) is rather comprehensive, for if (23.56) fails and

$$\sum_p \frac{\mathrm{Im}(f(p)p^{-it})}{p} \tag{23.57}$$

E:ConvSum

converges, then by Theorem 23.8 we have

$$\sum_{n\le x} f(n)n^{-it} = ax + o(x)$$

where

$$a = \prod_p \Big(1 - \frac{1}{p}\Big)\Big(1 + \frac{f(p)}{p^{1+it}} + \frac{f(p^2)}{p^{2+2it}} + \cdots\Big)$$

is nonzero, and by partial summation,

$$S(x) = \frac{a}{1+it}x^{1+it} + o(x)\,.$$

In the one remaining case, in which (23.56) fails for some t, and (23.57) does not hold for that t, then with more work it can be shown that

$$S(x) = \frac{x^{1+it}}{1+it}\prod_{p\le x}\Big(1 - \frac{1}{p}\Big)\Big(1 + \frac{f(p)}{p^{1+it}} + \frac{f(p^2)}{p^{2+2it}} + \cdots\Big) + o(x)\,. \tag{23.58}$$

E:AltEstS(x)

Proof That (a) implies (b) was the subject of the opening remarks of this section. That (b) implies (a) was established in a strong quantitative form in Theorem 23.14. To see that (c) and (d) are equivalent, we recall the decomposition (23.46). By (23.47) we know that $H(s) \asymp 1$ uniformly for $\sigma > 1$. Also, $\lim_{\sigma\to 1} D(s) = D(1+it)$, and $1 + D(1+it) = 0$ if and only if (23.56)(ii) holds. Finally,

$$|G(s)(\sigma-1)| \asymp \Big|\frac{G(s)}{\zeta(\sigma)}\Big| \asymp \exp\Big(-\sum_p \frac{\mathrm{Re}(1-f(p)p^{-it})}{p}\Big)\,.$$

Here the summands are nonnegative, so the expression is bounded, and tends to 0 if and only if the sum tends to infinity. But since the summands are nonnegative, this is equivalent to (23.56)(i). Thus (c) and (d) are equivalent. Next we show that (d) implies (b). Suppose first that (23.56)(i) holds for all t in an interval $[T_1, T_2]$. We observe that

$$F(s)(\sigma-1) \ll \exp\Big(-\sum_p \frac{\mathrm{Re}(1-f(p)p^{-it})}{p^\sigma}\Big)\,.$$

The function on the right hand side decreases to 0 as $\sigma \to 1^+$. Thus we obtain (b) for the interval $[T_1, T_2]$ by appealing to the following elementary consequence of compactness: If $r(\sigma, t)$ is continuous in t for each fixed $\sigma > 1$, and if for each fixed $t \in [T_1, T_2]$ the function $r(\sigma, t)$ is monotonically decreasing to 0 as $\sigma \to 1^+$, then $r(\sigma, t)$ tends to 0 as $\sigma \to 1^+$ uniformly for $t \in [T_1, T_2]$. Now suppose that (23.56)(i) fails for $t = t_0$, but that (23.56)(ii) holds for $t = t_0$. Then for $|t - t_0| \le 1$ we have

$$F(s)(\sigma - 1) \ll |s - 1 - it_0| \exp\Big(- \sum_p \frac{\operatorname{Re}(1 - f(p)p^{-it})}{p^\sigma}\Big) .$$

Again the right hand side decreases monotonically to 0 as $\sigma \to 1^+$, since (23.56)(i) holds for $0 < |t - t_0| \le 1$. Thus by the compactness principle again, we have (b) uniformly for $|t - t_0| \le 1$. Thus (d) implies (b). Since (b) clearly implies (c), we have shown that (a)–(d) are equivalent.

As for the last assertion, let $t_1 < t_2$ be fixed real numbers, and let $\mathcal{P}$ be the set of primes p for which $\arg p^{i(t_2 - t_1)} \in [2\pi/3, 4\pi/3] \pmod{2\pi}$. That is, if

$$I_k = [\exp(2\pi(k + 1/3)/(t_2 - t_1)), \exp(2\pi(k + 2/3)/(t_2 - t_1))],$$

then $\mathcal{P}$ consists of those primes such that $p \in I_k$ for some k. By the Prime Number Theorem we see that $\sum_{p \in I_k} 1/p \asymp 1/k$ for all large k. Hence $\sum_{p\in\mathcal{P}} 1/p = +\infty$. If $\operatorname{Re}(1 - f(p)p^{-it}) \le 1/2$, then $|\arg f(p)p^{-it}| \le \pi/3$. If this holds for both t_1 and t_2, then $|\arg p^{i(t_1 - t_2)}| \le 2\pi/3$. Thus if $p \in \mathcal{P}$, then the inequality $\operatorname{Re}(1 - f(p)p^{-it_j}) \le 1/2$ fails for at least one value of j, and so

$$\sum_{p\in\mathcal{P}} \frac{1}{p} \le 2 \sum_p \frac{\operatorname{Re}(1 - f(p)p^{-it_1})}{p} + 2 \sum_p \frac{\operatorname{Re}(1 - f(p)p^{-it_2})}{p} .$$

Consequently at most one of the sums on the right is convergent, and the proof is complete. □

Suppose that $f \in \mathcal{M}_0$. If there is a point of the unit circle $|z| = 1$ that is not a limit point of the numbers $f(p)$, then for any $t \ne 0$ there is a delta such that

$$\sum_{\substack{p \\ \operatorname{Re}(1 - f(p)p^{-it}) > \delta}} \frac{1}{p} = +\infty,$$

so (23.56)(i) holds for all $t \ne 0$. In closing we mention a commonly occurring situation.

Co:MFMVSpecCase **Corollary 23.16** *Suppose that* $f \in \mathcal{M}_0$*, and that there is a constant* $c > 0$ *such that* $|\operatorname{Im} f(p)| \le c \operatorname{Re}(1 - f(p))$*. Then*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} f(n)$$

exists.

Proof In view of the remark made prior to this Corollary, the condition (23.56)(i) holds for all $t \ne 0$. If (23.56) holds when $t = 0$, then by Theorem 23.15 the mean value tends to 0. Otherwise,

$$\begin{aligned}\sum_p \frac{|1 - f(p)|}{p} &\le \sum_p \frac{\operatorname{Re}(1 - f(p)) + |\operatorname{Im} f(p)|}{p}\\ &\le (c+1)\sum_p \frac{\operatorname{Re}(1 - f(p))}{p} < \infty,\end{aligned}$$

so the mean value exists and is non-zero, by Corollary 23.4. □

23.5.1 Exercises

1. Put $f(n) = 1$ for $N < n \le 2N$, $f(n) = 0$ otherwise, and let $F(s)$ be defined as in (23.26).
   (a) By Theorem 1.12, or otherwise, show that $F(s) \ll 1 + \tau/N$ uniformly for $\sigma \ge 1$.
   (b) Note that $S(x) \asymp x$ when $x = 2N$.
2. (Montgomery 1978) (a) Let $f_0(n) = i^{\Omega(n)}$. By Theorem 7.18, or otherwise, show that

$$S_0(x) = \sum_{n\le x} f_0(n) = cx(\log x)^{i-1} + O(x(\log x)^{-2})$$

   where

$$c = \frac{1}{\Gamma(i)} \prod_p \Big(1 - \frac{i}{p}\Big)^{-1}\Big(1 - \frac{1}{p}\Big)^i.$$

   (b) Let $F_0(s) = \sum_{n=1}^\infty f_0(n)n^{-s}$. By means of Theorem 6.7, or otherwise, show that $F_0(s) \ll \log\tau$ uniformly for $\sigma > 1$.
   (c) Let f be a totally multiplicative function with

$$f(p) = \begin{cases} i & (\text{if } \le x^{1/2} \text{ or } p > x),\\ e(\theta_p) & (\text{for } x^{1/2} < p \le x)\end{cases}$$

where the θ_p are to be determined.

(d) Explain why

$$S(x) = \sum_{\substack{n\le x \\ p|n \implies p\le x^{1/2}}} f_0(n) \;+ \sum_{x^{1/2}<p\le x} e(\theta_p)S_0(x/p).$$

(e) Deduce that there is a choice of the θ_p such that

$$|S(x)| = \Big|\sum_{\substack{n\le x \\ p|n \implies p\le x^{1/2}}} f_0(n)\Big| \;+ \sum_{x^{1/2}<p\le x} |S_0(x/p)|.$$

(f) Show that

$$\sum_{x^{1/2}<p\le x} |S_0(x/p)| \asymp \frac{x\log\log x}{\log x}\,.$$

(g) Show that $M(\alpha) \ll 1$ uniformly for $\alpha > 0$.

3. Recall that the 'negative binomial theorem' asserts that

$$(1-z)^{-r-1} = \sum_{n=0}^{\infty} \binom{n+r}{n} z^n$$

for $|z| < 1$. Here r is any complex number.

(a) Show that if $r > -1$, then $\binom{n+r}{n} \ge 0$ for all n.

(b) Suppose that f is totally multiplicative, that $|f(n)| \le 1$ for all n, and let F be defined as in (23.26). Show that if q is a positive real number, then

$$\int_{T_0-T}^{T_0+T} |F(\sigma+it)|^q\,dt \le 3\int_{-T}^{T} |\zeta(\sigma+it)|^q\,dt\,.$$

(c) Use (23.46) to show that if $f \in \mathcal{M}_0$ and $q > 0$, then

$$\int_{T_0-T}^{T_0+T} |F(\sigma+it)|^q\,dt \ll_q \int_{-T}^{T} |\zeta(\sigma+it)|^q\,dt\,.$$

4. (Turán) Let $f(n)$ be an integer-valued additive function, and let $N(x;q,a)$ denote the number of $n \le x$ such that $f(n) \equiv a \pmod q$.

(a) Show that $\lim_{x\to\infty} N(x;,q,a)/x = n(q,a)$ exists for all a and q.

(b) Show that $n(q,a) = 1/q$ for all a if and only if both the following hold:

(i) For each odd prime $p_1|q$,

$$\sum_{\substack{p \\ p_1|f(p)}} \frac{1}{p} = +\infty;$$

(ii) If $2|q$, then

$$\sum_{\substack{p \\ 2|f(p)}} \frac{1}{p} = +\infty$$

or both of the following hold: $f(2^k)$ is odd for all $k > 0$, and if $4|q$, then

$$\sum_{\substack{p \\ 4|f(p)}} \frac{1}{p} = +\infty.$$

23.6 Notes

S:NotesProbNoThy

Our use of probabilistic modelling is necessarily a little informal, since the size of sets of integers as measured by asymptotic density do not form a probability space. One of Kolmogorov's fundamental axioms states that if $E_1, E_2, \ldots,$ are pairwise disjoint events (i.e., sets) in a probability space, then

$$\mathsf{P}\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum_{n=1}^{\infty} \mathsf{P}(E_n)\,.$$

However, if $E_n = \{n\}$, then the asymptotic density of E_n is 0 for all n, while the density of the union of the E_n is 1.

23.7 References

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24

Exponential Sums II: Vinogradov's method

C:ExpSumII

24.1 Vinogradov's mean value theorem

S:Vmvt

In Chapter **??**—we derived nontrivial bounds for sums of the sort

$$\sum_{a\le n\le b} e(f(n))$$

from information regarding the first few derivatives of f. However, in studying the zeta function near the line $\operatorname{Re} s = 1$ we have to examine sums such as

$$\sum_{a\le n\le b} n^{-1-it},$$

and here the situation is much more delicate. To obtain a satisfactory treatment it is necessary to consider derivatives of

$$f(x) = \frac{t}{2\pi}\log x$$

of order r, where r depends on the interrelationship between a and t. It is possible to make use of Theorem **??** in this regard, although the dependence on r in the implicit constant there needs to be made explicit. The amount of saving which can be made in this way is rather poor because the exponent is dropping off like 2^{-r}, but Littlewood was able to show that the zeta function has a zero free region of the form

$$\sigma \ge 1 - c\frac{\log\log\tau}{\log\tau}.$$

We now introduce a much more efficient way of making use of higher derivatives, or rather, what is tantamount to the same thing, polynomial approximations to $f(x)$ of arbitrary degree. The underlying idea is to relate the individual sums to a mean value. Let

$$\boldsymbol{n}^{(k)} = (n, n^2, \dots, n^k)$$

and let

$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_k),$$
$$S(\boldsymbol{\alpha}, \mathcal{A}) = \sum_{n \in \mathcal{A}} e\big(\boldsymbol{n}^{(k)} \cdot \boldsymbol{\alpha}\big)$$

where $\mathcal{A}$ is a finite set of integers. We are interested in the mean value

$$J_k(\mathcal{A}, b) = \int_{\mathbb{T}^k} |S(\boldsymbol{\alpha}, \mathcal{A})|^{2b}\, \boldsymbol{d\alpha}.$$

Now

$$S(\boldsymbol{\alpha}, \mathcal{A})^b = \sum_{\boldsymbol{m}} r\big(\boldsymbol{m}, \mathcal{A}^b\big) e(\boldsymbol{m} \cdot \boldsymbol{\alpha})$$

where $r\big(\boldsymbol{m}, \mathcal{A}^b\big)$ denotes the number of solutions of the system

$$\begin{array}{ccccccc} n_1 & + & \cdots & + & n_b & = & m_1 \\ n_1^2 & + & \cdots & + & n_b^2 & = & m_2 \\ \vdots & & & & \vdots & & \vdots \\ n_1^k & + & \cdots & + & n_b^k & = & m_k \end{array} \qquad (24.1)$$

E:kxbeqns

with $n_i \in \mathcal{A}$. Thus by Parseval's identity,

$$J_k(\mathcal{A}, b) = \sum_{\boldsymbol{m}} r\big(\boldsymbol{m}, \mathcal{A}^b\big)^2.$$

When $\mathcal{B}$ and $\mathcal{C}$ are subsets of $\mathbb{R}^b$ containing only finitely many lattice points, let $N(\mathcal{B}, \mathcal{C}, \boldsymbol{l})$ denote the number of solutions of

$$\begin{array}{ccccccccccccc} m_1 & + & \cdots & + & m_b & = & n_1 & + & \cdots & + & n_b & + & l_1 \\ m_1^2 & + & \cdots & + & m_b^2 & = & n_1^2 & + & \cdots & + & n_b^2 & + & l_2 \\ \vdots & & & & \vdots & & \vdots & & & & \vdots & & \vdots \\ m_1^k & + & \cdots & + & m_b^k & = & n_1^k & + & \cdots & + & n_b^k & + & l_k \end{array}$$

with $\boldsymbol{m} \in \mathcal{B}$ and $\boldsymbol{n} \in \mathcal{C}$, and for brevity write $N(\mathcal{B}, \boldsymbol{l}) = N(\mathcal{B}, \mathcal{B}, \boldsymbol{l})$, $N(\mathcal{B}) = N(\mathcal{B}, \boldsymbol{0})$ and $N(\mathcal{B}, \mathcal{C}) = N(\mathcal{B}, \mathcal{C}, \boldsymbol{0})$. Then we can define the more general mean

$$J_k(\mathcal{A}, b, \boldsymbol{l}) = N\big(\mathcal{A}^b, \boldsymbol{l}\big),$$

so that

$$J_k(\mathcal{A}, b) = N\big(\mathcal{A}^b\big),$$

concerning which the following elementary observations are useful.

L:N(B)eqs+ineqs

Lemma 24.1 *In the above notation,*
(a) *If* $\mathcal{B} \subseteq \mathcal{C}$, *then* $N(\mathcal{B}, \boldsymbol{l}) \le N(\mathcal{C}, \boldsymbol{l})$,
(b) $N(\mathcal{B}, \boldsymbol{l}) \le N(\mathcal{B})$ *for all* $\boldsymbol{l}$,
(c) *If* $\mathcal{C} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_j$, *then* $N(\mathcal{C}) \le j \sum_{i=1}^{j} N(\mathcal{B}_i)$,
(d) *If* $\boldsymbol{d} = (d, d, \dots, d)$, *then* $N(\mathcal{B} + \boldsymbol{d}, \mathcal{C} + \boldsymbol{d}) = N(\mathcal{B}, \mathcal{C})$,
(e) $J_k(\mathcal{A}, b, \boldsymbol{l}) \le J_k(\mathcal{A}, b)$.

Proof (a) If $\mathcal{B} \subseteq \mathcal{C}$, then the solutions counted in $N(\mathcal{B}, \mathcal{B}, \boldsymbol{l})$ are also counted in $N(\mathcal{C}, \mathcal{C}, \boldsymbol{l})$.
(b) Let $r(\boldsymbol{m}, \mathcal{B})$ denote the number of solutions of (24.1) with $\boldsymbol{n} \in \mathcal{B}$. Then

$$N(\mathcal{B}, \boldsymbol{l}) = \int_{\mathbb{T}^k} \Big| \sum_{\boldsymbol{m}} r(\boldsymbol{m}, \mathcal{B}) e(\boldsymbol{m} \cdot \boldsymbol{\alpha}) \Big|^2 e(-\boldsymbol{l} \cdot \boldsymbol{\alpha}) \, \boldsymbol{d\alpha}$$
$$\le \int_{\mathbb{T}^k} \Big| \sum_{\boldsymbol{m}} r(\boldsymbol{m}, \mathcal{B}) e(\boldsymbol{m} \cdot \boldsymbol{\alpha}) \Big|^2 \boldsymbol{d\alpha} = N(\mathcal{B}),$$

as desired.
(c) In the above notation,

$$r(\boldsymbol{m}, \mathcal{C}) \le \sum_{i=1}^{j} r(\boldsymbol{m}, \mathcal{B}_i)$$

and so by Cauchy's inequality

$$r(\boldsymbol{m}, \mathcal{C})^2 \le j \sum_{i=1}^{j} r(\boldsymbol{m}, \mathcal{B}_i)^2.$$

It now suffices to sum this over $\boldsymbol{m}$, since

$$N(\mathcal{C}) = \sum_{\boldsymbol{m}} r(\boldsymbol{m}, \mathcal{C})^2, \qquad N(\mathcal{B}_i) = \sum_{\boldsymbol{m}} r(\boldsymbol{m}, \mathcal{B}_i)^2.$$

(d) Suppose that

$$\begin{array}{ccccccccccccc} m_1 & + & \cdots & + & m_b & = & n_1 & + & \cdots & + & n_b, \\ m_1^2 & + & \cdots & + & m_b^2 & = & n_1^2 & + & \cdots & + & n_b^2, \\ \vdots & & & & \vdots & & \vdots & & & & \vdots \\ m_1^k & + & \cdots & + & m_b^k & = & n_1^k & + & \cdots & + & n_b^k. \end{array}$$

By the binomial theorem we see that if $1 \le j \le k$, then

$$\sum_{i=1}^{b} (m_i + d)^j = \sum_{l=0}^{j} \binom{j}{l} d^{j-l} \sum_{i=1}^{b} m_i^l = \sum_{l=0}^{j} \binom{j}{l} d^{j-l} \sum_{i=1}^{b} n_i^l = \sum_{i=1}^{b} (n_i + d)^j.$$

Thus $N(\mathcal{B},\mathcal{C}) \le N(\mathcal{B}+\boldsymbol{d},\mathcal{C}+\boldsymbol{d})$. Hence also,

$$N(\mathcal{B}+\boldsymbol{d},\mathcal{C}+\boldsymbol{d}) \le N((\mathcal{B}+\boldsymbol{d})-\boldsymbol{d},(\mathcal{C}+\boldsymbol{d})-\boldsymbol{d}) = N(\mathcal{B},\mathcal{C}).$$

(e) is a special case of (b). □

Let $\theta_1,\ldots,\theta_q$ denote real or complex numbers and let

$$Q(z;\boldsymbol{\theta}) = \prod_{j=1}^{q}(z-\theta_j) = \sum_{k=0}^{q}(-1)^k\sigma_k z^{q-k} \tag{24.2}$$

E:DefQ

where $\sigma_0 = 1$ and $\sigma_1,\sigma_2,\ldots,\sigma_q$ are the *elementary symmetric functions* of the θ_j, which is to say that if $\mathcal{Q} = \{1,2,\ldots,q\}$, then

$$\sigma_r = \sigma_r(\boldsymbol{\theta}) = \sum_{\substack{\mathcal{S}\subseteq\mathcal{Q}\\ \operatorname{card}\mathcal{S}=r}}\ \prod_{j\in\mathcal{S}}\theta_j.$$

For $m = 1,2,\ldots$ we also form the *power sums* of the θ_j,

$$s_m = s_m(\boldsymbol{\theta}) = \sum_{j=1}^{q}\theta_j^m.$$

These are also symmetric polynomials in the θ_j, and are related to the σ_j by means of the *Newton–Girard formulæ*, which assert that

$$\sum_{k=0}^{r-1}(-1)^{r-1-k}\sigma_k s_{r-k} = r\sigma_r \tag{24.3}$$

E:NewGir1

for $1 \le r \le q$, and that

$$\sum_{k=0}^{q}(-1)^k\sigma_k s_{r-k} = 0 \tag{24.4}$$

E:NewGir2

for $r \ge q$. In this second identity, the quantity s_0 arises when $k = r = q$. It is to be understood that $s_0 = q$ even if one or more of the θ_j vanishes. We use the first of these identities (a proof of which is sketched in Exercise 1) to establish our next result.

L:pwrsums **Lemma 24.2** (a) *Suppose that $\theta_1,\ldots,\theta_q,\phi_1,\ldots,\phi_q$ are such that*

$$s_r(\boldsymbol{\theta}) = s_r(\boldsymbol{\phi}) \qquad (1 \le r \le q).$$

Then

$$Q(z;\boldsymbol{\theta}) = Q(z;\boldsymbol{\phi})$$

identically.

(b) *Suppose that p is a prime number with $p > q$, that u is a positive integer and that $\theta_1, \ldots, \theta_q, \phi_1, \ldots, \phi_q$ are integers such that*

$$s_r(\boldsymbol{\theta}) \equiv s_r(\boldsymbol{\phi}) \pmod{p^u} \qquad (1 \le r \le q).$$

Then

$$Q(z; \boldsymbol{\theta}) \equiv Q(z; \boldsymbol{\phi}) \pmod{p^u}$$

for all integers z.

Proof (a) For $r = 1$ we observe that $\sigma_1(\boldsymbol{\theta}) = s_1(\boldsymbol{\theta}) = s_1(\boldsymbol{\phi}) = \sigma_1(\boldsymbol{\phi})$. For $r > 1$ we argue by induction. In (24.3) we see that σ_r is expressed in terms of $s_1, s_2, \ldots, s_r$ and $\sigma_0, \sigma_1, \ldots, \sigma_{r-1}$. Hence by the inductive hypothesis the left hand side of (24.3) with respect to $\boldsymbol{\theta}$ is equal to the same expression with respect to $\boldsymbol{\phi}$.

(b) As in the preceding case we find that $\sigma_1(\boldsymbol{\theta}) \equiv \sigma_1(\boldsymbol{\phi}) \pmod{p^u}$. In the inductive step we find that $r\sigma_r(\boldsymbol{\theta}) \equiv r\sigma_r(\boldsymbol{\phi}) \pmod{p^u}$. Since $r \le q < p$, it follows that $\big(r, p^u\big) = 1$, and hence that $\sigma_r(\boldsymbol{\theta}) \equiv \sigma_r(\boldsymbol{\phi}) \pmod{p^u}$. □

L:J_kProps **Lemma 24.3** *Let $J_k(x, b) = J_k((0, x], b)$. Then*
(a) $J_k(x, b) \le b!x^b$ *when* $b \le k$,
(b) $J_k(x, b) \le k!x^{2b-k}$ *when* $b > k$,
(c) $J_k(x, b) \ge \lfloor x \rfloor^b$,
(d) $J_k(x, b) \ge (2b+1)^{-k} \lfloor x \rfloor^{2b-k(k+1)/2}$.

Proof (a) From (24.1) we see that $J_k(x, b)$ is the number of choices of $\boldsymbol{m}$, $\boldsymbol{n}$ in $(0, x]^b$ such that

$$s_r(\boldsymbol{m}) = s_r(\boldsymbol{n}) \qquad (1 \le r \le k). \tag{24.5}$$
E:sr(m)=sr(n)

Hence by Lemma 24.2 with $\boldsymbol{\theta} = \boldsymbol{m}$, $\boldsymbol{\phi} = \boldsymbol{n}$ we have

$$Q(z; \boldsymbol{m}) = Q(z; \boldsymbol{n})$$

identically. Hence the roots (counting multiplicity) coincide. Thus the n_i are permutations of the m_i.

(b) When $b \ge k$,

$$\begin{aligned} J_k(x, b) = \int_{\mathbb{T}^k} \Big| \sum_{n \le x} e\big(\boldsymbol{n}^{(k)} \cdot \boldsymbol{\alpha}\big) \Big|^{2b} \boldsymbol{d\alpha} &\le x^{2b-2k} \int_{\mathbb{T}^k} \Big| \sum_{n \le x} e\big(\boldsymbol{n}^{(k)} \cdot \boldsymbol{\alpha}\big) \Big|^{2k} \boldsymbol{d\alpha} \\ &\le x^{2b-2k} \cdot k!x^k \end{aligned}$$

by part (a).

(c) Since $J_k(x,b)$ is the number of solutions of (24.5), by taking $m_1 = n_1, m_2 = n_2, \dots, m_b = n_b$ we see that there exist at least $\lfloor x \rfloor^b$ solutions.

(d) For brevity put $N = \lfloor x \rfloor$. Then

$$\Big| \int_{\mathbb{T}^k} \Big| \sum_{n=1}^{N} e\big(\boldsymbol{n}^{(k)} \cdot \boldsymbol{\alpha}\big) \Big|^{2b} e(-\boldsymbol{l} \cdot \boldsymbol{\alpha})\, \boldsymbol{d\alpha} \Big| \le J_k(x,b). \qquad (24.6)$$ E:|S|2bFC

The integral on the left hand side is the number of solutions of

$$s_r(\boldsymbol{m}) - s_r(\boldsymbol{n}) = l_r \quad (1 \le r \le k)$$

with $\boldsymbol{m}$, $\boldsymbol{n}$ in $(0,x]^b$. Since $0 < s_r(\boldsymbol{m}) \le bN^r$ there are no solutions unless $\boldsymbol{l}$ satisfies

$$|l_r| \le bN^r \qquad (1 \le r \le k). \qquad (24.7)$$ E:|lr|<=

We sum both sides of (24.6) over all such $\boldsymbol{l}$, and note that on the left we are just counting all possible choices of $\boldsymbol{m}$ and $\boldsymbol{n}$, the number of which is N^{2b}. The number of $\boldsymbol{l}$ satisfying (24.7) is at most

$$(2b+1)^k N^{\frac{1}{2}k(k+1)}.$$

Thus

$$N^{2b} \le (2b+1)^k N^{\frac{1}{2}k(k+1)} J_k(x,b),$$

which gives the desired conclusion. □

Our treatment of $J_k(x,b)$ when $b > k$ is via a local or "p-adic" argument, and the following lemma, due originally to Linnik, is the sparking point of the method.

L:LinnikLocal **Lemma 24.4** *Suppose that p is a prime number with $p > k$. Let $A(p,\boldsymbol{h})$ denote the number of solutions of the simultaneous congruences*

$$\sum_{r=1}^{k} m_r^j \equiv h_j \pmod{p^j} \qquad (1 \le j \le k)$$

with $m_r \le p^k$ and the m_r distinct modulo p. Then

$$A(p,\boldsymbol{h}) \le k!\, p^{\frac{1}{2}k(k-1)}.$$

Proof Let $B(p,\boldsymbol{g})$ denote the number of solutions of

$$\sum_{r=1}^{k} m_r^j \equiv g_j \pmod{p^k} \qquad (1 \le j \le k) \qquad (24.8)$$ E:summrjmodpk

with $m_r \le p^k$ and the m_r distinct modulo p. Then for each $\boldsymbol{h}$, $A(p,\boldsymbol{h})$

is the sum of those $B(p, \boldsymbol{g})$ with $g_j \equiv h_j \pmod{p^j}$ and $1 \le g_j \le p^k$ for $1 \le j \le k$. The total number of possible choices for $\boldsymbol{g}$ is $p^{\frac{1}{2}k(k-1)}$. Thus it suffices to show that $B(p, \boldsymbol{g}) \le k!$. For a given $\boldsymbol{g}$ suppose that $m_1, \dots, m_k$ is a solution of (**??**) with $m_r \le p^k$ and the m_r distinct modulo p. Suppose that $n_1, \dots, n_k$ is another such solution. Then, by Lemma 24.2(b),

$$Q(z; \boldsymbol{m}) \equiv Q(z; \boldsymbol{n}) \pmod{p^k}$$

and so $Q(n_s; \boldsymbol{m}) \equiv 0 \pmod{p^k}$ whenever $1 \le s \le k$. Since

$$Q(z; \boldsymbol{m}) = \prod_{r=1}^{k} (z - m_r),$$

for each s there is an r such that $n_s \equiv m_r \pmod p$. Also, since the m_r are distinct modulo p it follows that m_r is unique, and so $n_s \equiv m_r \pmod{p^k}$. Thus $n_s = m_r$. Since the n_s are distinct modulo p, and so are distinct, it follows that the $\boldsymbol{n}$ are a permutation of the $\boldsymbol{m}$. □

We now have all the machinery we need to establish a useful version of the Vinogradov Mean Value Theorem.

T:VMVT **Theorem 24.5** *There is a positive number C such that when $k \ge 2$, r is a positive integer and x is a real number with $x \ge 1$ we have*

$$J_k(x, kr) \le D(k, r) x^{2rk - \frac{1}{2}k(k+1) + \eta(k,r)}$$

where

$$D(k, r) = \exp\big(Crk^2 \log k\big)$$

and

$$\eta(k, r) = \tfrac{1}{2}k^2 \Big(1 - \frac{1}{k}\Big)^r.$$

Proof We induct on r. The case $r = 1$ is immediate from 24.3(a) and the observations that $2k - \frac{1}{2}k(k+1) + \eta(k, 1) = k$ and that $k! \le k^k \le D(k, 1)$ provided that $C \ge 1$.

Suppose now that $r \ge 2$ and that the Theorem holds with r replaced by $r - 1$. Then $1 - \big(1 - \frac{1}{k}\big)^r \le \frac{r}{k}$, so

$$\frac{1}{2}k(k+1) - \eta(k, r) \le \min(k^2, rk).$$

Hence if $x \le \exp(C \max(k, r) \log k)$, then $x^{\frac{1}{2}k(k+1) - \eta(k,r)} \le D(k, r)$, and the trivial estimate $J_k(x, kr) \le x^{2kr}$ gives the Theorem. Thus we can suppose that

$$x > \exp(C \max(k, r) \log k). \tag{24.9}$$ E:xLB

Let

$$y = x^{1/k}$$

and choose z to be the least number such that the number of primes p with $y < p \le y + z$ is $\frac{1}{2}k(k^2 - 1)$. From (24.9) it follows that $y > k^C$, and so by the prime number theorem

$$z \le y$$

if C is a sufficiently large absolute constant. For brevity we write b for kr. Let $R_1(\boldsymbol{h})$ denote the number of solutions to the system

$$\sum_{r=1}^{b} m_r^j = h_j \qquad (1 \le j \le k) \tag{24.10}$$ E:summrj=hj

with $m_r \le x$ and $m_1, \ldots, m_k$ distinct, and let $R_2(\boldsymbol{h})$ denote the number of solutions in which the $m_1, \ldots, m_k$ are not distinct. Then

$$J_k(x, b) = \sum_{\boldsymbol{h}} \big(R_1(\boldsymbol{h}) + R_2(\boldsymbol{h})\big)^2 \le 2(S_1 + S_2)$$

where

$$S_i = \sum_{\boldsymbol{h}} R_i(\boldsymbol{h})^2.$$

We consider two cases: $S_1 \le S_2$, and $S_1 > S_2$. We deal first with the easy case, in which $S_2 \ge S_1$. In this case, $J_k(x, b) \le 4S_2$ and $R_2(\boldsymbol{h}) \le \binom{k}{2} R_3(\boldsymbol{h})$ where $R_3(\boldsymbol{h})$ is the number of solutions to the system (24.10) with $m_r \le x$ and $m_1 = m_2$. Let

$$f(\boldsymbol{\beta}) = \sum_{n \le x} e(\boldsymbol{n}^{(k)} \cdot \boldsymbol{\beta}).$$

Then

$$S_2 \le k^4 \int_{\mathbb{T}^k} \big| f(2\boldsymbol{\alpha})^2 f(\boldsymbol{\alpha})^{2b-4} \big| \, \boldsymbol{d\alpha}.$$

Hence by two applications of the simplest form of Hölder's inequality (or by a single application of the extended form found in Exercise 4)

$$S_2 \le k^4 \bigg(\int_{\mathbb{T}^k} |f(2\boldsymbol{\alpha})|^{2b} \, \boldsymbol{d\alpha} \bigg)^{\frac{1}{b}} \bigg(\int_{\mathbb{T}^k} |f(\boldsymbol{\alpha})|^{2b} \, \boldsymbol{d\alpha} \bigg)^{1 - \frac{2}{b}} = k^4 J_k(x, b)^{1 - \frac{1}{b}}.$$

Thus the Theorem holds in the first case provided that $C \geq 6$.

We now suppose that $S_2 < S_1$, which implies that

$$J_k(x,b) \leq 4S_1.$$

For a solution $\boldsymbol{m}$ of (24.10) counted by $R_1(\boldsymbol{h})$ let

$$P(\boldsymbol{m}) = \prod_{1 \leq i < j \leq k} (m_i - m_j).$$

Then $0 < |P| \leq x^{\frac{1}{2}k(k-1)}$. Since $y = x^{1/k}$, the number of prime divisors p of P with $p > y$ is at most $\frac{1}{2}k^2(k-1) < \frac{1}{2}k(k^2-1)$. Thus there is a prime p with

$$y < p \leq z$$

such that $p \nmid P$, and so for such a p the $m_1, \dots, m_k$ are distinct modulo p. Hence

$$R_1(\boldsymbol{h}) \leq \sum_{y < p \leq y+z} R_4(\boldsymbol{h}, p)$$

where $R_4(\boldsymbol{h}, p)$ denotes the number of solutions of (24.10) subject to $m_r \leq x$ and $m_1, \dots, m_k$ distinct modulo p. Let

$$I(p) = \sum_{\boldsymbol{h}} R_4(\boldsymbol{h}, p)^2.$$

Then $I(p)$ is the number of solutions of

$$s_j(\boldsymbol{m}) = s_j(\boldsymbol{n}) \qquad (1 \leq j \leq k)$$

with $m_1, \dots, m_b, n_1, \dots, n_b$ in $(0, x]$, $m_1, \dots, m_k$ distinct modulo p and $n_1, \dots, n_k$ distinct modulo p. Thus

$$\begin{aligned}
J_k(x,b) &\leq 4 \sum_{\boldsymbol{h}} R_1(\boldsymbol{h}))^2 \\
&\leq 4 \sum_{\boldsymbol{h}} \Big(\sum_{y < p \leq z} R_4(\boldsymbol{h}, p) \Big)^2 \\
&\leq 4 \sum_{\boldsymbol{h}} \frac{1}{2}k^3 \sum_{y < p \leq y+z} R_4(\boldsymbol{h}, p)^2 \\
&\leq 2k^3 \sum_{y < p \leq y+z} I(p) \\
&\leq k^6 \max_{\substack{p \\ y < p \leq y+z}} I(p).
\end{aligned}$$

Let

$$g(\boldsymbol{\alpha}, a) = \sum_{\substack{n \le x \\ n \equiv a \ (mod p)}} e\big(\boldsymbol{n}^{(k)} \cdot \boldsymbol{\alpha}\big).$$

Then

$$I(p) = \int_{\mathbb{T}^k} \Big| \sum_{\boldsymbol{a} \in \mathcal{A}} g(\boldsymbol{\alpha}, a_1) \cdots g(\boldsymbol{\alpha}, a_k) \Big|^2 \Big| \sum_{a=0}^{p-1} g(\boldsymbol{\alpha}, a) \Big|^{2b-2k} \boldsymbol{d\alpha}$$

where $\mathcal{A}$ denotes the set of k-tuples $\boldsymbol{a} = (a_1, \ldots, a_k)$ with $0 \le a_r < p$ and the a_r distinct. By Hölder's inequality

$$\Big| \sum_{a=0}^{p-1} g(\boldsymbol{\alpha}, a) \Big|^{2b-2k} \le p^{2b-2k-1} \sum_{a=0}^{p-1} |g(\boldsymbol{\alpha}, a)|^{2b-2k},$$

and so

$$I(p) \le p^{2b-2k} \max_{0 \le a < p} I_1(p, a)$$

where

$$I_1(p, a) = \int_{\mathbb{T}^k} \Big| \sum_{\boldsymbol{a} \in \mathcal{A}} g(\boldsymbol{\alpha}, a_1) \cdots g(\boldsymbol{\alpha}, a_k) \Big|^2 |g(\boldsymbol{\alpha}, a)|^{2b-2k} \boldsymbol{d\alpha},$$

and this is the number of solutions of the simultaneous equations

$$\sum_{i=1}^{k} \big(m_i^j - n_i^j\big) = \sum_{r=1}^{b-k} \big((pu_r + a)^j - (pv_r + a)^j\big) \qquad (1 \le j \le k)$$

with $m_i \le x$, $n_i \le x$, $-a/p < u_r \le (x-a)/p$, $-a/p < v_r \le (x-a)/p$, $m_1, \ldots, m_k$ distinct modulo p and $n_1, \ldots, n_k$ distinct modulo p. By Lemma 24.1(d) this is the number of solutions of

$$\sum_{i=1}^{k} \big((m_i - a)^j - (n_i - a)^j\big) = \sum_{r=1}^{b-k} p^j \big(u_r^j - v_r^j\big) \qquad (1 \le j \le k)$$

under the same conditions. Let $\mathcal{B}(p, a)$ denote the set of $2k$-tuples

$$(\boldsymbol{m}, \boldsymbol{n}) = (m_1, \ldots, m_k, n_1, \ldots, n_k)$$

such that $m_i \le x$, $n_i \le x$, $m_1, \ldots, m_k$ are distinct modulo p, $n_1, \ldots, n_k$ are distinct modulo p, and

$$\sum_{i=1}^{k} (m_i - a)^j \equiv \sum_{i=1}^{k} (n_i - a)^j \pmod{p^j} \qquad (1 \le j \le k).$$

For each such element $(\boldsymbol{m},\boldsymbol{n})$ put

$$h_j(\boldsymbol{m},\boldsymbol{n}) = p^{-j}\sum_{i=1}^{k}\big((m_i-a)^j-(n_i-a)^j\big).$$

Then by Lemma 24.1(e),

$$\begin{aligned} I_1(p,a) &= \sum_{(\boldsymbol{m},\boldsymbol{n})\in\mathcal{B}(p,a)} J_k\Big(\Big(-\frac{a}{p},\frac{x-a}{p}\Big],b-k,\boldsymbol{h}(\boldsymbol{m},\boldsymbol{n})\Big)\\ &\le \operatorname{card}\mathcal{B}(p,a)J_k\Big(\Big(-\frac{a}{p},\frac{x-a}{p}\Big],b-k\Big). \end{aligned}$$

By Lemma 24.4, since $p^k > y^k = x$ we have

$$\operatorname{card}\mathcal{B}(p,a) \le x^k k! p^{\frac12 k(k-1)}.$$

Therefore, by Lemma 24.1(d),

$$I_1(p,a) \le x^k k! p^{\frac12 k(k-1)} J_k\Big(1+\frac{x}{p},(r-1)k\Big)$$

and so by the inductive hypothesis

$$I_1(p,a) \le x^k k! p^{\frac12 k(k-1)} D(k,r-1)\Big(1+\frac{x}{p}\Big)^{2(r-1)k-\frac12 k(k-1)+\eta(r-1,k)}.$$

Hence

$$J_k(x,rk) \le \max_{y<p\le y+z} \lambda p^{k^2-\eta(r-1,k)} x^{2rk-k-\frac12 k(k+1)+\eta(r-1,k)}$$

where

$$\lambda = k^6 k!\Big(1+\frac{x}{p}\Big)^{2rk-2k-\frac12 k(k+1)+\eta(r-1,k)} D(k,r-1).$$

Each prime p here satisfies $p \le y+z \le 2y = 2x^{1/k}$ and so

$$J_k(x,rk) \le 2^{k^2}\lambda x^{2rk-\frac12 k(k+1)+\eta(r,k)}$$

and

$$2^{k^2}\lambda \le k^6 k! 2^{k^2}\big(1+2x^{1/k-1}\big)^{2rk}\exp\big(C(r-1)k^2\log k\big).$$

Now

$$\big(1+2x^{1/k-1}\big)^{2rk} \le \exp\big(4rkx^{-1/2}\big),$$

and by (24.9) this does not exceed e provided that $C \ge 6$. Thus

$$2^{k^2}\lambda \le \exp\big(6\log k + k\log k + k^2 + 1 + C(r-1)k^2\log k\big),$$

and this does not exceed

$$\exp\big(Crk^2\log k\big)$$

provided that $C \ge 6$. This completes the proof of the Theorem. □

24.1.1 Exercises

Exer:NewGir1 1. Let $Q(z) = Q(z;\boldsymbol{\theta})$, σ_r and s_r be defined as in (24.2) and the subsequent discussion.

(a) Put $P(z) = P(z;\boldsymbol{\theta}) = z^q Q(1/z)$. Show that

$$P(z) = \prod_{j=1}^{q}(1-\theta_j z) = \sum_{r=0}^{q}(-1)^r\sigma_r z^r.$$

(b) Deduce that

$$\frac{P'}{P}(z) = -\sum_{j=1}^{q}\frac{\theta_j}{1-\theta_j z}.$$

(c) Let R be determined by the equation $1/R = \max_j|\theta_j|$. Show that the above is

$$= -\sum_{m=1}^{\infty} s_m z^{m-1}$$

for $|z| < R$.

(d) Explain why

$$-\Big(\sum_{k=0}^{q}(-1)^k\sigma_k z^k\Big)\Big(\sum_{m=1}^{\infty}s_m z^{m-1}\Big) = \sum_{r=1}^{q}(-1)^r r\sigma_r z^{r-1}$$

for $|z| < R$.

(e) By considering pairs k, m in the above with $k+m=r$, deduce the first Newton–Girard formula, i.e., equation (24.3).

Exer:NewGir2 2. Let $Q(z)$ be defined as in (24.2). Observe that

$$0 = Q(\theta_j) = \sum_{k=0}^{q}(-1)^k\sigma_k\theta_j^{q-k}$$

where it is understood that $\theta_j^0 = 1$ even if $\theta_j = 0$. Suppose that $r \ge q$. Multiply both sides of the above by θ_j^{r-q}, and then sum over j to obtain the second Newton Girard formula, i.e., equation (24.4).

3. We say that a sequence u_n satisfies a linear recurrence of order q if there exist real or complex numbers $a_1, a_2, \ldots, a_q$ such that $u_n = a_1 u_{n-1} + a_2 u_{n-2} + \cdots + a_q u_{n-q}$ for all $n > q$. Thus (24.4) asserts that the s_m satisfy a linear recurrence of order q. More generally, for arbitrary real or complex constants $c_1, c_2, \ldots, c_q$ put
$$u_m = \sum_{j=1}^{q} c_j \theta_j^m.$$
Show that
$$\sum_{k=0}^{q} (-1)^k \sigma_k u_{r-k} = 0$$
for all $r \ge q$.

Exer:Holder3

4. Suppose that $p > 1, q > 1, r > 1$ are real numbers such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$.
   (a) Let a_n, b_n, c_n be nonnegative real numbers. By two applications of the simplest form of Hölder's inequality, or otherwise, show that
   $$\sum_{n=1}^{N} a_n b_n c_n \le \Big(\sum_{n=1}^{N} a_n^p\Big)^{1/p} \Big(\sum_{n=1}^{N} b_n^q\Big)^{1/q} \Big(\sum_{n=1}^{N} c_n^r\Big)^{1/r}.$$
   (b) Let u_n, v_n, w_n be arbitrary real or complex numbers. Show that
   $$\Big|\sum_{n=1}^{N} u_n v_n w_n\Big| \le \Big(\sum_{n=1}^{N} |u_n|^p\Big)^{1/p} \Big(\sum_{n=1}^{N} |v_n|^q\Big)^{1/q} \Big(\sum_{n=1}^{N} |w_n|^r\Big)^{1/r}.$$

5. Let
$$S(p, \boldsymbol{a}) = \sum_{n=1}^{p} e\big((a_1 n + a_2 n^2 + \cdots + a_k n^k) p^{-1}\big)$$
and
$$M(p) = \sum_{a_1=1}^{p} \cdots \sum_{a_k=1}^{p} |S(p, \boldsymbol{a})|^{2k}.$$
   (a) Show that if $p > k$, then $M(p) \le k! p^{2k}$.
   (b) Suppose that $p > k$ and $p \nmid a_k$. Show that
   $$|S(p, \boldsymbol{a})| \le (2k)^{\frac{1}{2}} p^{1-\frac{1}{k}}.$$

6. Show that if $p \nmid a$, then

$$\Big|\sum_{n=1}^{p} e\Big(\frac{an^k}{p}\Big)\Big| \le ((k, p-1)-1)p^{\frac{1}{2}}.$$

7. Show that $N(\mathcal{B}, \mathcal{C}, \boldsymbol{h}) \le N(\mathcal{B})^{\frac{1}{2}} N(\mathcal{C})^{\frac{1}{2}}$.

8. Show that in Linnik's lemma, either $A(p, \boldsymbol{h}) = 0$ or $A(p, \boldsymbol{h}) = k! p^{\frac{1}{2}k(k-1)}$. Show that if the condition that the m_j are distinct is omitted, then $A(p, \mathbf{0}) \ge p^{k(k-1)}$.

9. Show that $J_k(hN, b) \le h^{2b} J_k(N, b)$.
10. Suppose that

$$J_k(q, b) \ll q^{2b - \frac{k(k+1)}{2} + \eta}$$

and for a Dirichlet character q modulo q

$$W_k(q; \mathbf{a}, \chi) = \sum_{x=1}^{q} e\big(a_1 x + \cdots a_k^k)/q\big)\chi(x).$$

(a) Show that

$$\sum_{a_1=1}^{q} \cdots \sum_{a_k=1}^{q} |W_k(q, \mathbf{a}, \chi)|^{2b} \ll q^{2b+\eta}$$

(b) Show that if $(q, a_1, \ldots, a_k) = 1$, then

$$W_k(q; \mathbf{a}, \chi) \ll q^{1 - \frac{1-\eta}{2b} + \epsilon}$$

24.2 Vinogradov's method

S:Vm

Here we are mostly concerned to relate an individual exponential sum to a mean value of exponential sums. A very fruitful way of doing this is to perturb the endpoints of summation. Excercise **??** givesa a simple method for doing this. However the situation of particular interest to us is one of very great delicacy and the presence of the logarithmic factor in that exercise means that if the bound we are able to obtain for $S(\beta)$ is close to the trivial bound, then our resulting estimate is worse than trivial. Here we develop a method which has no loss of this kind.

We start by establishing such a connection for a rather special exponential sum, for which we require an auxiliary lemma.

L:TechIneq **Lemma 24.6** *Suppose that $X \ge 1$, $Y \ge 1$, and that $|\alpha - a/q| \le q^{-2}$ with $(q,a) = 1$. Then*

$$\sum_{n \le X} \min\Big(Y, \frac{1}{Y\|n\alpha\|^2}\Big) \ll \frac{XY}{q} + X + Y + q.$$

Proof Let $\theta = q^2\alpha - aq$, so that $|\theta| \le 1$. Then for $1 \le n \le \frac{1}{2}q$ we have $n\alpha = \frac{na}{q} + \frac{\theta'}{2q}$ where $\theta' = \frac{2n\theta}{q}$, so $|\theta'| \le 1$ and hence $\|n\alpha\| \ge \frac{1}{2}\|na/q\|$. Now suppose that $n = uq + v$ where u and v are integers with $u \ge 1$ and $|v| \le \frac{1}{2}q$. Then

$$n\alpha = ua + \frac{va}{q} + \frac{u\theta}{q} + \frac{v\theta}{q^2} = ua + \frac{va + \lfloor u\theta \rfloor}{q} + \frac{3\theta''}{2q}.$$

Thus, given u, we have

$$\|n\alpha\| \gg \Big\|\frac{va + \lfloor u\theta \rfloor}{q}\Big\|$$

for all but at most 3 values of v. Thus

$$\begin{aligned}\sum_{n \le X} \min\Big(Y, \frac{1}{Y\|n\alpha\|^2}\Big) &\ll \sum_{0 \le u \le \frac{X}{q} + \frac{1}{2}} \Bigg(Y + \sum_{0 \le v \le q} \min\Big(Y, \frac{1}{Y\|\frac{va + \lfloor u\theta \rfloor}{q}\|^2}\Big)\Bigg) \\ &\ll \sum_{0 \le u \le \frac{X}{q} + \frac{1}{2}} \Bigg(Y + \sum_{1 \le r \le \frac{1}{2}q} \min\Big(Y, \frac{q^2}{Yr^2}\Big)\Bigg),\end{aligned}$$

and for a suitable $R \ge 0$ this is

$$\ll \Big(1 + \frac{X}{q}\Big)\Big(Y + YR + \frac{q^2}{Y(R+1)}\Big).$$

The choice $R = \lfloor q/Y \rfloor$ gives the desired conclusion. □

T:VMTechEst **Theorem 24.7** *Let $f(n) = \boldsymbol{n}^{(k)} \cdot \boldsymbol{\alpha}$ and suppose that for $j = 1, 2, \ldots, k$ there are a_j, q_j so that $(a_j, q_,) = 1$, $|\alpha_j - a_j/q_j| \le q_j^{-2}$, and that $1 \le X \le X' \le 2X$, $1 \le Y \le Y' \le 2Y$. Let*

$$T = \sum_{X < l \le X'} \sum_{Y < m \le Y'} e(f(lm)).$$

Then for each $b \ge k$ we have

$$T \ll XY\Bigg(\frac{J_k(2X,b)J_k(2Y,b)}{(4XY)^{2b - \frac{1}{2}k(k+1)}}\Bigg)^{\frac{1}{4b^2}} \prod_{j=1}^{k} \Big(\frac{1}{q_j} + \frac{1}{X^j} + \frac{1}{Y^j} + \frac{q_j}{X^jY^j}\Big)^{\frac{1}{4b^2}}.$$

Proof The number of integers l such that $X < l \le 2X$ is $\le X+1 \le 2X$, so by Hölder's inequality

$$|T|^{2b} \le (2X)^{2b-1} \sum_{X<l\le 2X} \Big| \sum_{Y<m\le Y'} e(f(lm))\Big|^{2b}.$$

Now

$$\Big| \sum_{Y<m\le Y'} e(f(lm))\Big|^{2b} = \sum_{Y<m_1,\dots,m_{2b}\le Y'} e\Big(\sum_{j=1}^k \alpha_j l^j F_j(\boldsymbol{m})\Big)$$

where

$$F_j(\boldsymbol{m}) = m_1^j + \cdots + m_b^j - m_{b+1}^j - \cdots - m_{2b}^j.$$

Then

$$|T|^{2b} \le (2X)^{2b-1} \sum_{Y<m_1,\dots,m_{2b}\le 2Y} \Big| \sum_{X<l\le 2X} e\Big(\sum_{j=1}^k \alpha_j l^j F_j(\boldsymbol{m})\Big)\Big|.$$

Now Hölder's inequality, once more, gives

$$|T|^{4b^2} \le (4XY)^{4b^2-2b} \sum_{Y<m_1,\dots,m_{2b}\le 2Y} \Big| \sum_{X<l\le 2X} e\Big(\sum_{j=1}^k \alpha_j l^j F_j(\boldsymbol{m})\Big)\Big|^{2b}.$$

For given $h_1, \dots, h_k$ we collect those $\boldsymbol{m}$ for which $F_j(\boldsymbol{m}) = h_j$ for $1 \le j \le k$. We note that the contribution is 0 unless $|h_j| \le b(2Y)^j$ for all j, $1 \le j \le k$. Therefore,

$$|T|^{4b^2} \le (4XY)^{4b^2-2b} \sum_{\substack{h_1,\dots,h_k \\ |h_j|\le b(2Y)^j}} J_k(2Y, b, \boldsymbol{h}) \Big| \sum_{X<l\le 2X} e\Big(\sum_{j=1}^k \alpha_j h_j l^j\Big)\Big|^{2b}.$$

Let

$$H_j = 2\lfloor b(2Y)^j \rfloor.$$

Then

$$|T|^{4b^2} \le (2X)^{4b^2-2b} J_k(2Y, b) \times \sum_{\substack{h_1,\dots,h_k \\ |h_j|\le H_j}} \prod_{j=1}^k 2\Big(1 - \frac{|h_j|}{H_j}\Big) \Big| \sum_{X<l\le 2X} e\Big(\sum_{j=1}^k \alpha_j h_j l^j\Big)\Big|^{2b}.$$

On expanding the power to form a $2b$-fold sum, and taking the sum over the h_j inside, we find that the last factor above is

$$= \sum_{X<l_1,\ldots,l_{2b}\le 2X} \prod_{j=1}^{k} \Big(\frac{2}{H_j}\Big|\sum_{h=1}^{H_j} e\big(\alpha_j h F_j(\boldsymbol{l})\big)\Big|^2\Big).$$

By the estimate (**??**) for the sum of a geometric progression we see that the above is

$$\le 2^k \sum_{X<l_1,\ldots,l_{2b}\le 2X} \prod_{j=1}^{k} \min\Big(H_j, \frac{1}{H_j\|\alpha_j F_j(\boldsymbol{l})\|^2}\Big)$$
$$\le 2^k \sum_{\substack{g_1,\ldots,g_k\\ |g_j|\le G_j}} J_k(2X,b,\boldsymbol{g}) \prod_{j=1}^{k} \min\Big(H_j, \frac{1}{H_j\|\alpha_j g_j\|^2}\Big)$$

where $G_j = b(2X)^j$. By Lemma 24.6 this does not exceed, for some absolute constant C_1,

$$C_1^k J_k(2X,b) \prod_{j=1}^{k} \Big(\frac{G_j H_j}{q_j} + G_j + H_j + q_j\Big)$$
$$\le C_1^k J_k(2X,b) \prod_{j=1}^{k} \Big(4b^2(4XY)^j\Big(\frac{1}{q_j} + \frac{1}{X^j} + \frac{1}{Y^j} + \frac{q_j}{X^jY^j}\Big)\Big).$$

Therefore

$$|T|^{4b^2} \le (4XY)^{4b^2-2b} J_k(2X,b) J_k(2Y,b) C_1^k (4b^2)^k (4XY)^{\frac{1}{2}k(k+1)}\Delta$$

where

$$\Delta = \prod_{j=1}^{k} \Big(\frac{1}{q_j} + \frac{1}{X^j} + \frac{1}{Y^j} + \frac{q_j}{X^jY^j}\Big).$$

The desired conclusion now follows since $b \ge k$. □

L:CoefSumBnd **Lemma 24.8** *For integers h let c_h denote arbitrary complex numbers with $\sum_h |c_h| < \infty$, and let*

$$S(\beta) = \sum_h c_h e(h\beta).$$

Further let H be an integer and L and M be natural numbers. Then

$$\Big|\sum_{|h-H|\le M} c_h\Big| \le \Big(\frac{2M}{L} + 1\Big) \max_\beta |S(\beta)| + \sum_{M<|h-H|<M+L} |c_h|.$$

We note that $S(\beta)$ is uniformly approximated by the continuous functions $s_N(\beta) = \sum_{h=-N}^{N} c_h e(h\beta)$, so $S(\beta)$ is continuous. Also, $S(\beta)$ has period 1, and the circle group $\mathbb{T}$ is compact, so the supremum of $|S(\beta)$ is attained, so we are free to refer to $\max_\beta |S(\beta)|$.

Proof By replacing $S(\beta)$ by $S(\beta)e(-H\beta)$ we may assume that $H = 0$. We note the basic formulæ that define the Fejér kernel $\Delta_M(\beta)$, namely that

$$\Delta_M(\beta) = \sum_{h=-M+1}^{M-1} \Big(1 - \frac{|h|}{M}\Big) e(h\beta) = \frac{1}{M}\Big(\frac{\sin \pi M\beta}{\sin \pi\beta}\Big)^2.$$

We note that $\Delta_M(\beta) \ge 0$ for all β, and hence that

$$\int_0^1 |\Delta_M(\beta)|\, d\beta = \int_0^1 \Delta_M(\beta)\, d\beta = \widehat{\Delta}_M(0) = 1.$$

(If these properties are not familiar, see Exercise 1.) Let

$$K(\beta) = \frac{(M+L)\Delta_{M+L}(\beta) - M\Delta_M(\beta)}{L} = \sum_{-M-L+1}^{M+L-1} \widehat{K}(h) e(h\beta)$$

where

$$\widehat{K}(h) = \begin{cases} 1 & (|h| \le M), \\ (M + L - |h|)/L & (M \le |h| \le M + L), \\ 0 & (|h| > M + L). \end{cases}$$

Since $|\widehat{K}_M(h)| \le 1$ for $M < |h| < M + L$ it follows by the triangle inequality that

$$\Big|\sum_{h=-M}^{M} c_h - \sum_{h=-M-L+1}^{M+L-1} c_h \widehat{K}(h)\Big| \le \sum_{M<|h|<M+L} |c_h|.$$

On the other hand,

$$\Big|\sum_h c_h \widehat{K}(h)\Big| = \Big|\int_0^1 S(\beta)K(-\beta)\, d\beta\Big| \le \int_0^1 |K(-\beta)|\, d\beta \max_\beta |S(\beta)|$$

and $|K(-\beta)| = |K(\beta)| \le \frac{M+L}{L}\Delta_{M+L}(\beta) + \frac{M}{L}\Delta_M(\beta)$, so

$$\int_0^1 |K(\beta)|\, d\beta \le \frac{M+L}{L} + \frac{M}{L},$$

which gives the stated result. □

Note that in Lemma 24.8, if the c_h are bounded, then a good choice for L would be

$$L \asymp (M \sup_\beta |S(\beta)|)^{\frac{1}{2}}.$$

However we would need to deal with sums over h on the right under somewhat more general conditions, and the following lemma shows how to do this.

L:SumBizcn **Lemma 24.9** *Suppose that $c_1, c_2, \ldots$ are complex numbers, and that $0 < X \le N^{\frac{1}{2}}$, $\sum_{n=1}^\infty |c_n| < \infty$ and*

$$|c_n| \ll \sum_{\substack{d|n \\ X<d\le 2X}} 1.$$

Then for each Y, Z with $0 \le Y \le Z \le N$ we have

$$\sum_{Y<n\le Z} c_n \ll N^{\frac{1}{2}} \Big(1 + \max_\beta \Big|\sum_{n=1}^\infty c_n e(n\beta)\Big|\Big)^{\frac{1}{2}}.$$

Proof Let H be one of the integers closest to $\frac{1}{2}(Y+Z)$ and choose $M \ge 1$ minimally so that $H - M \le Y$ and $H + M \ge Z$. Note that $M \le N$. Then $\sum_{Y<n\le Z} c_n$ differs from $\sum_{|h-H|\le M} c_h$ by an amount $\ll \max_{n\le N} d(n) \ll N^{\frac{1}{2}}$. For convenience we put $c_n = 0$ when $n \le 0$. Now Lemma 24.8 gives

$$\sum_{Y<n\le Z} c_n \ll (ML^{-1}+1) \max_\beta \Big|\sum_{n=1}^\infty c_n e(n\beta)\Big| + N^{\frac{1}{2}} + \sum_{M<|h-H|<M+L} |c_h|.$$

The last sum here is bounded by two sums of the form

$$\sum_{K<h\le K+L'} |c_h|$$

where K is a non-negative integer and $L' \le L$. Such a sum is

$$\ll \sum_{K<h\le K+L'} \sum_{\substack{d|h \\ X<d\le 2X}} 1 \le \sum_{X<d\le 2X} \Big(\frac{L}{d}+1\Big) \ll L + N^{\frac{1}{2}}.$$

Hence

$$\sum_{Y<n\le Z} c_n \ll (NL^{-1}+1) \max_\beta \Big|\sum_{n=1}^\infty c_n e(n\beta)\Big| + L + N^{\frac{1}{2}}.$$

Let $S = \max_\beta \big|\sum_{n=1}^\infty c_n e(n\beta)\big|$. If $S \le N$, then the choice $L = \lfloor (NS)^{\frac12} \rfloor$ $+1$ gives the lemma. If $S > N$, then the trivial bound

$$\sum_{Y<n\le Z} c_n \ll \sum_{X<d\le 2X} \frac{N}{d} \ll N$$

suffices. □

L:SumBizcn2 **Lemma 24.10** *Suppose that M, M', N and N' are natural numbers with $M \le N$ and $N < N' \le 2N$, that X is a positive real number with $X \le (2M)^{\frac12}$, and that c_m $(m \in \mathbb{N})$ and d_h $(h \in \mathbb{Z})$ are complex numbers such that*

$$c_m \ll \sum_{\substack{d|m \\ X<d\le 2X}} 1, \qquad \sum_{m=1}^\infty |c_m| < \infty, \qquad |d_h| \le 1.$$

Then

$$\Big|\sum_{M<m\le 2M} c_m \sum_{N<n\le N'} d_n\Big|$$
$$\ll NM^{\frac12}\left(1 + \max_{n\in(M+N,2M+2N]} \max_\beta \Big|\sum_{m=1}^\infty c_m d_{n-m} e(m\beta)\Big|\right)^{\frac12}$$

Proof The product

$$\sum_{M<m\le 2M} c_m \sum_{N<n\le N'} d_n$$

can be rearranged to give

$$\sum_{M<m\le 2M} \sum_{N+m<n\le N'+m} c_m d_{n-m} = \sum_{M+N<n\le 2M+N'} \sum_{\substack{M<m\le 2M \\ n-N'\le m\le n-N}} c_m d_{n-m}.$$

Now we apply the preceding lemma, with N replaced by $2M$, to the inner sum. Thus the inner sum is

$$\ll M^{\frac12}\left(\max_\beta \Big|\sum_{m=1}^\infty c_m d_{n-m} e(m\beta)\Big|\right)^{\frac12},$$

from which the conclusion follows. □

T:VT1 **Theorem 24.11** *Suppose that $N < N' \le 2N$, $3 \le Q \le N^{\frac12}$, $k \ge 2$, $R > 0$, that f is a $k+1$ times continuously differentiable real-valued function defined on the interval $[N-3Q^2, 2N+4Q^2]$, and that*

$$\frac{|f^{(k+1)}(u)|}{(k+1)!} \le (2Q)^{-2k-2} R^{-1}$$

for all u. Suppose further that $b \geq k$*, that*

$$0 < R_j \leq Q^j,$$

that for each integer $n \in [N+Q^2, 2N+2Q^2]$ *there are* $a_2/q_2, \ldots, a_k/q_k$ *such that* $(a_j, q_j) = 1$*,*

$$R_j \leq q_j \leq Q^{2j} R_j^{-1},$$

and

$$\Big|\frac{f^{(j)}(n)}{j!} - \frac{a_j}{q_j}\Big| \leq \frac{1}{q_j^2}.$$

Then the exponential sum

$$S = \sum_{N<n\leq N'} e(f(n))$$

satisfies

$$S \ll N\biggl(\frac{1}{Q} + \frac{1}{\sqrt{R}} + \Bigl(\frac{J_k(2Q,b)}{(2Q)^{2b-\frac{1}{2}k(k+1)}}\Bigr)^{\frac{1}{4b^2}} \Bigl(\prod_{j=2}^{k} \frac{1}{R_j}\Bigr)^{\frac{1}{8b^2}}\biggr).$$

Proof Define

$$c_n = \sum_{\substack{l>Q,\, m\leq 2Q \\ lm=n}} 1.$$

Let $M = Q^2$. Now

$$\sum_{M<n\leq 2M} c_n$$

is a sum over the ordered pairs of integers l, m with $Q < l \leq 2Q$, $Q < m \leq 2Q$ and $M/m < l \leq 2M/l$, and this certainly counts every pair l, m with $Q < l \leq \sqrt{2}Q$, $Q < m \leq \sqrt{2}Q$ and for $Q \geq 3$ there are always $\gg Q^2$ such pairs. Hence, by the preceding lemma

$$Q^2 S \ll NQ\biggl(1 + \max_{n\in(Q^2+N, 2Q^2+2N]} \max_{\beta} \Big|\sum_{m=1}^{\infty} c_m e(f(n-m) + m\beta)\Big|\biggr)^{\frac{1}{2}}.$$

The innermost sum is

$$\sum_{Q<l\leq 2Q} \sum_{Q<m\leq 2Q} e(f(n-lm) + lm\beta).$$

By Taylor's theorem with Lagrange's form of the remainder,

$$f(n-lm) = \sum_{j=0}^{k} (-lm)^j \frac{f^{(j)}(n)}{j!} + x^{k+1} \frac{f^{(k+1)(n-\theta lm}(\theta x)}{(k+1)!}$$

with $0 < \theta < 1$. Let $\alpha_1 = -f'(n) + \beta$ and $\alpha_j = (-1)^j \frac{f^{(j)}(n)}{j!}$ $(2 \le j \le k)$. Then

$$lm\beta + f(n - lm) = f(n) + \sum_{j=1}^{k} (lm)^j \alpha_j + O(R^{-1})$$

and so

$$\sum_{Q<l\le 2Q} \sum_{Q<y\le 2Q} e(f(n-lm) + lm\beta)$$

$$= e(f(n)) \sum_{Q<l\le 2Q} \sum_{Q<y\le 2Q} e\Big(\sum_{j=1}^{k} (lm)^l \alpha_j\Big) + O\big(Q^2 R^{-1}\big).$$

For $2 \le j \le k$, the hypothesis of Theorem 24.7, with a_j replaced by $(-1)^j a_j$, is satisfied and we may certainly take $q_1 = 1$. Hence, by Theorem 24.7,

$$\sum_{Q<l\le 2Q} \sum_{Q<y\le 2Q} e(f(n-lm) + lm\beta)$$

$$\ll \frac{Q^2}{R} + Q^2 \Big(\frac{J_k(2Q,b)}{(2Q)^{2b-\frac12 k(k+1)}}\Big)^{\frac{1}{2s^2}} \prod_{j=2}^{k} \Big(\frac{4}{R_j}\Big)^{\frac{1}{4b^2}}.$$

We note that $k \ll 4b^2$. Thus

$$S \ll \frac{N}{Q}\Bigg(1 + \frac{Q^2}{R} + Q^2 \Big(\frac{J_k(2Q,b)}{(2Q)^{2b-\frac12 k(k+1)}}\Big)^{\frac{1}{2b^2}} \prod_{j=2}^{k} \Big(\frac{1}{R_j}\Big)^{\frac{1}{4b^2}}\Bigg)^{\frac12}.$$

□

T:VT2 **Theorem 24.12** *Suppose that* $0 < \delta < 1$, $N < N' \le 2N$, $R > 0$, $3 \le Q \le N^{\frac12}$, $k \ge 2$, *that* f *is a* $k+1$ *times continuously differentiable real-valued function on the interval* $\big[N - 3Q^2, 2N + 4Q^2\big]$, *and that*

$$\frac{|f^{(k+1)}(u)|}{(k+1)!} \le (2Q)^{-2k-2} R^{-1}.$$

Suppose further that

$$0 < R_j \le Q^j,$$

that for each integer $n \in [N + Q^2, 2N + 2Q^2]$ *there are* $a_2/q_2, \dots, a_k/q_k$ *such that* $(a_j, q_j) = 1$,

$$R_j \le q_j \le Q^{2j} R_j^{-1}$$

and

$$\Big|\frac{f(j)(n)}{j!} - \frac{a_j}{q_j}\Big| \le \frac{1}{q_j^2},$$

and that there are $h \ge \delta k$ values of j with $2 \le j \le k$ such that $R_j \ge Q^{\delta j}$. Then there is a number $\lambda > 0$ depending only on δ such that the exponential sum

$$S = \sum_{N<n\le N'} e(f(n))$$

satisfies

$$S \ll NQ^{-\lambda k^{-2}} + NR^{-\frac{1}{2}}.$$

Proof By Theorem 24.5,

$$\left(\frac{J_k(2Q,kr)}{(2Q)^{2kr-\frac{1}{2}k(k+1)}}\right)^{\frac{1}{4k^2r^2}} \le \exp\Big(\frac{C\log k}{4r}\Big)Q^{\frac{1}{8r^2}(1-1/k)^r}.$$

The sum of δj over the h values of j for which $R_j \ge Q^{\delta j}$ is bounded below by

$$\sum_{j=2}^{h} \delta j = \frac{\delta}{2}(h+3)h > \frac{\delta^3 hk^2}{2}.$$

Thus

$$S \ll NQ^{-1} + NR^{-\frac{1}{2}} + \exp\Big(\frac{C\log k}{4r}\Big)NQ^{\frac{1}{8r^2}\left((1-1/k)^r - \frac{\delta^3}{2}\right)}.$$

Now we take $r = \varkappa k$ where $\varkappa$ is sufficiently large in terms of δ so that

$$(1-1/k)^r < \frac{\delta^3}{4}.$$

□

T:VT3 **Theorem 24.13** *There is a positive number C such that if t is a real number and r, N, N' are natural numbers with*

$$t > N^{\frac{1}{9}}, \quad N^{r-1} < t \le N^r, \quad N < N' \le 2N,$$

then

$$\sum_{N<n\le N'} (n+\alpha)^{it} \ll N^{1-Cr^{-2}}$$

uniformly for $0 \le \alpha \le 1$.

Proof Let $f(x) = \frac{t}{2\pi}\log(x+\alpha)$. Then, for $j \ge 1$

$$\frac{f^{(j)}(x)}{j!} = (-1)^{j-1}\frac{t}{2\pi j(x+\alpha)^j}.$$

Suppose that

$$Q \le \frac{1}{2}N^{\frac{1}{2}}.$$

Then for $N - 3Q^2 \le x \le 2N + 4Q^2$,

$$\Big|\frac{f^{(k+1)}(x)}{(k+1)!}\Big| \le N^r\Big(\frac{4}{N}\Big)^{k+1}.$$

Also, since $t > N^{r-1} \ge N^{r/9}$, if $N + Q^2 \le n \le 2N + 2Q^2$, then $n + \alpha \le 2N + N/2 + 1 < 4N$ and

$$\frac{N^{r/9}}{(8\pi N)^j} \le \frac{t}{2\pi j(4N+1)^j} \le \Big|\frac{f^{(j)}(n)}{j!}\Big| \le N^{r-j}. \qquad (24.11)$$ E:UsefulIneq

Our aim is to apply Theorem 24.12. Thus we need to introduce a parameter R, and it will be suitable provided it satisfies

$$N^r\Big(\frac{4}{N}\Big)^{k+1} \le (2Q)^{-2k-2}R^{-1}. \qquad (24.12)$$ E:RIneq

We also need to find suitable rational approximations a_j/q_j. To this end for each $n \in [N + Q^2, 2N + 2Q^2]$ and each j with $2 \le j \le k$ let

$$Q_j = 2\Big|\frac{j!}{f^{(j)}(n)}\Big|.$$

Then by Dirichlet's theorem choose q_j, a_j so that $(a_j, q_j) = 1$, $q_j \le Q$, and

$$\Big|\frac{f^{(j)}(n)}{j!} - \frac{a_j}{q_j}\Big| \le \frac{1}{q_jQ_j}.$$

If $a_j = 0$, then we would have

$$\Big|\frac{f^{(j)}(n)}{j!}\Big| \le \frac{1}{2}\Big|\frac{f^{(j)}(n)}{j!}\Big|,$$

which is impossible. Thus

$$1 \le |a_j| \le q_j\Big|\frac{f^{(j)}(n)}{j!}\Big| + \frac{1}{Q_j} = \Big(q_j + \frac{1}{2}\Big)\Big|\frac{f^{(j)}(n)}{j!}\Big|,$$

and so

$$\Big|\frac{j!}{f^{(j)}(n)}\Big| - \frac{1}{2} \le q_j \le Q_j = 2\Big|\frac{j!}{f^{(j)}(n)}\Big|.$$

From the inequalities (24.11) we now deduce the further inequalities

$$\frac{1}{2}N^{j-r} \le q_j \le (16\pi N)^j N^{-r/9}.$$

The upper bound is immediate, as is the lower one provided that $j \ge r$. But $q_j \ge 1$, so the lower bound above is trivial when $j < r$. Now we will be in a position to apply the preceding theorem with $R = N^\delta$, $R_j = R^j$ provided that (24.12) holds and that

$$N^{j\delta} \le \frac{1}{2}N^{j-r}, \qquad (16\pi N)^j N^{-r/9} \le Q^{2j} N^{-j\delta} \tag{24.13}$$ E:MoreIneqs

hold for an appreciable range of j. To this end we take

$$\delta = \frac{1}{100}, \qquad Q = N^{\frac{1}{2}-\delta}.$$

Then (24.12) holds provided that $16^{k+1}N^{r+\delta} \le N^{\delta(2k+2)}$, and this follows as long as $k\delta > r$ and $N \ge 16^{1/\delta}$. We may certainly suppose the latter of these inequalities since otherwise the conclusion is trivial. Thus for (24.12) it suffices that

$$r \le k\delta. \tag{24.14}$$ E:rlekdelta

The inequalities (24.13) will follow provided that

$$N^{2j\delta} \le N^{j-r}, \qquad N^{3j\delta} \le N^{r/9}$$

and $N \ge 56^{1/\delta}$, which again we may certainly suppose. The two inequalities above reduce to

$$\frac{r}{1-2\delta} \le j \le \frac{r}{27\delta},$$

which is to say that

$$\frac{50}{49}r \le j \le \frac{100}{18}r.$$

It remains to make a suitable choice for k in terms of r, and $k = 100r$ suffices. □

24.2.1 Exercises

Exer:Fejer 1. (a) By expanding the mod-square and collecting terms, show that

$$\Big|\sum_{m=0}^{N-1} e(mx)\Big|^2 = \sum_{n=-N+1}^{N-1}(N-|n|)e(nx) = N+2\sum_{n=1}^{N-1}(N-n)\cos 2\pi nx.$$

(b) Use the formula for the sum of a geometric series to show that

$$\sum_{m=0}^{N-1} e(mx) = e\big(\tfrac{N-1}{2}x\big)\frac{\sin \pi N x}{\sin \pi x}.$$

(c) Let the Fejér kernel $\Delta_N(x)$ be defined by the formula

$$\Delta_N(x) = \frac{1}{N}\Big(\frac{\sin \pi N x}{\sin \pi x}\Big)^2.$$

Show that the above is

$$= \sum_{n=-N}^{N} (1 - |n|/N) e(nx).$$

(d) Show that

$$0 \le \Delta_N(x) \le \min\Big(N, \frac{1}{4N\|x\|^2}\Big)$$

for all x.

(e) Let $L^1(\mathbb{T})$ denote the set of functions f with period 1 such that $\int_0^1 |f(x)|\,dx < \infty$. For $f \in L^1(\mathbb{T})$ let

$$\widehat{f}(n) = \int_0^1 f(x) e(-nx)\,dx$$

denote the Fourier coefficients of f, and set

$$\sigma_N(f;x) = \sum_{-N}^{N} (1 - |n|/N) \widehat{f}(n) e(nx).$$

Show that the above is

$$= \int_0^1 \Delta_N(x-u) f(u)\,du.$$

(f) Suppose that $f \in L^1(\mathbb{T})$. Show that

$$f(x) - \sigma_N(f;x) = \int_0^1 \Delta_N(x-u)(f(x) - f(u))\,du.$$

(g) Let $C(\mathbb{T})$ denote the set of continuous functions with period 1. Suppose that $f \in C(\mathbb{T})$. Show that for any $\varepsilon > 0$ there is an $N_0 = N_0(\varepsilon, f)$ such that

$$\big|f(x) - \sigma_N(f;x)\big| < \varepsilon$$

for all x, if $N > N_0$.

2. Let $c_h\, h \in \mathbb{Z}$ denote arbitrary complex numbers with $\sum_h |c_h| < \infty$, and let

$$S(\beta) = \sum_h c_h e(h\beta).$$

Show that for any integer H and natural number N we have

$$\sum_{H<h\le H+N} c_h = \int_0^1 S(\beta) \sum_{H<h\le H+N} e(-h\beta)\,d\beta.$$

Hence, or otherwise, show that

$$\Big|\sum_{H<h\le H+N} c_h\Big| \le (\log(eN)) \sup_\beta |S(\beta)|.$$

3. Let $S^*(\boldsymbol{\alpha}) = \max_{M\le N}\Big|\sum_{n\le M} e(\boldsymbol{n}^{(k)}\cdot\boldsymbol{\alpha})\Big|$.

   (a) Prove that

   $$\int_{\mathbb{T}^k} S^*(\boldsymbol{\alpha})\,\boldsymbol{d\alpha} < (\log 4N)^{2b} J_k(N,b).$$

   (b) Suppose that $|\alpha_j - \beta_j| \le \frac{1}{2kN^j}$ for $j = 1,\ldots,k$. Show that

   $$e^{-\pi} S^*(\boldsymbol{\alpha}) < S^*(\boldsymbol{\beta}) < e^{\pi} S^*(\boldsymbol{\alpha}).$$

   (c) Suppose that $k\ge 3$, that $|\alpha_k - a/q| \le q^{-2}$, $(a,q)=1$ and that $N\le q\le N^{k-1}$. Show that $S(\boldsymbol{\alpha}) \ll_k N^{1-(4k^2\log k)^{-1}}$.

   (d) Suppose that $k\ge 3$, that $|\alpha_k - a/q| \le q^{-2}$, $(a,q)=1$ and that $N^{\frac14 k} \le q \le N^{\frac34 k}$. Show that there is a positive constant c such that $S(\boldsymbol{\alpha}) \ll_k N^{1-ck^{-2}}$.

24.3 The Korobov-Vinogradov zero-free region

S:KVZFR

The Hurwitz zeta function $\zeta(s,w)$ is defined for $\operatorname{Re} s > 1$ and $0 \le \operatorname{Re} w \le 1$ by

$$\zeta(s,w) = \sum_{n=0}^{\infty} \frac{1}{(n+w)^s}. \tag{24.15}$$

The following lemma can be established in exact analogy to Theorem 1.12.

T:Hurcont **Lemma 24.14** *Suppose that* $\operatorname{Re} s > 0$*,* $0 < \alpha \le 1$ *and* $x \ge 1$*. Then*

$$\zeta(s,\alpha) - \frac{1}{s-1} = \sum_{0\le n\le x} \frac{1}{(n+\alpha)^s} + \frac{(x+\alpha)^{1-s}-1}{s-1} + \frac{\{x\}+\alpha}{(x+\alpha)^s} - s\int_x^\infty \frac{\{u\}}{(u+\alpha)^{s+1}}\,du.$$

T:VZetaEst **Theorem 24.15** *There is a positive number C such that if $0 < \alpha \le 1$ and σ and t are real numbers with $\sigma \ge \frac{1}{2}$, then*

$$\zeta(s,\alpha) - \frac{1}{s-1} - \alpha^{-s} \ll (\log \tau)^{\frac{2}{3}} \tau^{C\Theta^{\frac{3}{2}}}$$

where $\tau = |t| + 4$ and $\Theta = \max(0, 1-\sigma)$, and the implicit constant is absolute.

To deduce the following corollary we need only observe that when χ is non-principal modulo q we have

$$L(s,\chi) = q^{-s} \sum_{a=1}^{q} \chi(a)\zeta(s, a/q)$$

and then

$$\sum_{a=1}^{q} \chi(a) a^{-s} \ll \frac{q^{\Theta} - 1}{\Theta}$$

with the interpretation that this is $\log q$ when $\Theta = 0$.

T:VinZL **Corollary 24.16** *There is a positive number C such that if σ and t are real numbers with $\sigma \ge \frac{1}{2}$ and $\Theta = \max(0, 1-\sigma)$, then*

$$\zeta(s) - \frac{1}{s-1} \ll (\log \tau)^{\frac{2}{3}} \tau^{C\Theta^{\frac{3}{2}}} \qquad (24.16)$$ E:vinzeta

If in addition χ is a non-principle character modulo q, then

$$L(s,\chi) \ll \frac{q^{\Theta} - 1}{\Theta} + q^{1-\sigma} (\log \tau)^{\frac{2}{3}} \tau^{C\Theta^{\frac{3}{2}}} \qquad (24.17)$$ E:VinL

Proof If necessary by taking complex conjugates, we may assume that $t \ge 0$ and by Lemma 24.14 with $x = 1$ the result is immediate when $t \le 4$. Thus we may suppose that $t \ge 4$. Moreover, the conclusion is trivial when $\sigma \ge 2$.

The lemma with $x = t$ gives

$$\zeta(s,\alpha) - \frac{1}{s-1} - \alpha^{-s} = \sum_{n \le t} (n+\alpha)^{-s} + O(t^{-\sigma}).$$

Suppose that $N < N' \le 2N$, $N' \le t$. Then, by Theorem, 24.13 there is a positive constant c_1 such that

$$\sum_{N < n \le N'} (n+\alpha)^{-it} \ll N \exp\Big(- c_1 \frac{(\log N)^3}{(\log t)^2} \Big).$$

Thus, by partial summation, whenever $N < N'' \le 2N$ and $N'' \le t$,

$$\sum_{N<n\le N'} (n+\alpha)^{-\sigma-it} \ll N^{\Theta} \exp\Big(-c_1 \frac{(\log N)^3}{(\log t)^2}\Big).$$

We now let N take on the successive values $2^0, 2^2, 2^2, \dots, 2^k$ where k is chosen so that $2^k < t \le 2^{k+1}$. Thus

$$\sum_{n\le t}(n+\alpha)^{-s} \ll 1 + \sum_{j=0}^{k} \exp\Big(\Theta j \log 2 - c_1(\log 2)^3 \frac{j^3}{(\log t)^2}\Big).$$

When $j^2 > \frac{2\Theta(\log t)^2}{c_1(\log 2)^3}$ the general term is

$$\ll \exp\Big(-\frac{c_1}{2}(\log 2)^3 \frac{j^3}{(\log t)^2}\Big)$$

and when $j^2 \le \frac{2\Theta(\log t)^2}{c_1(\log 2)^3}$ is

$$\exp\big(c_2 \Theta^{\frac{3}{2}} \log t\big) \exp\Big(-c_1(\log 2)^3 \frac{j^3}{(\log t)^2}\Big).$$

Thus, in general, each term is

$$\ll \exp\big(c_2 \Theta^{\frac{3}{2}} \log t\big) \exp\Big(-\frac{c_1}{2}(\log 2)^3 \frac{j^3}{(\log t)^2}\Big).$$

Hence

$$\sum_{n\le t} n^{-s} \ll 1 + \exp\big(c_2 \Theta^{\frac{3}{2}} \log t\big) \sum_{j=0}^{k} \exp\Big(-c_1(\log 2)^3 \frac{j^3}{(\log t)^2}\Big).$$

By monotonicity

$$\sum_{j=0}^{\infty} \exp\big(-\delta j^3\big) \le 1 + \int_0^{\infty} \exp\big(-\delta u^3\big)\, du \ll \delta^{-1/3}$$

uniformly for $0 < \delta \le 1$. Thus

$$\sum_{j=0}^{k} \exp\Big(-c_1(\log 2)^3 \frac{j^3}{(\log t)^2}\Big) \ll (\log \tau)^{\frac{2}{3}}$$

and the theorem follows. □

L:6.3rescaled **Lemma 24.17** *Suppose that* $0 < r < R < \Upsilon$ *and* f *is analytic in a* *domain containing the closed disc centered at* s_0 *and of radius* Υ. *Then for* $|s - s_0| \le r$ *we have*

$$\left| \frac{f'}{f}(s) - \sum_{k=1}^{K} \frac{1}{s - s_k} \right| \ll \frac{1}{(R-r)\log \frac{\Upsilon}{R}} \log \frac{M}{|f(s_0)|}$$

where the sum is over all zeros s_k *of* f *for which* $|s_k| \le R$ *and* M *is the maximum modulus of* f *on the circle centered at* s_0 *and of radius* Υ.

Proof This follows by the proof of Lemma 6.3 re-scaled by a factor of Υ and keeping the constant explicit. □

T:Vzfr,Ests **Theorem 24.18** *There is a positive number* C *such that whenever*

$$\sigma > 1 - \frac{1}{C(\log \tau)^{\frac{2}{3}}(\log\log \tau)^{\frac{1}{3}}}$$

and $\tau = |t| + 4$ *we have*

$$\zeta(s) \ne 0,$$

$$\frac{\zeta'}{\zeta}(s) \ll (\log \tau)^{\frac{2}{3}}(\log\log \tau)^{\frac{1}{3}},$$

$$|\log \zeta(s)| \le \frac{2}{3}\log\log \tau + \frac{1}{3}\log\log\log \tau + O(1),$$

and

$$\frac{1}{\zeta(s)} \ll (\log \tau)^{\frac{2}{3}}(\log\log \tau)^{\frac{1}{3}}.$$

Proof We first establish the zero free region. Let $\beta + i\gamma$ be a zero of ζ with $\gamma \ge 30$. Put $t = \gamma$, $\tau = t + 4$ and let

$$\Upsilon = \frac{(\log\log \tau)^{\frac{2}{3}}}{c_1(\log \tau)^{\frac{2}{3}}}$$

where c_1 is chosen large enough to ensure that $\Upsilon \le \frac{1}{2}$. Suppose that

$$\sigma_0 = 1 + \frac{1}{4}\Upsilon,$$

and put $s_0 = \sigma_0 + i\gamma$, $s_0' = \sigma_0 + 2i\gamma$, We have

$$\frac{1}{|\zeta(s_0)|}, \frac{1}{|\zeta(s_0')|} \le \frac{\zeta(2\sigma_0)}{\zeta(\sigma_0)} \ll \log \tau$$

and by (24.16),

$$|\zeta(s)| \le (\log \tau)^{c_2} \qquad (|s - s_0| \le \Upsilon \text{ or } |s - s_0'| \le \Upsilon).$$

If $1-\beta \ge \Upsilon/12$, then we are done. Hence we may suppose that $\beta > 1-\Upsilon/12$. Suppose that $1<\sigma<\sigma_0$. Then by Lemma 24.17 with $r=\Upsilon/3$, $R=\Upsilon/2$ we find that

$$-\operatorname{Re}\frac{\zeta'}{\zeta}(\sigma+2i\gamma) < c_3(\log\tau)^{\frac{2}{3}}(\log\log\tau)^{\frac{1}{3}},$$
$$-\operatorname{Re}\frac{\zeta'}{\zeta}(\sigma+i\gamma) < c_3(\log\tau)^{\frac{2}{3}}(\log\log\tau)^{\frac{1}{3}} - \frac{1}{\sigma-\beta}.$$

Moreover,

$$-\frac{\zeta'}{\zeta}(\sigma) < \frac{1}{\sigma-1} + C_3.$$

Then, by the inequality

$$0 \le -3\frac{\zeta'}{\zeta}(\sigma) - 4\operatorname{Re}\frac{\zeta'}{\zeta}(\sigma+i\gamma) - \operatorname{Re}\frac{\zeta'}{\zeta}(\sigma+2i\gamma)$$

we find that

$$\frac{4}{\sigma-\beta} - \frac{3}{\sigma-1} \le 3C_4(\log\tau)^{\frac{2}{3}}(\log\log\tau)^{\frac{1}{3}}.$$

Also $-\Upsilon/4 < 1+6(1-\beta)-\sigma_0 < \Upsilon/4$. Thus we may take $\sigma = 1+6(1-\beta)$. Then the left hand side is

$$= \Big(\frac{4}{7}-\frac{1}{2}\Big)\frac{1}{1-\beta} = \frac{1}{14(1-\beta)},$$

and the result follows.

To bound the logarithmic derivative we follow the essentials of the proof of Theorem 6.7 but with

$$s_1 = 1 + \frac{1}{c_1(\log\tau)^{\frac{2}{3}}(\log\log\tau)^{\frac{1}{3}}} + it,$$

and in place of Lemma 6.4 we use Lemma 24.17 with

$$4r = 2R = \Upsilon = \frac{1}{c_1}\Big(\frac{\log\log\tau}{\log\tau}\Big)^{\frac{2}{3}}$$

and $s_0 = s_1$. The bounds for $\log\zeta(s)$ and $1/\zeta(s)$ then follow as in the proof of Theorem 6.7. □

T:VzfrL **Theorem 24.19** *There is a positive number C such that if χ is a Dirichlet character modulo q, then the region*

$$\mathcal{R}_q = \Big\{ s : \sigma > 1 - \frac{1}{C\big(\log q + (\log\tau)^{2/3}(\log\log\tau)^{1/3}\big)} \Big\}$$

contains no zero pf $L(s,\chi)$ unless χ is a quadratic character, in which case $L(s,chi)$ has at most one, necessarily real zero $\beta < 1$ in $\mathcal{R}_q$

Proof If χ is principal, then the conclusion follows at once from Theorem 24.18. Hence we may suppose that $q \ge 3$. If $4\log q > \log\tau$, then the theorem follows at once by Theorem 11.3. Hence we may suppose also that $4\log q \le \log\tau$, and in particular $|t| \ge 50$, so that in the proof we need only consider zeros $\rho = \beta + i\gamma$ with $|\gamma \ge 50$. Now we follow the proof of Theorem 11.3, but we use (24.17) and Lemma 24.17 in place of Lemma 11.1, following the pattern of the proof of Theorem 24.18.

We suppose first that

$$\log q \le (\log\tau)^{2/3}(\log\log\tau)^{1/3}.$$

Now when χ is complex we can follow the exact analogue of the proof of the first assertion of Theorem 24.18. Let ρ be a zero with $|\gamma| \ge 50$ and as before let $t = \gamma$, $\tau = 4||t|$. We take

$$\Upsilon = \frac{(\log\log\tau)^{\frac{2}{3}}}{c_1(\log\tau)^{\frac{2}{3}}}$$

where c_1 is chosen large enough to ensure that $\Upsilon \le \frac{1}{2}$ and as before in Lemma 24.17 take $s_0 = 1 + \frac{1}{4}\Upsilon$, $r = \Upsilon/3$, $R = \Upsilon/2$. Then for $\beta > 1 - \Upsilon/12$ and $1 < \sigma < 1 + \Upsilon/4$ we have the inequalities

$$\begin{aligned} -\operatorname{Re}\frac{L'}{L}(\sigma + 2i\gamma, \chi^2) &< c_2(\log\tau)^{\frac{2}{3}}(\log\log\tau)^{\frac{1}{3}}, \\ -\operatorname{Re}\frac{L'}{L}(\sigma + i\gamma, \chi) &< c_2(\log\tau)^{\frac{2}{3}}(\log\log\tau)^{\frac{1}{3}} - \frac{1}{\sigma-\beta}, \\ -\frac{L'}{L}(\sigma, \chi_0) &< c_2\log\log q + \frac{1}{\sigma - 1}. \end{aligned}$$

and we can proceed as in the usual way.

When $\chi_2 = \chi_0$, we have to replace the first of the above inequalities by

$$-\operatorname{Re}\frac{L'}{L}(\sigma + 2i\gamma, \chi^2) < \frac{1-\sigma}{(1-\sigma)^2 + 4\gamma^2} + c_2(\log\tau)^{\frac{2}{3}}(\log\log\tau)^{\frac{1}{3}}$$

and again the proof proceeds as that of Theorem 11.3.

That leaves the case when $(\log\tau)^{2/3}(\log\log\tau)^{1/3} < \log q \le \frac{1}{4}\log\tau$. Now we choose

$$\Upsilon = \frac{(\log q)^2}{c_1(\log\tau)^2}$$

and we find that the first two inequalities above are replaced by

$$-\operatorname{Re}\frac{L'}{L}(\sigma+2i\gamma,\chi^2) < c_2\log q,$$
$$-\operatorname{Re}\frac{L'}{L}(\sigma+i\gamma,\chi) < c_2\log q-\frac{1}{\sigma-\beta},$$

in the case when χ is complex and with the same adjustment as before when $\chi^2=\chi_0$. □

T:VestsL **Theorem 24.20** *Let χ be a non-principal character modulo q, let C be the constant of Theorem 24.19, and suppose that*

$$\sigma \ge 1-\frac{1}{2C\big(\log q+(\log\tau)^{2/3}(\log\log\tau)^{1/3}\big)}.$$

If $L(s,\chi)$ has no exceptional zero, or if β_1 is an exceptional zero of $L(s,\chi)$ but $|s-\beta_1|\ge 1/\log q$, then

$$\frac{L'}{L}(s,\chi) \ll (\log q)+(\log\tau)^{2/3}(\log\log\tau)^{1/3}, \tag{24.18}$$

$$|\log L(s,\chi)| \le \log\big(\log q+(\log\tau)^{2/3}(\log\log\tau)^{1/3}\big)+O(1) \tag{24.19}$$

and

$$\frac{1}{L(s,\chi)} \ll (\log q)+(\log\tau)^{2/3}(\log\log\tau)^{1/3}. \tag{24.20}$$

We do not have anything new to add when there is an exceptional zero β_1 with $|s-\beta_1|<1/\log q$

24.4 Improvements in the Distribution of Prime Numbers

S:IDP

We now apply the new zero-free regions to the distribution of primes.

T:PET **Theorem 24.21** *There is a constant $c>0$ such that*

$$\psi(x)=x+O\left(x\exp\left(-\frac{c(\log x)^{3/5}}{(\log\log x)^{1/5}}\right)\right), \tag{24.21}$$ E:I6.12

$$\vartheta(x)=x+O\left(x\exp\left(-\frac{c(\log x)^{3/5}}{(\log\log x)^{1/5}}\right)\right), \tag{24.22}$$ E:I6.13

and

$$\pi(x) = \mathrm{li}(x) + O\left(x \exp\left(-\frac{c(\log x)^{3/5}}{(\log\log x)^{1/5}}\right)\right). \tag{24.23}$$

E:I6.14

Proof The proof follows that of Theorem 6.9, but using Theorem 24.18 in place of Theorems 6.6 ad 6.7 and with T taken to be

$$T = \frac{c_1(\log x)^{3/5}}{(\log\log x)^{1/5}}$$

with an suitable positive constant c_1. □

For primes in arithmetic progressions there is, as usual, a trade-off between the error term, the range for q, and any possible exceptional zero.

T:PAPET **Theorem 24.22** *There is a constant c_1 such that if χ is a character modulo q and*

$$q \le \exp\left(\frac{\log x}{c_1 \log\log x}\right),$$

then

$$\psi(x,\chi) = E_0(\chi)x - E_1(\chi)\frac{x^{\beta_1}}{\beta_1} + O\left(x \exp\left(-\frac{\log x}{c_1\big(\log q + (\log x)^{2/5}(\log\log x)^{1/5}\big)}\right)\right) \tag{24.24}$$

E:I11.23

where $E_0(\chi) = 0$ unless χ is principal in which case it is 1 and $E_1(\chi) = 0$ unless $L(s,\chi)$ has an exceptional zero β_1 in which case it is 1.

Proof Here one can follow the proof of Theorem 11.16 but with

$$T = \exp\left(\frac{\log x}{c_2\big(\log q + (\log x)^{2/5}(\log\log x)^{1/5}\big)}\right)$$

for some constant c_2. □

In applying this to $\psi(x,q,a)$ there is some limitation on the size of q imposed by the requirement to have an error that is small compared with

$$\frac{x}{\phi(q)}.$$

T:VPage **Corollary 24.23** *There is a constant c_1 such that if*

$$q \le \exp\left(c_1\sqrt{\log x}\right)$$

and $(q,a)=1$, *then*

$$\psi(x,q,a)=\frac{x}{\phi(q)}-E_1(\chi)\frac{\chi_1(a)x^{\beta_1}}{\phi(q)\beta_1}$$
$$+O\left(x\exp\left(-\frac{\log x}{c_1\big(\log q+(\log x)^{2/5}(\log\log x)^{1/5}\big)}\right)\right). \quad (24.25)$$ E:I11.29

When one further applies the consequence, Corollary 11.15, of Siegel's theorem, Theorem 11.14, one has

T:VSiegel1 **Corollary 24.24** *Let* c_1 *be the same constant as in Corollary 24.22. For any positive* A *there is an* $x_0(A)$ *such that if* $q\le(\log x)^A$, *then*

$$\psi(x,\chi)=E_0(\chi)x+O\left(x\exp\left(-\frac{(\log x)^{3/5}}{c_1(\log\log x)^{1/5}}\right)\right). \quad (24.26)$$ E:I11.31

T:VSiegel2 **Corollary 24.25** *Let* c_1 *be the same constant as in Corollary 24.22. For any positive* A *there is an* $x_0(A)$ *such that if* $q\le(\log x)^A$ *and* $(q,a)=1$, *then*

$$\psi(x,q,a)=\frac{x}{\phi(q)}+O\left(x\exp\left(-\frac{(\log x)^{3/5}}{c_1(\log\log x)^{1/5}}\right)\right) \quad (24.27)$$ E:I11.31a

and

$$\pi(x,q,a)=\frac{\mathrm{li}(x)}{\phi(q)}+O\left(x\exp\left(-\frac{(\log x)^{3/5}}{c_1(\log\log x)^{1/5}}\right)\right) \quad (24.28)$$ E:I11.37

Note that c_1 is effective. However, in the current state of knowledge, $x_0(A)$ is not. Alternatively one can make the implicit constant in the O-notation ineffective so that the theorem holds for all $x\ge 2$.

Further improvements are dependent on the distribution and density of zeros near the 1-line and are dependent on the main results of Chapter 28.

24.5 Notes

S:ExpSumII Notes

§24.1 The Vinogradov mean value theorem first appeared in [IV35,IV36] Vinogradov (1935, 1936). The main motivation at that stage was additive number theory, and especially with the aim of improving the known results in Waring's Problem, at least for larger exponents, and this it did spectacularly. There is some indication that Mordell's [LM32] Mordell (1932) work on complete sums was suggestive. The first application to the Riemann zeta function was not made by Vinogradov, but by [NC36] Chudakov (1936). Theorem 24.7 is already in the form given in ????. It is crucial in estimating the kind of sums useful for application to the Riemann zeta-function that the growth of the number $D(k, r)$ should not be too rapid in terms of k and r.

Lemma 24.4 is in [YL43] Linnik (1943).

[RV81] Vaughan (1981) conjectured that

$$J_k(X, b) \sim C_{k,b} \min\left(X^{2b-k(k+1)/2}, X^b\right),$$

and a weaker form of this,

$$J_k(X, b) \ll_{k,b,\varepsilon} X^{\varepsilon} \min\left(X^{2b-k(k+1)/2}, X^b\right)$$

was established by [JB16] Bourgain, Demeter& Guth (2016). Not quite so strong bounds had been established earlier by [TW12] Wooley (2012). However none of these more recent methods give estimates for $D(k, r)$ which are suitable for application here.

§24.2 There is a long succession of papers, [NC46] Chudakov (1946), [TT50] Tatuzawa (1950), first edition (1951) of [ET86] Titchmarsh (1986), [PT53] Turán (1953), [LS57] Schoenfeld (1957), each with small improvements on what went before. The best methods we have currently for deducing bounds from the Vinogradov mean value theorem for the sums that are the subject of Theorem 24.13 were developed by [NK58] Korobov (1958) and [IV58] Vinogradov (1958) independently. However in both these papers claims are made about the error in the prime number theorem, namely that

$$\psi(x) - x \ll x \exp(-c(\log x)^{3/5})$$

which have never been substantiated. There is a very scathing comment by [AI64] Ingham (1964) in a review "it is highly desirable that the claim to the stronger and neater result should be substantiated or withdrawn without further delay". It never has been, although in 1980 Vinogradov does simply write "my method gives" (24.21), tacitly admitting by neglect that this is the best that can he can do. There are full accounts of the [NK58] Korobov (1958) and [IV58] Vinogradov (1958) methods in [AW63] Walfisz (1963) (actually written by Richert), and [HR67] Richert (1967) makes the application to the Hurwitz zeta function.

The kernel $K(\beta)$ defined in the proof of Lemma 24.8 is an example of a de la Vallée Poussin kernel. The upper bound (**??**) suffices for our purposes, but [HM15] Mehta (2015) determined the exact value of this quantity.

§24.4 The application of these methods to primes in arithmetic progressions is not often done, but once one has the bound for the Hurwitz zeta function it is routine and potentially useful. There is an excellent account of the state of play prior to 1958 in [KP57] ?, Chapter VIII, including earlier versions of the results of this section.

24.6 References

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25
Appoximate Functional Equations

C:AFE

We recall that Theorem 10.2 asserts that if s is a complex number, $s \neq 0$, $s \neq 1$, and z is a complex number with $\operatorname{Re} z \geq 0$, then

$$\begin{aligned}\zeta(s)\Gamma(s/2)\pi^{-s/2} &= \pi^{-s/2}\sum_{n=1}^{\infty} n^{-s}\Gamma(s/2, \pi n^2 z) \\ &+ \pi^{(s-1)/2}\sum_{n=1}^{\infty} n^{s-1}\Gamma((1-s)/2, \pi n^2/z) \\ &+ \frac{z^{(s-1)/2}}{s-1} - \frac{z^{s/2}}{s}\,.\end{aligned} \tag{25.1}$$ E:zetaFE

Here $\Gamma(s, a)$ is the incomplete gamma function,

$$\Gamma(s, a) = \int_a^\infty e^{-z} z^{s-1}\, dz\,. \tag{25.2}$$ E:DefIncGamFcn

The integrand is regular at $z = 0$ only when s is a positive integer. To avoid issues of the singularity at 0 we insist that a, and the path joining a to ∞ should lie in the slit complex plane $\mathbb{C} \setminus (-\infty, 0]$. For any fixed such a, $\Gamma(s, a)$ is an entire function of s. When we discussed this formula in Chapter 10, we simply took $z = 1$ (which is essentially what Riemann did). However, the formula (25.1) is not useful when $|t|$ is large and z is real, since then the factors $\Gamma\big(s/2, \pi n^2 z\big)$ are much larger than $\Gamma(s/2)$. For example, $|\Gamma(1 + it, 1)| \asymp 1/|t|$ while $\Gamma(1 + it)$ is exponentially small. More precisely, in Appendix C we used Stirling's formula to show that

$$|\Gamma(s)| \asymp \tau^{\sigma-1/2} e^{-\pi\tau/2} \tag{25.3}$$ E:GamEst

when $|t| \geq 1$ and $|\sigma|$ is uniformly bounded. Fortunately, $\Gamma(s, z) \ll |\Gamma(s)|$ for such s when z is of the form $z = ae^{i\phi}$ where $a > 0$ and

$$\phi = \arctan t\,. \tag{25.4}$$ E:Defphi

Throughout our discussion, the second argument of the incomplete gamma function will have this argument. We note that from this definition it follows that

$$\cos\phi = \frac{1}{\sqrt{t^2+1}}, \qquad \sin\phi = \frac{t}{\sqrt{t^2+1}}\,. \tag{25.5}$$ E:trigvals

We note some useful estimates concerning ϕ. It is clear from the above that

$$\frac{1}{|t|+1} < \cos\phi < \frac{1}{|t|} \tag{25.6}$$ E:cosphiEst1

uniformly for all $t \ne 0$. Since $\tau = |t| + 4$, we also see that

$$\frac{1}{\tau} < \cos\phi \ll \frac{1}{\tau} \tag{25.7}$$ E:cosphiEst1.5

uniformly for all t. From this latter estimate we deduce that

$$\big|(\cos\phi)^{-s}\big| = (\cos\phi)^{-\sigma} \asymp \tau^{\sigma} \tag{25.8}$$ E:cosphiEst2

uniformly for $-A \le \sigma \le A$, with the implicit constant depending on A.

Suppose that $t > 0$. Then

$$\phi = \arctan t = \int_0^t \frac{du}{u^2+1} = \frac{\pi}{2} - \int_t^\infty \frac{du}{u^2+1} > \frac{\pi}{2} - \int_t^\infty \frac{du}{u^2} = \frac{\pi}{2} - \frac{1}{t}\,.$$

Thus $\frac{\pi}{2}t - 1 < \phi t < \frac{\pi}{2}t$. The function $\phi t = t\arctan t$ is an even function of t, so it follows that $\frac{\pi}{2}|t| - 1 < \phi t < \frac{\pi}{2}|t|$, so

$$\phi t = \frac{\pi}{2}\tau + O(1) \tag{25.9}$$ E:phitIneq

uniformly for all real t. Hence

$$\big|e^{i\phi s}\big| = e^{-\phi t} \asymp e^{-\pi\tau/2} \tag{25.10}$$ E:e^iphis

uniformly for all s.

To estimate

$$\Gamma\big(s, ae^{i\phi}\big) = \int_{ae^{i\phi}}^\infty e^{-z} z^{s-1}\,dz$$

we integrate along the ray $z = ue^{i\phi}$ for $a \le u < \infty$. Thus the above is

$$= e^{i\phi s}\int_a^\infty \exp\big(-ue^{i\phi}\big)u^{s-1}\,du\,. \tag{25.11}$$ E:IncGamForm0

By the further change of variable $v = u\cos\phi$ we see that the above is

$$= \frac{e^{i\phi s}}{(\cos\phi)^s}\int_{a\cos\phi}^\infty e^{-v}v^{\sigma-1}e^{it(\log v - v)}\,dv$$
$$= \frac{e^{i\phi s}}{(\cos\phi)^s}\int_{a\cos\phi}^\infty r(v)e^{i\theta(v)}\,dv \tag{25.12}$$ E:IncGamForm

where

$$r(v) = e^{-v}v^{\sigma-1},$$
$$\theta(v) = t(\log v - v)\,. \tag{25.13}$$ E:Defr,theta

On combining (25.3), (25.8), (25.10), and (25.12), we find that

$$\big|\Gamma\big(s, ae^{i\phi}\big)\big| \asymp |\Gamma(s)|\tau^{1/2}\bigg|\int_{a\cos\phi}^\infty r(v)e^{i\theta(v)}\,dv\bigg|\,. \tag{25.14}$$ E:MainReduc

L:IncGamEst **Lemma 25.1** *Suppose that ϕ is given by* (25.4)*, and that A is a positive constant. Then*

$$\Gamma\big(s, ae^{i\phi}\big) = \Gamma(s)\bigg(\chi_{[0,\tau]}(a) + O\Big(e^{-a/\tau}\Big(\frac{a}{\tau}\Big)^\sigma \min\Big(1, \frac{\tau^{1/2}}{|a-\tau|}\Big)\Big)\bigg) \tag{25.15}$$ E:IncGamEst

uniformly for $a \ge 0$, $0 < \sigma \le A$, $|t| \ge 1$.

When $|t| \le 1$ we have available the trivial estimates

$$\big|\Gamma\big(s, ae^{i\phi}\big)\big| \ll \Gamma(\sigma, a\cos\phi) \le \Gamma(\sigma)\,. \tag{25.16}$$ E:IncGamTrivEst

Proof First suppose that $a \ge \tau + \tau^{1/2}$. Let $r(v)$ and $\theta(v)$ be defined as in (25.13). We note that

$$\frac{r(v)}{\theta'(v)} = \frac{e^{-v}v^\sigma}{t(1-v)}\,.$$

If $V > 1$, then the maximum modulus of the above on the interval $[V, \infty)$, and the its total variation on the same interval are both

$$\ll \frac{e^{-V}V^\sigma}{t(V-1)}\,.$$

By Theorem **??**

$$\int_{a\cos\phi}^\infty r(v)e^{i\theta(v)}\,dv \ll \frac{e^{-a\cos\phi}(a\cos\phi)^\sigma}{\tau(a\cos\phi - 1)}$$

since $a\cos\phi > a/\tau > 1$. From (25.7) and (25.8) it follows that the above is

$$\ll |\Gamma(s)|e^{-a/\tau}\Big(\frac{a}{\tau}\Big)^{\sigma}\frac{\tau^{1/2}}{a-\tau}, \tag{25.17}$$ E:Case1Est

which is the desired bound in this case.

Now suppose that $\tau - \tau^{1/2} \le a \le a + \tau^{1/2} = b$. We write the integral in (25.14) as

$$\int_{a\cos\phi}^{b} r(v)e^{i\theta(v)}\,dv + \int_{b}^{\infty} r(v)e^{i\theta(v)}\,dv\,.$$

From (25.17) we know that the second integral above is $\ll \tau^{-1/2}$. From the information that $\tau - \tau^{1/2} \le a \le \tau + \tau^{1/2}$, by means of a little calculation we deduce that $1 - \tau^{-1/2} \le a\cos\phi \le 1 + 17\tau^{-1/2}$. Now $r(v) \asymp 1$ when $v \asymp 1$, so the first integral above is

$$\ll \int_{a\cos\phi}^{b} r(v)\,dv \le \int_{1-\tau^{-1/2}}^{1+17\tau^{-1/2}} r(v)\,dv \ll \int_{1-\tau^{-1/2}}^{1+17\tau^{-1/2}} 1\,dv \ll \tau^{-1/2}\,.$$

By (25.14) it follows that $\Gamma\big(s, ae^{i\phi}\big) \ll |\Gamma(s)|$ in this case.

Finally, we suppose that $0 \le a \le \tau - \tau^{1/2}$. By taking $a = 0$ in (25.12) we obtain a formula for $\Gamma(s)$. On subtracting, we find that

$$\Gamma\big(s, ae^{i\phi}\big) = \Gamma(s) - \frac{e^{i\phi s}}{(\cos\phi)^s}\int_0^{a\cos\phi} r(v)e^{i\theta(v)}\,dv\,.$$

We treat this last integral as we did in the first case, and thus find that

$$\Gamma\big(s, ae^{i\phi}\big) = \Gamma(s)\bigg(1 + O\Big(e^{-a/\tau}\Big(\frac{a}{\tau}\Big)^{\sigma}\frac{\tau^{1/2}}{\tau - a}\Big)\bigg),$$

so we have the stated result. □

We now combine the functional equation as expressed by the identity identity (25.1) with Lemma 25.1, to obtain a useful *approximate functional equation*. In what follows, we put

$$\Delta(s) = \frac{\Gamma\big(\frac{1-s}{2}\big)}{\Gamma\big(\frac{s}{2}\big)}\pi^{s-\frac{1}{2}} = 2^s\pi^{s-1}\Gamma(1-s)\sin\frac{\pi s}{2}\,. \tag{25.18}$$ E:DefDelta

Thus the functional equation for the zeta function in its asymmetric form (as in Corollary 10.4) asserts that $\zeta(s) = \Delta(s)\zeta(1-s)$. From (25.3) it follows that

$$|\Delta(s)| \asymp \tau^{1/2-\sigma} \tag{25.19}$$ E:DeltaEst

when $|t| \ge 1$ and σ is uniformly bounded.

T:zetaAFE **Theorem 25.2** *Suppose that $0<\delta\le 1/2$ is fixed, that $\delta\le\sigma\le 1-\delta$, and that $2\pi xy=\tau$ where $x\ge 1$, $y\ge 1$. Then*

$$\zeta(s)=\sum_{n\le x}n^{-s}+\Delta(s)\sum_{n\le y}n^{s-1} + O\big((x^{-\sigma}+\tau^{1/2-\sigma}y^{\sigma-1})\log\tau\big)\,. \tag{25.20}$$
E:zetaAFE

Note that the last term in the first sum above is of size $x^{-\sigma}$, and that the last term in the second sum is of size $\tau^{1/2-\sigma}y^{\sigma-1}$. Thus the error term is larger than these quantities by only one logarithm.

Proof We divide both sides of (25.1) by $\Gamma(s/2)\pi^{-s/2}$ to see that

$$\zeta(s)=\sum_{n=1}^\infty n^{-s}\frac{\Gamma(s/2,\pi n^2 z)}{\Gamma(s/2)}+\Delta(s)\sum_{n=1}^\infty n^{s-1}\frac{\Gamma((1-s)/2,\pi n^2/z)}{\Gamma((1-s)/2)} + O\Big(\frac{\big|z^{(s-1)/2}\big|+\big|z^{s/2}\big|}{|\Gamma(s/2)|\,\tau}\Big). \tag{25.21}$$
E:zetaAFE0

We take $z=re^{i\phi}$ where $r=\tau/(2\pi x^2)$ and $\phi=\arctan(t/2)$. The error term above is

$$\ll \big(x^{-\sigma}+\tau^{1/2-\sigma}y^{\sigma-1}\big)\tau^{-1/2},$$

which is inconsequential in (25.20). By Lemma 25.1, the first sum above is $\sum_{n\le x}n^{-s}$ plus an error term that is

$$\begin{aligned}
&\ll \sum_{n=1}^\infty n^{-\sigma}\Big(\frac{n^2r}{\tau}\Big)^{\sigma/2}\min\Big(1,\frac{(\tau/2)^{1/2}}{|\pi n^2 r-\tau/2|}\Big)e^{-2\pi n^2 r/\tau}\\
&\ll x^{-\sigma}\tau^{-1/2}\sum_{n\le x-1}\frac{1}{1-n^2/x^2}\ +x^{-\sigma}\ +x^{-\sigma}\tau^{-1/2}\sum_{n\ge x+1}\frac{e^{-n^2/x^2}}{n^2/x^2-1}\\
&\ll x^{1-\sigma}\tau^{-1/2}\sum_{n\le x-1}\frac{1}{x-n}\ +x^{-\sigma}\ +x^{1-\sigma}\tau^{-1/2}\sum_{n\ge x+1}\frac{e^{-n^2/x^2}}{n-x}\\
&\ll \tau^{1/2-\sigma}y^{\sigma-1}\log x+x^{-\sigma}.
\end{aligned}$$

This is admissible in (25.20). Similarly, the second sum in (25.21) is

$\sum_{n\le y} y^{s-1}$ plus an amount that is

$$\begin{aligned}
&\ll \sum_{n=1}^{\infty} n^{\sigma-1}\Big(\frac{n^2}{r\tau}\Big)^{(1-\sigma)/2} \min\Big(1, \frac{(\tau/2)^{1/2}}{|\tau/2-\pi n^2/r|}\Big) e^{-2\pi n^2/(r\tau)} \\
&\ll y^{\sigma-1}\tau^{-1/2} \sum_{n\le y-1} \frac{1}{1-n^2/y^2} + y^{\sigma-1} + y^{\sigma-1}\tau^{-1/2} \sum_{n\ge y+1} \frac{e^{-n^2/y^2}}{n^2/y^2-1} \\
&\ll y^{\sigma}\tau^{-1/2} \sum_{n\le y-1} \frac{1}{y-n} + y^{\sigma-1} + y^{\sigma}\tau^{-1/2} \sum_{n\ge y+1} \frac{e^{-n^2/y^2}}{n-y} \\
&\ll y^{\sigma}\tau^{-1/2}\log y + y^{\sigma-1}.
\end{aligned}$$

We multiply by $\Delta(s)$, and appeal to (25.19) to see that the error term here is

$$\ll (y/\tau)^{\sigma}\log y + y^{\sigma-1}\tau^{1/2-\sigma} \ll x^{-\sigma}\log y + y^{\sigma-1}\tau^{1/2-\sigma}.$$

This is admissible in (25.20), so the proof is complete. □

Despite its elegance, the approximate functional equation is not immediately useful for estimating mean values, since the number of terms in the sums depends on t, while our basic tools concern one fixed sum. To overcome this difficulty we average over the parameters. Set

$$x = \sqrt{2\pi\tau}\cdot r, \qquad y = \sqrt{2\pi\tau}/r$$

in the approximate functional equation (25.20), divide both sides by r, integrate both sides from 1/2 to 2, and finally divide both sides by $\log 4$. The result is that

$$\begin{aligned}\zeta(s) = \sum_n w(n/\sqrt{\tau})n^{-s} + \Delta(s)\sum_n w(n/\sqrt{\tau})n^{s-1} \\ + O\big(\tau^{-\sigma/2}\log\tau\big)\end{aligned} \tag{25.22}$$ E:Wtdafe

where

$$w(u) = \begin{cases} 1 & (0\le u\le\sqrt{\pi/2}), \\ \dfrac{\log(\sqrt{8\pi}/u)}{\log 4} & (\sqrt{\pi/2}\le u\le\sqrt{8\pi}), \\ 0 & (\sqrt{8\pi}\le u)\,. \end{cases} \tag{25.23}$$ E:Defw

Let

$$A(s,x) = \sum_{n\le x} n^{-s} \tag{25.24}$$ E:DefA(s,x)

denote a partial sum of the Dirichlet series that defines $\zeta(s)$. Then

$$\sum_n w(n/\sqrt{\tau})n^{-s} = \int_{1^-}^{\infty} w(x/\sqrt{\tau})\, dA(s,x),$$

which by integration by parts is

$$= \Big[w(x/\sqrt{\tau})A(s,x) \Big|_{1^-}^{\infty} - \int_1^{\infty} A(s,x)\, dw(x/\sqrt{\tau})\,.$$

Now $A(s,x) = 0$ for $x < 1$, and $w(x/\sqrt{\tau}) = 0$ for $x > \sqrt{8\pi\tau}$, so the contributions of the endpoints vanishes. The function $w(u)$ is continuous and piecewise differentiable, so the above is

$$= \frac{-1}{\sqrt{\tau}} \int_1^{\infty} A(s,x) w'(x/\sqrt{\tau})\, dx\,.$$

As $w'(u) = -1/(u \log 4)$ for $\sqrt{\pi/2} < u < \sqrt{8\pi}$, and $w'(u) = 0$ otherwise, the above is

$$= \frac{1}{\log 4} \int_{\sqrt{\pi\tau/2}}^{\sqrt{8\pi\tau}} A(s,x)\, \frac{dx}{x} = \int A(s,x)\, d\mu_\tau(x),$$

say. Here μ_τ is a probability measure (i.e., a nonnegative measure with total mass 1), so the identity above expresses the left hand side as a weighted average of the partial sums $A(s,x)$. The support of μ_τ depends on t, but by integrating over a longer interval we can obtain an upper bound that holds uniformly for t in an interval. Suppose that k is a positive integer. Then by Hölder's Inequality (with exponents $2k$ and $2k/(2k-1)$, whose reciprocals sum to 1) we find that

$$\begin{aligned} \Big| \sum_n w(n/\sqrt{\tau})n^{-s} \Big|^{2k} &\le \Big(\int |A(s,x)|^{2k}\, d\mu_\tau(x) \Big) \Big(\int 1\, d\mu_\tau(x) \Big)^{2k-1} \\ &= \int |A(s,x)|^{2k}\, d\mu_\tau(x)\,. \end{aligned} \tag{25.25}$$

E:Sumub1

Hence the upper bound

$$\Big| \sum_n w(n/\sqrt{\tau})n^{-s} \Big|^{2k} \le \int_{T^{1/2}}^{8T^{1/2}} |A(s,x)|^{2k}\, \frac{dx}{x} \tag{25.26}$$

E:Sumub2

holds uniformly for $T \le t \le 2T$. We do not obtain sharp constants by arguing in this way, but sometimes we can obtain useful bounds.

We now turn to L functions. If χ is a primitive character modulo q with $q > 1$, then (by Theorem 10.7) we know that

$$\begin{aligned} &L(s,\chi)\Gamma((s+\kappa)/2)(q/\pi)^{(s+\kappa)/2} \\ &\quad = (q/\pi)^{(s+\kappa)/2}\sum_{n=1}^{\infty}\chi(n)n^{-s}\Gamma\big((s+\kappa)/2, \pi n^2 z/q\big) \qquad (25.27) \\ &\quad = \varepsilon(\chi)(q/\pi)^{(1-s+\kappa)/2)}\sum_{n=1}^{\infty}\overline{\chi}(n)n^{s-1}\Gamma\big((1-s+\kappa)/2, \pi n^2/(qz)\big) \end{aligned}$$

E:LfcnFE

for z with $\operatorname{Re} z > 0$. As usual,

$$\kappa = \kappa(\chi) = \begin{cases} 0 & \text{if } \chi(-1) = 1, \\ 1 & \text{if } \chi(-1) = -1 \end{cases} \qquad \varepsilon(\chi) = \frac{\tau(\chi)}{i^\kappa \sqrt{q}}\,. \qquad (25.28)$$

E:LfePars

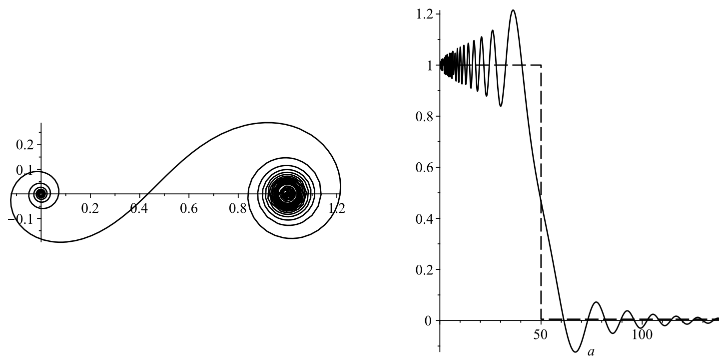
In general, the incomplete gamma function $\Gamma(s,a)$ undergoes a change of behaviour when $|a|$ passes $|s|$; evidence of this can be seen in Lemma 25.1. The crude estimate (25.15) sufficed to give Theorem 25.2 because the numbers n^2 pass τ so quickly. However, in the formula above n^2 is replaced by n^2/q, which passes τ comparatively slowly if q is large. This would not be a problem if the weight $w_0(u) = w_0(s,u) = \Gamma\big(s, ue^{i\theta}\big)/\Gamma(s)$ were to move rather smoothly from near 1 to near 0. To estimate a weighted sum $\sum_{n=1}^N w(n)a_n$ in terms of its partial sums $A(x) = \sum_{n\le x} a_n$ we integrate by parts, as we did in treating the zeta function. This works especially well if $w(x)$ is monotonic, or at least has bounded variation. Unfortunately, this avenue is not immediately available, because $w_0(u)$ has large variation (see Exercise 1). This is illustrated in Figure 25.1 where $w_0(u)$ is first depicted as a curve in the complex plane, and then its real part is graphed, along with its asymptotic shape.
Fortunately, we can avoid this disaster by introducing a simple averaging. Since we are working in the multiplicative group of positive real numbers, instead of the usual arithmetic mean, $\frac{1}{b-a}\int_a^b f(x)\,dx$, we use its multiplicative analogue, $\frac{1}{\log b/a}\int_a^b f(x)\,\frac{dx}{x}$. We put

$$w_1(s,a) = \frac{1}{\Gamma(s)\log 4}\int_{1/2}^{2}\Gamma\big(s, axe^{i\phi}\big)\,\frac{dx}{x} \qquad (25.29)$$

E:Defw1

where ϕ is defined by (25.4). In Figure 25.2 we see that the situation appears to be considerably improved. We now establish that this is indeed the case.

L:w1Est **Lemma 25.3** *Suppose that* $0 < \sigma \le A$, *that* ϕ *is given by* (25.4), *and*



(a) (b)

Figure 25.1 Graphs of (a) $w_0(\frac{1}{2}+50i, a)$ and (b) $\operatorname{Re} w_0(\frac{1}{2}+50i, a)$, for $0 \le a \le 150$. Fi:w0

that w_1 is defined as above. For $a > 0$,

$$\frac{\partial}{\partial a} w_1(s,a) \ll e^{-a/(2\tau)} a^{\sigma-1} \tau^{-\sigma}$$

where the implicit constant may depend on A.

It follows immediately from the above that

$$\begin{aligned}\operatorname{Var}_{[0,\infty)} w_1(s,a) &= \int_0^\infty \Big| \frac{\partial}{\partial a} w_1(s,a) \Big| \, da \ll \int_0^\infty e^{-a/(2\tau)} \Big(\frac{a}{\tau}\Big)^\sigma \, \frac{da}{a} \\ &= \int_0^\infty e^{-x} (2x)^\sigma \, \frac{dx}{x} = \Gamma(\sigma) 2^\sigma,\end{aligned}$$

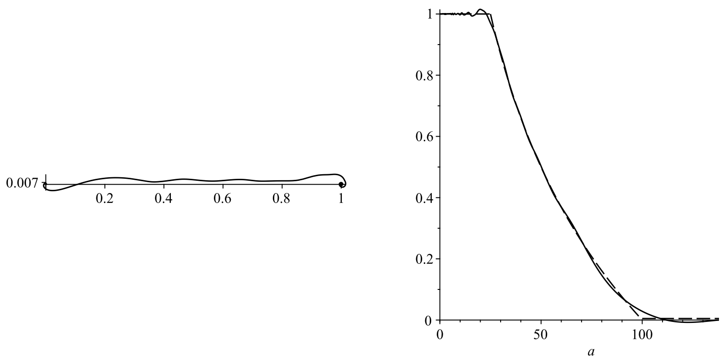
which is not only finite, but also uniformly bounded for $0 < \delta \le \sigma \le A$.

Proof By Leibniz's rule we know that

$$\frac{\partial}{\partial a} w_1(s,a) = \frac{1}{\Gamma(s)\log 4} \int_{1/2}^2 \frac{\partial}{\partial a} \Gamma\big(s, axe^{i\phi}\big) \, \frac{dx}{x}\,.$$

By the parameterization (25.11), this is

$$\begin{aligned}&= \frac{e^{i\phi s}}{\Gamma(s)\log 4} \int_{1/2}^2 \frac{\partial}{\partial a} \int_{ax}^\infty \exp\big(-ue^{i\phi}\big) u^{s-1} \, du \, \frac{dx}{x} \\ &= \frac{-e^{i\phi s}}{\Gamma(s)\log 4} \int_{1/2}^2 \exp\big(-axe^{i\phi}\big) (ax)^{s-1} \, dx \\ &= \frac{-e^{i\phi s}}{a\Gamma(s)\log 4} \int_{a/2}^{2a} \exp\big(-ue^{i\phi}\big) u^{s-1} \, du\,.\end{aligned}$$



(a) (b)

Figure 25.2 Graphs of (a) $w_1(\frac{1}{2}+50i,a)$ and (b) $\operatorname{Re} w_1(\frac{1}{2}+50i,a)$ and its asymptotic shape, for $0 \le a \le 150$. Fi:w1

By (25.11) again, this is

$$= \frac{\Gamma\big(s, 2ae^{i\phi}\big) - \Gamma\big(s, \frac{1}{2}ae^{i\phi}\big)}{a\Gamma(s)\log 4}\,.$$

The right hand side of (25.15) is

$$\ll |\Gamma(s)|e^{-a/\tau}\Big(\frac{a}{\tau}\Big)^{\sigma}.$$

We apply this twice to obtain the stated bound. □

T:LafeEst **Theorem 25.4** *Put*

$$A(s,\chi;x) = \sum_{n\le x}\chi(n)n^{-s}\,. \tag{25.30}$$ E:DefA

Suppose that χ is a primitive character modulo q with $q>1$, and that $0<\delta\le 1/2$. Then

$$L(s,\chi) \ll \int_0^\infty \big(|A(s,\chi;u)| + |A(1-s,\overline{\chi};u)|\big)e^{-u^2/(q\tau)}\Big(\frac{u^2}{q\tau}\Big)^{\delta/2}\frac{du}{u}\,.$$

uniformly for $\delta\le\sigma\le 1-\delta$.

By the change of variable $v = u^2/(q\tau)$ we find that

$$\int_0^\infty \exp\Big(\frac{-u^2}{q\tau}\Big)\Big(\frac{u^2}{q\tau}\Big)^{\delta/2}\frac{du}{u} = 2\int_0^\infty e^{-v}v^{\delta/2-1}\,dv = 2\Gamma(\delta)\,.$$

Thus our bound for $|L(s,\chi)|$ is comparable to a weighted average of the quantity $|A(s,\chi;u)| + |A(1-s,\overline{\chi};u)|$. The bulk of the weight in this average is placed on sums of length $u \ll \sqrt{q\tau}$.

To prepare for the proof of the above, we make some preliminary remarks. The functional equation of $L(s,\chi)$ in its symmetric form asserts that

$$L(s,\chi)\Gamma((s+\kappa)/2)(q/\pi)^{(s+\kappa)/2} = \varepsilon(\chi)L(1-s,\overline{\chi})\Gamma((1-s+\kappa)/2)(q/\pi)^{(1-s+\kappa)/2}\,.$$

(See Corollary 10.8.) Let $\Delta(s,\chi)$ denote the quotient of the cofactors,

$$\begin{aligned}\Delta(s,\chi) &= \frac{\varepsilon(\chi)\Gamma((1-s+\kappa)/2)(q/\pi)^{(1-s+\kappa)/2}}{\Gamma((s+\kappa)/2)(q/\pi)^{(s+\kappa)/2}} \\ &= \varepsilon(\chi)2^s\pi^{s-1}q^{1/2-s}\Gamma(1-s)\sin\frac{\pi}{2}(s+\kappa)\,.\end{aligned} \tag{25.31}$$ E:DefDelta(s,chi)

Thus the asymmetric form of the functional equation (as in Corollary 10.9) asserts that $L(s,\chi) = \Delta(s,\chi)L(1-s,\overline{\chi})$. Since $|\varepsilon(\chi)| = 1$ it follows from (25.3) that

$$|\Delta(s,\chi)| \asymp (q\tau)^{1/2-\sigma} \tag{25.32}$$ E:Delta(s,chi)Est

uniformly for $-A \le \sigma \le A$, $|t| \ge 1$. On dividing both sides of (25.27) by $\Gamma((s+\kappa)/2)(q/\pi)^{(s+\kappa)/2}$, we find that

$$\begin{aligned}L(s,\chi) &= \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}\cdot\frac{\Gamma\big((s+\kappa)/2,\pi n^2z/q\big)}{\Gamma((s+\kappa)/2)} \\ &\quad+\Delta(s,\chi)\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)}{n^{1-s}}\cdot\frac{\Gamma\big((1-s+\kappa)/2,\pi n^2/(qz)\big)}{\Gamma((1-s+\kappa)/2)}\,.\end{aligned} \tag{25.33}$$ E:LfcnFE2

If we take $z = re^{i\phi}$ with ϕ given by (25.4), then the above reads

$$\begin{aligned}L(s,\chi) &= \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}w_0((s+\kappa)/2),\pi n^2r/q) \\ &\quad+\Delta(s,\chi)\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)}{n^{1-s}}w_0((1-s+\kappa)/2,\pi n^2/(rq))\,.\end{aligned} \tag{25.34}$$ E:afew0

Here the factor $e^{-i\phi}$ that arises in the second sum is appropriate, since the imaginary part of $1-s$ is the negative of that of s. We divide both sides of this by r, integrate from 1/2 to 2, and divide by $\log 4$ to find that

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} w_1((s+\kappa)/2), \pi n^2/q)$$
$$+ \Delta(s,\chi) \sum_{n=1}^{\infty} \frac{\overline{\chi}(n)}{n^{1-s}} w_1((1-s+\kappa)/2, \pi n^2/q)\,. \qquad (25.35) \quad \text{E:afew1}$$

In calculating the second term we have used the fact that

$$\int_{1/2}^{2} f(c/r)\,\frac{dr}{r} = \int_{1/2}^{2} f(cr)\,\frac{dr}{r}\,.$$

Proof The first sum on the right hand side of (25.35) is

$$\int_0^\infty w_1((s+\kappa)/2, \pi u^2/q)\, dA(s,\chi;u)\,. \qquad (25.36) \quad \text{E:InitInt}$$

When we integrate this by parts we must consider the size of

$$w_1((s+\kappa)/2, \pi u^2/q) A(s,\chi;u)$$

for u near 0 and as $u \to \infty$. The lower endpoint is easy to treat, since $A(s,\chi,u) = 0$ when $u < 1$. As $u \to \infty$ we have $A(s,\chi;u) \ll u^A$ for some A. From (25.15) we see that $w_1((s+\kappa)/2, \pi u^2/q) \ll \exp(-cu^2)$ for some $c > 0$. Here A and c may depend on various parameters, but are independent of u. Thus the product of the two bounds tends to 0 as $u \to \infty$. Hence by integration by parts the expression (25.36) is

$$-\int_0^\infty A(s,\chi,u)\, dw_1((s+\kappa)/2, \pi u^2/q)$$
$$= -\int_0^\infty A(s,\chi,u) \frac{\partial}{\partial u} w_1((s+\kappa)/2, \pi u^2/q)\, du$$
$$= \frac{-2\pi}{q} \int_0^\infty A(s,\chi,u) u \frac{\partial}{\partial a} w_1((s+\kappa)/2, a)\Big|_{a=\pi u^2/q} du\,.$$

By Lemma 25.3 this is

$$\ll (q\tau)^{-(\sigma+\kappa)/2} \int_0^\infty |A(s,\chi;u)| \exp\Big(\frac{-\pi u^2}{2q\tau}\Big) u^{\sigma+\kappa-1}\, du\,.$$

By considering separately the ranges $0 < u \le \sqrt{q\tau}$ and $\sqrt{q\tau} \le u < \infty$ we find that $\exp(-\pi u^2/(2q\tau))(u^2/(q\tau))^{(\sigma+\kappa}/2 \ll \exp(-u^2/(q\tau))(u^2/(q\tau))^{\delta/2}$ uniformly for $u > 0$. Thus the above is

$$\ll \int_0^\infty |A(s,\chi;u)| e^{-u^2/(q\tau)} \Big(\frac{u^2}{q\tau}\Big)^{\delta/2} \frac{du}{u}\,.$$

□

Suppose that k is a fixed positive integer. Then by Hölder's Inequality,

$$\begin{aligned}\Big|\int_0^\infty |A(s,\chi;u)|e^{-u^2/(q\tau)}\Big(\frac{u^2}{q\tau}\Big)^{\delta/2}\,\frac{du}{u}\Big|^{2k} &\le \Big(\int_0^\infty |A(s,\chi;u)|^{2k}e^{-u^2/(q\tau)}\Big(\frac{u^2}{q\tau}\Big)^{\delta/2}\,\frac{du}{u}\Big)\\ &\quad\times\Big(\int_0^\infty e^{-u^2/(q\tau)}\Big(\frac{u^2}{q\tau}\Big)^{\delta/2}\,\frac{du}{u}\Big)^{2k-1}\\ &\ll \int_0^\infty |A(s,\chi;u)|^{2k}e^{-u^2/(q\tau)}\Big(\frac{u^2}{q\tau}\Big)^{\delta/2}\,\frac{du}{u}\,. \qquad (25.37)\end{aligned}$$

E:Bndqtau

If $Q/2 \le q \le Q$ and $T/2 \le |t| \le T$, then the above is

$$\ll \int_0^\infty |A(s,\chi;u)|^{2k}e^{-u^2/(2QT)}\Big(\frac{u^2}{QT}\Big)^{\delta/2}\,\frac{du}{u}\,. \qquad (25.38)$$

E:BndQT

This latter form will be useful in forming moment estimates.

??.1 Exercises

Exer:VarIncGam

1. Suppose that $\sigma \ge \delta > 0$ and that $t > 0$.

   (a) Suppose that $0 \le \phi < \pi/2$. Show that the variation of $\Gamma\big(s, ue^{i\phi}\big)$ as u runs from 0 to ∞ is

   $$\frac{e^{-\phi t}}{(\cos\phi)^\sigma}\Gamma(\sigma)\,.$$

   (b) Show that the above is minimized by taking $\phi = \arctan t/\sigma$.

   (c) Conclude that

   $$\operatorname{Var}_{[0,\infty)}\Gamma\big(s, ue^{i\phi}\big) \gg \tau^{1/2}|\Gamma(s)|$$

   uniformly in all choices of ϕ.

2. Suppose that $c > 0$. Show that (25.35) is a special case of the more

general identity

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} w_1((s+\kappa)/2), \pi n^2/(cq))$$
$$+ \Delta(s,\chi) \sum_{n=1}^{\infty} \frac{\overline{\chi}(n)}{n^{1-s}} w_1((1-s+\kappa)/2, c\pi n^2/q)\,.$$

3. If $\int_0^\infty |f(x)|x^{\sigma-1}\,dx < \infty$, then we call the function

$$F(s) = \int_0^\infty f(x)x^{s-1}\,dx$$

the *Mellin transform* of f, and usually one can recover f from F by means of the *inverse Mellin transform*,

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)x^{-s}\,ds$$

for suitable values of c. In Chapter 5 we several useful pairs f, F of this kind. Suppose now that $\int_0^\infty |f(x)|x^{\sigma-1}\,dx < \infty$, and that $\int_0^\infty |g(x)|x^{\sigma-1}\,dx < \infty$ and set

$$F(s) = \int_0^\infty f(x)x^{s-1}\,dx, \qquad G(s) = \int_0^\infty g(x)x^{s-1}\,dx\,.$$

Define h by the convolution formula

$$h(x) = (f*g)(x) = \int_0^\infty f(y)g(x/y)\,\frac{dy}{y}\,.$$

(a) Show that

$$\int_0^\infty |h(x)|x^{\sigma-1}\,dx \le \int_0^\infty |f(x)|x^{\sigma-1}\,dx \int_0^\infty |g(x)|x^{\sigma-1}\,dx\,.$$

(b) Put

$$H(s) = \int_0^\infty h(x)x^{s-1}\,dx\,.$$

Show that $H(s) = F(s)G(s)$.

4. (a) By integrating by parts, or otherwise, show that if $\operatorname{Re} z > 0$ and $\operatorname{Re} s > 0$, then

$$\int_0^\infty \Gamma(z,a)a^{s-1}\,da = \frac{\Gamma(s+z)}{s}\,.$$

(b) Suppose that $\operatorname{Re} z > 0$ and that $\operatorname{Re} s > 0$. For $c > 0$ let

$$f(z,a) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s+z)}{s} a^{-s}\, ds\,.$$

(i) By using Theorem C.4, or otherwise, show that $\frac{\partial}{\partial a} f(z,a) = -e^{-a}a^z$.

(ii) Show that $\frac{\partial}{\partial a}\Gamma(z,a) = -e^{-a}a^z$.

(iii) Show that when z is fixed, $\lim_{a\to\infty} f(z,a) = 0$.

(iv) Show that when z is fixed, $\lim_{a\to\infty} \Gamma(z,a) = 0$.

(v) (Mellin) Deduce that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s+z)}{s} a^{-s}\, ds = \Gamma(z,a)$$

for $\operatorname{Re} z > 0$, $\operatorname{Re} a > 0$.

5. For $\operatorname{Re} z > 0$ and w in the slit complex plane $|\arg w| < \pi$, let

$$\gamma(z,w) = \Gamma(z) - \Gamma(z,w) = \int_0^w e^{-u}u^{z-1}\, du$$

be the complementary incomplete gamma function, and set $g(z,w) = w^{-z}\gamma(z,w) = \int_0^1 e^{-wv}v^{z-1}\, dv$.

(a) Show that

$$g(z,w) = \frac{e^{-w}}{z} + \frac{w}{z} g(z+1,w)\,.$$

(b) Show that

$$g(z,w) = e^{-w}\sum_{k=0}^{K} \frac{w^k}{z(z+1)\cdots(z+k)} \ + \ \frac{w^{K+1}g(z+K+1,w)}{z(z+1)\cdots(z+K)}\,.$$

(c) Deduce that

$$g(z,w) = e^{-w}\sum_{k=0}^{\infty} \frac{w^k}{z(z+1)\cdots(z+k)}\,.$$

(d) Note that

$$\frac{g(z,w)}{\Gamma(z)} = e^{-w}\sum_{k=0}^{\infty} \frac{w^k}{\Gamma(z+k+1)}$$

is an entire function of z, and of w.

25.1 Notes

S: MeanLargeVals Notes

Section 21.1. Concerning majorant inequalities, antecedents of Theorem 26.13 are found in the work of Wiener (unpubished — see Theorem 12.6.12 of Boas (1954), Erdős and Fuchs (1956), Wiener and Wintner (1956), and Halász (1968). Logan (1988) showed that the constant 3 is best possible. For a general discussion of majorant principles, see Shapiro (1975).

Section 21.2. Tem79 Temme (1979) gave a detailed account of the asymptotics of the incomplete gamma function $\Gamma(s, a)$. Rub05 Rubinstein (2005) developed a variety of tools for the computation of various sorts of L-functions.

Section 21.3. Should mention the mean value theorem of Nigel Watt (NW95), which depends on Kloosterman sums. See Heath-Brown's review. Section 21.4. For an account of the Phragmń–Lindelöf Theoem see, for example, Theorem 5.1.9 in BS2A Simon (2015).

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26
Mean Values of Dirichlet Polynomials

C:MeanLargeVals

26.1 Mean value estimates

S:MVEst

Suppose that $\lambda_1 < \lambda_2 < \cdots < \lambda_N$. Then

$$\int_0^T \Big|\sum_{n=1}^N a_n e(\lambda_n t)\Big|^2 dt = T\sum_{n=1}^N |a_n|^2 + \sum_{\substack{1\le m,n\le N\\ m\ne n}} a_m\overline{a_n}\frac{e((\lambda_m-\lambda_n)T)-1}{2\pi i(\lambda_m-\lambda_n)}. \qquad (26.1)$$

E:Int|S|2

Here the last term is a bounded function of T, so that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \Big|\sum_{n=1}^N a_n e(\lambda_n t)\Big|^2 dt = \sum_{n=1}^N |a_n|^2.$$

This gives us a sort of asymptotic Parseval identity, but we want something more quantitative for a given finite T. We note that if λ_n is very close to λ_{n+1}, then the terms $a_n e(\lambda_n t)$ and $a_{n+1}e(\lambda_{n+1}t)$ will reinforce each other over certain long stretches of t, and then subtract from each other over in other long stretches of t. Thus any bound that is to depend on the numbers $|a_n|^2$ must also involve some information concerning the spacing of the numbers λ_n.

T:MV1

Theorem 26.1 *Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_N$ are distinct real numbers, and that $\delta > 0$ has the property that $|\lambda_m - \lambda_n| \ge \delta$ for $m \ne n$. For any real $T \ge 0$, any positive integer N, and any real or complex numbers a_n there is a real number θ, $-1 \le \theta \le 1$, such that*

$$\int_0^T \Big|\sum_{n=1}^N a_n e(\lambda_n t)\Big|^2 dt = (T+\theta/\delta)\sum_{n=1}^N |a_n|^2.$$

By replacing a_n by $a_n e(\lambda_n T_0)$, we find that

$$\int_{T_0}^{T_0+T} \Big|\sum_{n=1}^N a_n e(\lambda_n t)\Big|^2 dt = (T+\theta/\delta)\sum_{n=1}^N |a_n|^2$$

for some θ. Thus our estimate depends on the length of the interval of integration, but not on its position.

First Proof Let S_+ and S_- be functions with the properties described in Theorem **??**, with $I=[0,T]$. Then

$$\int_0^T \Big|\sum_{n=1}^N a_n e(\lambda_n t)\Big|^2 dt \le \int_{\mathbb{R}} \Big|\sum_{n=1}^N a_n e(\lambda_n t)\Big|^2 S_+(t)\,dt$$

by Theorem **??**(b). On expanding the right hand side and integrating term by term, we see that the above is

$$= \sum_{m,n} a_m\overline{a_n}\widehat{S}_+(\lambda_n-\lambda_m) \le (T+1/\delta)\sum_{n=1}^N |a_n|^2$$

by Theorem **??**(c),(d). Similarly,

$$\int_0^T \Big|\sum_{n=1}^N a_n e(\lambda_n t)\Big|^2 dt \ge \int_{\mathbb{R}} \Big|\sum_{n=1}^N a_n e(\lambda_n t)\Big|^2 S_-(t)\,dt$$
$$= \sum_{m,n} a_m\overline{a_n}\widehat{S}_-(\lambda_n-\lambda_m) \ge (T-1/\delta)\sum_{n=1}^N |a_n|^2.$$

□

Second Proof We estimate the contribution of the non-diagonal terms on the right hand side of (26.1) *via* two applications of Hilbert's inequality in the form of Theorem **??**. Specifically, by taking $y_m = a_m e(\lambda_m T)$ and $x_n = \overline{a_n}e(-\lambda_n T)$ we deduce that

$$\Big|\sum_{\substack{1\le m,n\le N\\ m\ne n}} a_m\overline{a_n}\frac{e((\lambda_m-\lambda_n)T)}{2\pi i(\lambda_m-\lambda_n)}\Big| \le \frac{1}{2\delta}\sum_{n=1}^N |a_n|^2.$$

Similarly,

$$\Big|\sum_{\substack{1\le m,n\le N\\ m\ne n}} \frac{a_m\overline{a_n}}{2\pi i(\lambda_m-\lambda_n)}\Big| \le \frac{1}{2\delta}\sum_{n=1}^N |a_n|^2,$$

so the desired estimate now follows by the triangle inequality. □

Cor:MVDir1 **Corollary 26.2** *For any positive real T, any positive integer N, and any real or complex numbers $a_1, a_2, \ldots, a_N$,*

$$\int_0^T \Big|\sum_{n=1}^N a_n n^{-it}\Big|^2 dt = (T + O(N)) \sum_{n=1}^N |a_n|^2. \tag{26.2}$$ E:MVEst1

Proof We note that

$$\log(n+1) - \log n = \int_n^{n+1} \frac{1}{u}\, du \ge \frac{1}{n+1}\,.$$

Thus we can appeal to Theorem 26.1 with $\delta = 1/(2\pi N)$. □

Exam:T+O(N) **Example 26.1** Put $D(s) = \sum_{n=1}^N n^{-s}$. By integrating by parts we find that

$$D(s) = \frac{N^{1-s} - 1}{1-s} + 1 - s\int_1^N \frac{\{u\}}{u^{s+1}}\, du\,.$$

Hence in particular,

$$D(it) = \frac{N^{1-it}}{1-it} + O(\tau \log 2N)$$

where $\tau = |t| + 4$. Thus $|D(it)| \asymp N$ for $0 \le t \le 1$, and so $\int_0^1 |D(it)|^2\, dt \asymp N^2 = N\sum_{n=1}^N |a_n|^2$ since $a_n = 1$ for all n in the present case. Thus the error term $O(N)\sum_{n=1}^N |a_n|^2$ in (26.2) is best-possible.

The proofs of Theorem 26.1 are short and elegant, due to the substantial material found in Appendices **??** and **??**. In some cases of interest, the λ_n are irregularly spaced, so that one λ_n might be closer to its nearest neighbor than another one. In such a situation, we have a more delicate weighted estimate, although with a slightly inferior constant.

T:MV2 **Theorem 26.3** *Let $\lambda_1, \lambda_2, \ldots$ be distinct real numbers, and put*

$$\delta_n = \min_{\substack{m \\ m \ne n}} |\lambda_m - \lambda_n|\,.$$

For any real $T \ge 0$ and any real or complex numbers a_n with $\sum_{n=1}^\infty |a_n| < \infty$ there is a real number θ, $-1 \le \theta \le 1$, such that

$$\int_0^T \Big|\sum_{n=1}^\infty a_n e(\lambda_n t)\Big|^2 dt = T\sum_{n=1}^\infty |a_n|^2 + \frac{3}{2}\theta \sum_{n=1}^\infty \frac{|a_n|^2}{\delta_n}\,.$$

Proof We proceed as in the second proof of Theorem 26.1, but now appeal to Theorem ?? in place of Theorem ?? to see that for any positive N there is a θ_N such that

$$\int_0^T \Big|\sum_{n=1}^N a_n e(\lambda_n t)\Big|^2 dt = T\sum_{n=1}^N |a_n|^2 + \frac{3}{2}\theta_N \sum_{n=1}^N \frac{|a_n|^2}{\delta_n}\,.$$

The stated result then follows by allowing N to tend to infinity. □

Cor:MVDir2 **Corollary 26.4** *For any positive real T and any real or complex numbers $a_1, a_2, \ldots$ with $\sum_{n=1}^\infty |a_n| < \infty$,*

$$\int_0^T \Big|\sum_{n=1}^\infty a_n n^{-it}\Big|^2 dt = T\sum_{n=1}^\infty |a_n|^2 + O\Big(\sum_{n=1}^\infty n|a_n|^2\Big).$$

Proof This follows from Theorem 26.3 with $\delta_n = 1/(2\pi(n+1))$. □

The following extension of the above is occasionally useful.

Cor:MVDir3 **Corollary 26.5** *Suppose that $T > 0$, and that $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ are sequences with $\sum_{n=1}^\infty |a_n| < \infty$ and $\sum_{n=1}^\infty |b_n| < \infty$, and $a_1 b_1 = 0$. Then*

$$\int_0^T \Big|\sum_{n=1}^\infty \big(a_n n^{-it} + b_n n^{it}\big)\Big|^2\, dt = \sum_{n=1}^\infty \big(|a_n|^2 + |b_n|^2\big)\big(T + O(n)\big).$$

Proof This follows from Theorem 26.3 in the same way as Corollary 26.4, but now some of the frequencies are of the form $\frac{1}{2\pi}\log n$ while others are of the form $-\frac{1}{2\pi}\log n$. □

In Corollary 26.4 it would typically be the case that the a_n are nonzero for most n. However, when the a_n are nonzero only for n in a sparse set, we can obtain a slightly better estimate, as follows.

Cor:MVDir4 **Corollary 26.6** *Suppose that $a_1, a_2, \ldots$ are real or complex numbers such that $\sum_{n=1}^\infty |a_n| < \infty$. and let d_n be an integer such that $d_n \le n/2$ and with the property that $a_m = 0$ whenever $0 < |m - n| < d_n$. Then for any $T > 0$,*

$$\int_0^T \Big|\sum_{n=1}^\infty a_n n^{-it}\Big|^2 dt = T\sum_{n=1}^\infty |a_n|^2 + O\Big(\sum_{n=1}^\infty \frac{n|a_n|^2}{d_n}\Big).$$

This extension of Corollary 26.4 can be applied to either or both of the sums in Corollary 26.5.

Proof It suffices to apply Theorem 26.3 with $\delta_n = d_n/(4\pi n)$. □

Primes and primepowers are somewhat sparse, but only irregularly so. When considering a sum over primepowers whose coefficients (on the primepowers) are slowly varying, the following estimate is useful.

T:primepowergaps

Theorem 26.7 *Let $\mathcal{Q}$ denote the set of primepowers. For $n \in \mathcal{Q}$, let d_n be the minimum of $|m-n|$ for $m \in \mathcal{Q}$, $m \ne n$. Then for $U \ge 4$,*

$$\sum_{\substack{n\in\mathcal{Q}\\ U\le n\le 2U}} \frac{1}{d_n} \ll \frac{U\log\log U}{(\log U)^2}.$$

Since $d_n \ge 1$ for all n, it is trivial that the above sum is $\ll U/\log U$. If it were the case that $d_n \gg \log n$ for all $n \in \mathcal{Q}$, then the sum would be $\asymp U/(\log U)^2$. If twin primes occur with the conjectured frequency, then the above estimate is best possible.

Proof For the purposes of this proof, let $\mathcal{P}$ denote the set of primes, and $\mathcal{Q}^*$ the set of proper primepowers, which is to say the set of numbers of the form p^k with $k > 1$. Thus $\mathcal{Q}$ is a disjoint union of $\mathcal{P}$ and $\mathcal{Q}^*$. For primes p set $d_0(p)$ be the minimum of $|p-p'|$ for $p' \in \mathcal{P}$, $p' \ne p$. We show first that

$$\sum_{U\le p\le 2U} \frac{1}{d_0(p)} \ll \frac{U\log\log U}{(\log U)^2}. \tag{26.3}$$

E:d0(p)sum

The number of summands in the above sum is $\ll U/\log U$, so the contribution made by those primes for which $d_0(p) > \log U$ is $\ll U/(\log U)^2$. The contributions to the above sum made by primes for which $d_0(p) \le \log U$ is

$$\ll \sum_{r\le\log U} \frac{1}{r}\#\{p\in[U,2U] \,:\, p+r\in\mathcal{P}\}.$$

By the sieve estimate of Corollary 3.14 the above is

$$\ll \sum_{r\le\log U} \frac{c(r)U}{(\log U)^2}$$

where $c(r) = \prod_{p|r,p>2}\left(\frac{p-1}{p-2}\right)$. From (2.32) it follows that $\sum_{r\le R} c(r) \ll R$, so (26.3) follows by partial summation.

For all primepowers $n \in \mathcal{Q}$ we define $d_1(n)$ as follows. If $n \in \mathcal{Q}^*$, then set $d_1(n) = 1$; let p_1 denote the largest prime $< n$, and p_2 denote the least prime $> n$, and set $d_1(p_1) = d_1(p_2) = 1$. The number of $n \in \mathcal{Q}^*$ such that $U \le n \le 2U$ is $\ll U^{1/2}/\log U$. It p is a prime such that the greatest primepower $< p$ is a prime, and the least primepower $> p$ is also a prime, then we set $d_1(p) = d_0(p)$. Thus

$$\sum_{\substack{n\in Q\\ U\le n\le 2U}} \frac{1}{d_1(n)} = \sum_{U\le p\le 2U} \frac{1}{d_0(p)} + O\Big(\frac{U^{1/2}}{\log U}\Big) \ll \frac{U\log\log U}{(\log U)^2}.$$

Since $d_1(n) \le d_n$ for all primepowers n, the desired bound follows. □

In Chapter **??** we found that it is fruitful to estimate the mean square of a trigonometric polynomial at well-spaced points in terms of the continuous mean square. Similarly, it is useful to estimate the mean square of a Dirichlet polynomial at well-spaced points.

T:DiscreteDP1 **Theorem 26.8** *Let*

$$D(s) = \sum_{n=1}^{N} a_n n^{-s}. \tag{26.4}$$ E:DefD

Suppose that $0 < \delta \le 1$, $T \ge 1$, *and that* $t_1, t_2, \ldots, t_R$ *are real numbers such that* $A \le t_1 < t_2 < \cdots < t_R \le A + T$ *and* $t_{r+1} - t_r \ge \delta$ *for* $r = 1, 2, \ldots, R-1$. *If* $D(s)$ *is defined as in* (26.4), *then*

$$\sum_{r=1}^{R} |D(it_r)|^2 \ll (\log N + 1/\delta)(T + N) \sum_{n=1}^{N} |a_n|^2$$

uniformly for any complex numbers a_n *and any real number* A.

Proof Let $\mathfrak{M}_r = [t_r - \delta/2, t_r + \delta/2]$ for $r = 1, 2, \ldots, R$. By Lemma **??** it follows that

$$|D(it_r)|^2 \le \frac{1}{\delta}\int_{\mathfrak{M}_r} |D(it)|^2\,dt + \int_{\mathfrak{M}_r} |D(it)D'(it)|\,dt\,.$$

The intervals $\mathfrak{M}_r$ are disjoint and all lie in the interval $[A-\delta, A+T+\delta]$, so

$$\sum_{r=1}^{R} |D(it_r)|^2 \le \frac{1}{\delta}\int_{A-\delta}^{A+T+\delta} |D(it)|^2\,dt + \int_{A-\delta}^{A+T+\delta} |D(it)D'(it)|\,dt\,.$$

By the Cauchy–Schwarz inequality the above is

$$\le \frac{1}{\delta}\int_{A-\delta}^{A+T+\delta} |D(it)|^2\,dt + \Big(\int_{A-\delta}^{A+T+\delta} |D(it)|^2\,dt\Big)^{1/2}\Big(\int_{A-\delta}^{A+T+\delta} |D'(it)|^2\,dt\Big)^{1/2}.$$

By Corollary 26.2 it follows that this is

$$\ll \frac{T+N}{\delta}\sum_{n=1}^N |a_n|^2 + (T+N)\Big(\sum_{n=1}^N |a_n|^2\Big)^{1/2}\Big(\sum_{n=1}^N |a_n \log n|^2\Big)^{1/2}.$$

This gives the stated result. □

We now extend the above to allow points $s_r = \sigma_r + it_r$ whose abscissæ are not all equal.

T:DiscreteDP2 **Theorem 26.9** *Let $D(s)$ be defined as in* (26.4)*, let A be a real number, and suppose that $T \ge 2$. Let $s_1, s_2, \ldots, s_R$ be distinct complex numbers in the half-strip $\sigma_r \ge 0$, $A \le t_r \le A+T$, and suppose that $\delta \le 1$ has been chosen so that $|t_{r_1} - t_{r_2}| \ge \delta$ whenever $r_1 \ne r_2$. Then*

$$\sum_{r=1}^R |D(s_r)|^2 \ll (\log N + 1/\delta)(T+N)\sum_{n=1}^N |a_n|^2\Big(1+\log\frac{\log 2N}{\log 2n}\Big).$$

Proof Let

$$S(s;u) = \sum_{2\le n\le N} a_n n^{-s},$$

so that if $\sigma \ge 0$, then

$$D(s) = a_1(1-N^{-\sigma}) + D(it)N^{-\sigma} + \sigma\int_2^N S(it;u)u^{-\sigma-1}\,du.$$

Hence

$$|D(s)|^2 \ll |a_1|^2 + |D(it)|^2 + \Big(\sigma\int_2^N |S(it;u)|u^{-\sigma-1}\,du\Big)^2.$$

By the Cauchy–Schwarz inequality the last term above is

$$\le \Big(\sigma^2\int_2^N \frac{\log u}{u^{2\sigma+1}}\,du\Big)\Big(\int_2^N \frac{|S(it,u)|^2}{u\log u}\,du\Big).$$

By integrating by parts twice we find that

$$\sigma^2\int_1^N \frac{\log u}{u^{2\sigma+1}}\,du = \frac14 - \frac{1}{4N^{2\sigma}} - \frac{\sigma}{2}\log N \le \frac14.$$

Thus

$$|D(s)|^2 \ll |a_1|^2 + |D(it)|^2 + \int_2^N \frac{|S(it;u)|^2}{u\log u}\,du \qquad (26.5)$$ E:D(sigma)est

uniformly for $\sigma \ge 0$. When we take $s = s_r$ and sum over r, the contribution of $|a_1|^2$ is $\ll |a_1|^2 R \ll |a_1|^2 T/\delta$. The contribution of $|D(it_r)|^2$ is exactly as it was in Theorem 26.8, and

$$\sum_{r=1}^{R} \int_2^N \frac{|S(it;u)|^2}{u \log u}\, du \ll (\log N + 1/\delta)(T+N) \int_2^N \sum_{2 \le n \le u} |a_n|^2 (u \log u)^{-1} du.$$

Here the integral on the right is

$$= \sum_{n=2}^{N} |a_n|^2 \int_n^N \frac{du}{u \log u} = \sum_{n=2}^{N} |a_n|^2 \log \frac{\log N}{\log n}.$$

This gives the result. □

When $N > T$ we do not obtain good bounds for the mean square in terms of $\sum_{n=1}^{N} |a_n|^2$, but we now show that quite reasonable upper bounds can be obtained in terms of short sums of the coefficients.

T:gallagher1

Theorem 26.10 (Gallagher) *Let $\mathcal{L}$ be a countable set of real numbers and for $\lambda \in \mathcal{L}$ let $a(\lambda)$ be complex numbers such that $\sum_{\lambda \in \mathcal{L}} |a(\lambda)| < \infty$. For real t and $\delta > 0$ let*

$$S(t) = \sum_{\lambda \in \mathcal{L}} a(\lambda) e(\lambda t), \qquad A_\delta(x) = \delta^{-1} \sum_{\substack{\lambda \in \mathcal{L} \\ |\lambda - x| < \delta/2}} a(\lambda)\,.$$

Then

$$\int_{-\infty}^{\infty} |S(t)|^2 \Big(\frac{\sin \pi \delta t}{\pi \delta t}\Big)^2 dt = \int_{-\infty}^{\infty} |A_\delta(x)|^2\, dx.$$

Proof Let

$$F_\delta(x) = \begin{cases} \delta^{-1} & \text{when } |x| < \delta/2, \\ 0 & \text{when } |x| \ge \delta/2\,. \end{cases}$$

Then

$$A_\delta(x) = \sum_\lambda a(\lambda) F_\delta(x - \lambda)\,.$$

We note that $A_\delta \in L^1(\mathbb{R})$, since by the triangle inequality

$$\|A_\delta\|_{L^1(\mathbb{R})} \le \sum_\lambda \|a(\lambda) F_\delta(x - \lambda)\|_{L^1(\mathbb{R})} = \sum_\lambda |a(\lambda)| < \infty\,.$$

Let $\widehat{A}_\delta(t)$ denote the Fourier transform of $A_\delta(x)$, i.e.,

$$\widehat{A}_\delta(t) = \int_{-\infty}^{\infty} A_\delta(x) e(-xt)\, dx\,.$$

Then

$$\widehat{A}_\delta(t) = \sum_{\lambda\in\mathcal{L}} a(\lambda)\int_{-\infty}^{\infty} F_\delta(x-\lambda)e(-xt)dx$$
$$= \sum_{\lambda\in\mathcal{L}} a(\lambda)e(-\lambda t)\widehat{F}_\delta(t) = S(-t)\widehat{F}_\delta(t)\,.$$

Next we show that $A_\delta \in L^2(\mathbb{R})$. To see why this is so, we first note that

$$|A_\delta(x)|^2 = \sum_{\substack{\lambda_i\in\mathcal{L}\\ |x-\lambda_i|\le\delta/2}} a(\lambda_1)\overline{a(\lambda_2)} \le \sum_{\substack{\lambda_i\in\mathcal{L}\\ |x-\lambda_i|\le\delta/2}} |a(\lambda_1)|\,|a(\lambda_2)|\,.$$

If $|\lambda_1-\lambda_2|>\delta$, then there is no x for which both inequalities $|x-\lambda_1|\le\delta/2$, $|x-\lambda_2|\le\delta/2$ are satisfied. If $|\lambda_1-\lambda_2|\le\delta$, then the set of x for which $|x-\lambda_1|\le\delta/2$ and $|x-\lambda_2|\le\delta/2$ is an interval of length $\delta-|\lambda_1-\lambda_2|$. Thus

$$\int_{-\infty}^{\infty}|A_\delta(x)|^2\,dx \le \sum_{\lambda_1,\lambda_2\in L}\max(0,\delta-|\lambda_1-\lambda_2|)|a(\lambda_1)|\,|a(\lambda_2)|$$
$$\le \delta\Big(\sum_{\lambda\in L}|a(\lambda)|\Big)^2 < \infty\,.$$

By Plancherel's identity for functions in $L^1(\mathbb{R})\cap L^2(\mathbb{R})$, it follows that

$$\int_{-\infty}^{\infty}|A_\delta(x)|^2\,dx = \int_{-\infty}^{\infty}|\widehat{A}_\delta(t)|^2\,dt = \int_{-\infty}^{\infty}|S(-t)\widehat{F}_\delta(t)|^2\,dt\,.$$

Since

$$\widehat{F}_\delta(t) = \frac{\sin\pi\delta t}{\pi\delta t}$$

we have the stated result. □

When the above is applied to ordinary Dirichlet series, this gives

Cor:gallagher2 **Corollary 26.11** (Gallagher) *Suppose that* $\sum_{n=1}^\infty |a_n|<\infty$, *that* $\delta>0$, *and that* $\kappa=e^{2\pi\delta}$. *Then*

$$\int_{-\infty}^{\infty}\Big|\sum_{n=1}^{\infty}\frac{a_n}{n^{it}}\Big|^2\Big(\frac{\sin\pi\delta t}{\pi\delta t}\Big)^2 dt = \frac{1}{2\pi\delta^2}\int_0^\infty\Big|\sum_{\substack{n=1\\ y<n<\kappa y}}^{\infty} a_n\Big|^2\frac{dy}{y}. \qquad (26.6)$$ E:GallEst2

Proof Take $\lambda_n=\frac{\log n}{2\pi}$ in Theorem 26.10, and set $y=e^{2\pi x-\pi\delta}$. Thus $dx=\frac{dy}{2\pi dy}$. □

Cor:gallagher3 **Corollary 26.12** (Gallagher) *Suppose that* $\sum_{n=1}^\infty |a_n| < \infty$*, and let* $\kappa = \exp\big(T^{-1}\big)$ *where* $T \ge 1$*. Then*

$$\int_{-T}^{T} \Big| \sum_{n=1}^\infty \frac{a_n}{n^{it}} \Big|^2 \, dt \ll T^2 \int_0^\infty \Big| \sum_{y < n \le \kappa y} a_n \Big|^2 \frac{dy}{y} \,. \tag{26.7}$$

E:GallEst3

The sum on the right hand side above is empty when $y < 1/\kappa$, so in effect the range of integration is $[1/\kappa, \infty)$.

Proof We take $\delta = 1/(2\pi T)$ in Corollary 26.11. Put $g(u) = \frac{\sin u}{u}$ for $u \ne 0$, and $g(0) = 1$. Then $|g(u)| \gg 1$ uniformly for $0 \le u \le 1/2$, so the left hand side of (26.7) is majorized by the left hand side of (26.6). $\square$

In addition to the mean value estimates we have already considered, the following *majorant principle* is sometimes useful.

T:majorant **Theorem 26.13** *Suppose that* $\lambda_1, \lambda_2, \ldots$ *are real numbers, and that real or complex numbers* a_j *and* A_j *have the property that* $|a_j| \le A_j$ *for all* j*. Suppose also that* $\sum_{j=1}^\infty A_j < \infty$*. Then for any* $T > 0$*,*

$$\int_{-T}^{T} \Big| \sum_{j=1}^\infty a_j e\big(\lambda_j t\big) \Big|^2 \, dt \le 3 \int_{-T}^{T} \Big| \sum_{j=1}^\infty A_j e\big(\lambda_j t\big) \Big|^2 \, dt \,. \tag{26.8}$$

E:majorant0

Proof For $T > 0$ let

$$K_T(t) = \max(0, 1 - |t|/T) \,. \tag{26.9}$$

E:DefK_T

We show first that

$$\int_{-T}^{T} K_T(t) \Big| \sum_{j=1}^\infty a_j e\big(\lambda_j t\big) \Big|^2 \, dt \le \int_{-T}^{T} K_T(t) \Big| \sum_{j=1}^\infty A_j e\big(\lambda_j t\big) \Big|^2 \, dt \,. \tag{26.10}$$

E:wtdmajorant1

To see why this is so, and observe that by integration by parts it is immediate that

$$\widehat{K}_T(\alpha) = \int_{-\infty}^{\infty} K_T(t) e(-t\alpha) \, dt = \frac{1}{T} \Big(\frac{\sin \pi T \alpha}{\pi \alpha} \Big)^2 \ge 0 \,.$$

By multiplying out the modulus-squared we see that the left hand side of (26.10) is

$$= \sum_{j,k} a_j \overline{a_k} \int_{-\infty}^{\infty} K_T(t) e((\lambda_j - \lambda_k)t) \, dt = \sum_{j,k} a_j \overline{a_k} K_T(\lambda_k - \lambda_j) \,.$$

Since $\widehat{K}_T(\alpha) \ge 0$ for all α, the above is

$$\le \sum_{j,k} A_j A_k \widehat{K}_T(\lambda_k - \lambda_j) = \int_{-\infty}^{\infty} K_T(t) \Big| \sum_{j=1}^{\infty} A_j e\big(\lambda_j t\big)\Big|^2 \, dt\,.$$

Thus we have (26.10). We note that if we apply (26.10) with a_j replaced by $a_j e(\lambda_j U)$, then we have

$$\begin{aligned}\int_{-\infty}^{\infty} K_T(t-U) \Big| \sum_{j=1}^{\infty} a_j e\big(\lambda_j t\big)\Big|^2 \, dt \\ \le \int_{-\infty}^{\infty} K_T(t) \Big| \sum_{j=1}^{\infty} A_j e\big(\lambda_j t\big)\Big|^2 \, dt\end{aligned} \tag{26.11}$$ E:wtdmajorant2

for any real number U. We note that the function

$$w(t) = K_T(t+T) + K_T(t) + K_T(t-T)$$

majorizes the characteristic function of the interval $[-T, T]$. Thus by three applications of (26.11), with $U = -T$, $U = 0$, and $U = T$, we find that

$$\int_{-T}^{T} \Big| \sum_{j=1}^{\infty} a_j e\big(\lambda_j t\big)\Big|^2 \, dt \le 3 \int_{-T}^{T} K_T(t) \Big| \sum_{j=1}^{\infty} A_j e\big(\lambda_j t\big)\Big|^2 \, dt,$$

and (26.8) follows from this. □

To see how we might use Theorem 26.13, suppose that we have two Dirichlet series,

$$\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \qquad A(s) = \sum_{n=1}^{\infty} A_n n^{-s} \tag{26.12}$$ E:DefDSs

with $|a_n| \le A_n$ for all n and suppose that σ is chosen so that $\sum_{n=1}^{\infty} A_n n^{-\sigma} < \infty$. Then

$$\int_{-T}^{T} |\alpha(\sigma + it)|^2 \, dt \le 3 \int_{-T}^{T} |A(\sigma + it)|^2 \, dt\,.$$

Specializing this still further, we see in particular that if $|a_n| \le 1$ for all n, and $1 < \sigma \le 2$, then

$$\int_{-1}^{1} |\alpha(\sigma + it)|^2 \, dt \le 3 \int_{-1}^{1} |\zeta(\sigma + it)|^2 \, dt \asymp \frac{1}{\sigma - 1}\,. \tag{26.13}$$ E:SpecMaj

26.1.1 Exercises

1. (Goldston 1981) Let $S(t) = \sum_{\mu \in \mathcal{M}} c(\mu)e(\mu t)$ where $\mathcal{M}$ is a countable set of real numbers and $\sum_{\mu \in \mathcal{M}} |c(\mu)| < \infty$. Suppose that $T > 0$, $\delta > 0$, and take $I = [0, T]$ in Theorem E.2.

   (a) Explain why
$$\int_0^T |S(t)|^2\,dt \le \int_{-\infty}^{\infty} S_+(t)|S(t)|^2\,dt.$$

   (b) Deduce that
$$\int_0^T |S(t)|^2\,dt \le \Big(T + \frac{1}{\delta}\Big) \sum_{\mu \in \mathcal{M}} |c(\mu)|^2 + \big(\max_t \big|\widehat{S}_+(t)\big|\big) \sum_{\substack{\mu,\nu \in \mathcal{M} \\ |\mu - \nu| < \delta}} |c(\mu)c(\nu)|.$$

   (c) Show similarly that
$$\int_0^T |S(t)|^2\,dt \ge \Big(T + \frac{1}{\delta}\Big) \sum_{\mu \in \mathcal{M}} |c(\mu)|^2 - \big(\max_t \big|\widehat{S}_-(t)\big|\big) \sum_{\substack{\mu,\nu \in \mathcal{M} \\ |\mu - \nu| < \delta}} |c(\mu)c(\nu)|.$$

   (d) Write $S_\pm(x) = \chi_I(x) + \big(S_\pm(x) - \chi_I(x)\big)$, and deduce that
$$\big|\widehat{S}_\pm(t)\big| \le \|S_\pm\|_{L^1(\mathbb{R})} \le \|\chi_I\|_{L^1(\mathbb{R})} + \|S_\pm - \chi_I\|_{L^1(\mathbb{R})} = T + \frac{1}{\delta}.$$

   (e) Conclude that there is a number θ, $|\theta| \le 1$, such that
$$\int_0^T |S(t)|^2\,dt = \Big(T + \frac{\theta}{\delta}\Big)\Big(\sum_{\mu \in \mathcal{M}} |c(\mu)|^2 + \sum_{\substack{\mu,\nu \in \mathcal{M} \\ |\mu - \nu| < \delta}} |c(\mu)c(\nu)|\Big).$$

2. Let $\mathcal{Q}$ and d_n be defined as in Theorem 26.7.

   (a) Show that
$$\sum_{\substack{n \in \mathcal{Q} \\ U \le n \le 2U}} d_n \ll U.$$

   (b) By Cauchy's inequality, or otherwise, deduce that
$$\sum_{\substack{U \le n \le 2U \\ n \in \mathcal{Q}}} \frac{1}{d_n} \gg \frac{U}{(\log U)^2}.$$

3. Suppose that f is analytic in a domain that contains the closed disc $|z| \le R$.

   (a) Show that if $0 < r \le R$, then
   $$f(0) = \frac{1}{2\pi}\int_0^{2\pi} f\big(re^{i\theta}\big)\,d\theta.$$

   (b) Deduce that
   $$\frac{1}{2}R^2 f(0) = \frac{1}{2\pi}\int_0^R\int_0^{2\pi} f\big(re^{i\theta}\big)\,rdr\,d\theta.$$

   (c) Deduce that
   $$f(0) = \frac{1}{\pi R^2}\iint_{\mathcal{D}} f(\sigma + it)\,d\sigma\,dt$$
   where $\mathcal{D} = \{s := \sigma^2 + t^2 \le R^2\}$.

   (d) Deduce that
   $$|f(0)| \le \frac{1}{\pi R^2}\iint_{\mathcal{D}} |f(s)|\,d\sigma\,dt.$$

   (e) Suppose that $D(s)$ is defined as in (26.4), that the complex numbers $s_1, s_2, \ldots, s_R$ lie in the rectangle $0 \le \sigma_r \le 1/\log N$, $A \le t_r \le A + T$, and that $|s_{r_1} - s_{r_2}| \ge 1/\log N$ whenever $r_1 \ne r_2$. Show that
   $$\sum_{r=1}^{R} |D(s_r)|^2 \ll (T + N)(\log N)\sum_{n=1}^{N}|a_n|^2.$$

   (f) Suppose that $D(s)$ and the s_r are defined as above, except that now the s_r lie in the rectangle $0 \le \sigma_r \le 1$, $A \le t_r \le A + T$. Show that
   $$\sum_{r=1}^{R} |D(s_r)|^2 \ll (T + N)(\log N)^2\sum_{n=1}^{N}\frac{|a_n|^2}{\log(n+1)}.$$

4. JBC83 (Conrey, 1983)

   (a) Suppose that f is a bounded function of bounded variation on the interval $[a, b]$. Show that
   $$i(\log y)\int_a^b f(t)y^{it}\,dt = \Big[f(t)y^{it}\Big|_a^b - \int_a^b y^{it}\,df(t)$$
   for any real number y. (Material in Appendix A may be useful here.)

(b) With f as above, show that

$$(\log y)\int_a^b f(t)y^{it}\,dt \ll \sup_{a\le t\le b}|f(t)| + \operatorname{var}_{[a,b]} f.$$

(c) Suppose that functions f and g are bounded and of bounded variation on an interval $[a,b]$. Show that

$$\operatorname{var}_{[a,b]} fg \le \sup_{a\le t\le b}|f(t)|\ \operatorname{var}_{[a,b]} g + \operatorname{var}_{[a,b]} f \sup_{a\le t\le b}|g(t)|.$$

(d) Show that if m and n are positive integers with $m < n \le 2m$, then $\log\frac{n}{m} \asymp \frac{n-m}{m}$.

(e) Show that if m and n and positive integers with $n > 2m$, then $\log\frac{n}{m} \gg 1$.

(f) Deduce that if m and n are positive integers with $m < n$, then $\log\frac{n}{m} \gg \min(1, \frac{n-m}{m})$.

(g) Show that if N is a positive integer and $1 \le m \le N$, then

$$\sum_{\substack{1\le n\le N\\ n\ne m}} \frac{1}{|\log n/m|} \ll N\log N.$$

(h) By means of Theorem **??**, or otherwise, show that

$$\sum_{\substack{1\le m,n\le N\\ m\ne n}} \frac{a_m a_n}{|\log n/m|} \ll N(\log N)\sum_{n=1}^N |a_n|^2$$

uniformly for arbitrary real or complex numbers a_n.

(i) Let $f_1, f_2, \ldots, f_N$ be functions defined on an interval $I = [A, A+T]$, and suppose that C is a number such that $\sup_I |f_n| \le C$ and $\operatorname{var}_I f_n \le C$ for all n. Note that

$$\int_I \Big|\sum_{n=1}^N a_n f_n(t) n^{-it}\Big|^2\,dt = \sum_{n=1}^N |a_n|^2 \int_I |f_n(t)|^2\,dt + \sum_{\substack{1\le m,n\le N\\ m\ne n}} a_m\overline{a_n}\int_I f_m(t)\overline{f_n(t)}|\Big(\frac{n}{m}\Big)^{it}\,dt.$$

For given m and n with $m \ne n$, put $f(t) = f_m(t)\overline{f_n(t)}$. Show that $\sup_I |f(t)| \le C^2$ and that $\operatorname{var}_I f \le C^2$.

(j) Show that in the above situation,

$$\int_I \Big|\sum_{n=1}^N a_n f_n(t) n^{-it}\Big|^2\,dt = \sum_{n=1}^N |a_n|^2 \int_I |f_n(t)|^2\,dt + O(N\log N)\sum_{n=1}^N |a_n|^2.$$

This has many uses. For example, if $N(t)$ is an increasing function with $N(T) = N$, then the integral

$$\int_A^{A+T} \Big|\sum_{n\le N(t)} a_n n^{-it}\Big|^2\,dt$$

can be estimated, with an error term that is only one logarithm larger than in Corollary 26.2.

5. Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_N$ are distinct real numbers and that $\delta > 0$ has the property that $|\lambda_m - \lambda_n| \ge \delta$ whenever $m \ne n$. Put

$$A(t) = \sum_{n=1}^N a_n e(\lambda_n t), \qquad B(t) = \sum_{n=1}^N b_n e(\lambda_n t)\,.$$

(a) Show that for any $T \ge 0$ there is a number θ with $|\theta| \le 1$ such that

$$\int_0^T A(t)\overline{B(t)}\,dt = T\sum_{n=1}^N a_n\overline{b_n} + \frac{\theta}{\delta}\Big(\sum_{n=1}^N |a_n|^2\Big)^{1/2}\Big(\sum_{n=1}^N |b_n|^2\Big)^{1/2}.$$

(b) Suppose that δ_n is defined as in Theorem 26.3. Show that for any $T \ge 0$ there is a θ with $|\theta| \le 1$ such that

$$\int_0^T A(t)\overline{B(t)}\,dt = T\sum_{n=1}^N a_n\overline{b_n} + \frac{3}{2}\theta\Big(\sum_{n=1}^N \frac{|a_n|^2}{\delta_n}\Big)^{1/2}\Big(\sum_{n=1}^N \frac{|b_n|^2}{\delta_n}\Big)^{1/2}.$$

6. We consider the situation of Theorem 26.1.

(a) Show that if $\lambda_{n+1} - \lambda_n = \delta$ for all n, and $\delta T = 1$, then

$$\frac{1}{T}\int_0^T \Big|\sum_{n=1}^N a_n e(\lambda_n t)\Big|^2\,dt = \sum_{n=1}^N |a_n|^2\,.$$

(b) Show that if $\lambda_{n+1} - \lambda_n \ge \delta$ for all n, $\varepsilon > 0$, and $\delta T \ge 1 + \varepsilon$, then

$$\frac{1}{T}\int_0^T \Big|\sum_{n=1}^N a_n e(\lambda_n t)\Big|^2 \, dt \ge C(\varepsilon)\sum_{n=1}^N |a_n|^2 \qquad (26.14)$$ E:MSLB

where $C(\varepsilon) = \varepsilon/(1+\varepsilon)$.

Ing36

7. (Ingham, 1936) The object of this exercise is to show that if $\lambda_{n+1} - \lambda_n \ge \delta$ for all n and $\delta T = 1$, then the inequality (26.14) is false when $C(\varepsilon)$ is replaced by any constant $C > 0$. For $|z| < 1$ and $0 < \alpha < 1/2$, write

$$(1+z)^{-\alpha} = \sum_{m=0}^\infty b_m z^m \,.$$

For $0 < r < 1$ and $\lambda > 0$, put

$$f_r(t) = e(\lambda t/2)(1 + re(t))^{-\alpha} + e(-\lambda t/2)(1 + re(-t))^{-\alpha} \,.$$

(a) Show that

$$f_r(t) = \sum_{m=0}^\infty b_m r^m \big(e((m+\lambda/2)t) + e(-(m+\lambda/2)t)\big) \,.$$

(Note that the frequencies in the above are spaced by at least 1 if $\lambda \ge 1$.)

(b) Show that

$$\int_{-1/}^{1/2} |(1 + re(t))|^{2\alpha} \, dt = \sum_{m=0}^\infty |b_m|^2 r^{2m} \,.$$

(c) Note that $(1+z)^{-\alpha} = z^{-\alpha/2}\big(z^{1/2} + z^{-1/2}\big)^{-\alpha}$, and deduce that

$$\lim_{r\to 1^-} (1 + re(t))^{-\alpha} = e(-\alpha t/2)(2\cos \pi t)^{-\alpha}$$

for $-1/2 < t < 1/2$.

(d) Show that $|g(re(t))|^2 < (\cos \pi t)^{-2\alpha}$ when $-1/2 < t < 1/2$ and $0 \le r < 1$.

(e) Explain how you know that

$$\begin{aligned}\lim_{r\to 1^-}\int_{-1/2}^{1/2} |g(re(t))|^2 \, dt &= \int_{-1/2}^{1/2} \lim_{r\to 1^-} |g(re(t))|^2 \, dt \\ &= \int_{-1/2}^{1/2} (2\cos \pi t)^{-2\alpha} \, dt \,.\end{aligned}$$

(f) Show that the right hand side above is

$$\sim \frac{2(2\pi)^{-2\alpha}}{1-2\alpha}$$

as $\alpha \to (1/2)^-$.

(g) Show that

$$\begin{aligned}\lim_{r\to 1^-} &\int_{-1/2}^{1/2} |f_r(t)|^2\,dt \\ &= \int_{-1/2}^{1/2} \big|\big(e(\tfrac{1}{2}(\lambda-\alpha)t\big) + e(\tfrac{1}{2}(\alpha-\lambda)t)\big)(2\cos\pi t)^{-\alpha}\big|^2\,dt \\ &= \int_{-1/2}^{1/2} (2\cos\pi(\lambda-\alpha)t)^2(2\cos\pi t)^{-2\alpha}\,dt\,.\end{aligned}$$

(h) Take $\lambda = \alpha+1$. Then the above integral is

$$\int_{-1/2}^{1/2} (2\cos\pi t)^{2-2\alpha}\,dt\,.$$

Explain why this is bounded, and show that it tends to $4/\pi$ as $\alpha \to (1/2)^-$. Note that the frequencies are the numbers $\pm\big(m+\frac{\alpha+1}{2}\big)$ for $m = 0, 1, 2, \ldots$. Thus the gaps between the frequencies are all 1, except for the gap between $-(\alpha+1)/2$ and $(\alpha+1)/2$, which is nearly $3/2$.

(i) Explain why the asymptotic mean square of f is

$$2\sum_{m=0}^{\infty} b_m^2 = 2^{1-2\alpha}\int_{-1/2}^{1/2} (\cos\pi t)^{-2\alpha}\,dt\,.$$

(j) Theorem 26.1 relates to finite sums, but our construction thus far has involved infinite sums. Explain why

$$\sum_{m=0}^{M}\Big(1-\frac{m}{M}\Big)b_m\big(e((m+\lambda/2)t) + e(-(m+\lambda/2)t)\big)$$

behaves similarly to the infinite sum, when M is large.

8. In the situation of Theorem 26.8, apart from the ordering of the t_r, the hypothesis as to spacing is equivalent to asserting that no interval $[t, t+\delta)$ contains more than 1 of the t_r. Suppose we weaken this hypothesis by asserting only that no interval $[k\delta, (k+1)\delta)$ contains more than 1 of the t_r. Show that the same conclusion still follows (with a larger implicit constant).

Exer:Sobolev2d 9. Suppose that $f(x_1, x_2)$ has continuous partial derivatives through the second order, and let $f_1 = \frac{\partial f}{\partial x_1}$, $f_2 = \frac{\partial f}{\partial x_2}$, and $f_{12} = \frac{\partial^2 f}{\partial x_1 \partial x_2}$.

(a) Let $U = [0, 1]$. Show that if $(a_1, a_2) \in U^2$, then

$$|f(a_1, a_2)| \le \int_{U^2} |f(x_1, x_2)| + |f_1(x_1, x_2)| + |f_2(x_1, x_2)| + |f_{12}(x_1, x_2)|\, dx_1\, dx_2\,.$$

(b) Show that

$$|f(1/2, 1/2)| \le \int_{U^2} |f(x_1, x_2)| + \frac{1}{2}|f_1(x_1, x_2)| + \frac{1}{2}|f_2(x_1, x_2)| + \frac{1}{4}|f_{12}(x_1, x_2)|\, dx_1\, dx_2\,.$$

(c) Suppose that $\delta_1 > 0$ and that $\delta_2 > 0$. Show that

$$|f(a_1, a_2)| \le \int_{a_1 - \frac{1}{2}\delta_1}^{a_1 + \frac{1}{2}\delta_1} \int_{a_2 - \frac{1}{2}\delta_2}^{a_2 + \frac{1}{2}\delta_2} \frac{|f(x_1, x_2)|}{\delta_1 \delta_2} + \frac{|f_1(x_1, x_2)|}{2\delta_2} + \frac{|f_2(x_1, x_2)|}{2\delta_1} + \frac{f_{12}(x_1, x_2)|}{4}\, dx_2\, dx_1\,.$$

10. Show that

$$\int_0^T \Big| \operatorname{Re} \sum_{n=1}^N a_n n^{-it} \Big|^2 = \sum_{n=1}^N |a_n|^2 \Big(\frac{1}{2}T + O(n)\Big)\,.$$

11. Ing36 (Ingham, 1936) Suppose that $\{\lambda_n\}$ is a sequence of real numbers with $\lambda_{n+1} > \lambda_n$ for all n, and put

$$S(t) = \sum_{-\infty}^{\infty} a_n e(\lambda_n t)$$

where $\sum_n |a_n| < \infty$.

(a) Show that if $\lambda_n = n$ for all n, then

$$\max_n |a_n| \le \int_{-1/2}^{1/2} |S(t)|\, dt\,.$$

Our object, in what follows, is to explore what might be said in this direction for more general well-spaced λ_n.

(b) Set $K(t) = \cos \pi t$ for $-1/2 \le t \le 1/2$ and $K(t) = 0$ for $|t| > 1/2$. Show that

$$\widehat{K}(\alpha) = \frac{2 \cos \pi\alpha}{\pi(1 - 4\alpha^2)}\,.$$

(c) Show that

$$\int_{-\infty}^{\infty} K(t)e(ct)S(t)\,dt = \sum_{n=-\infty}^{\infty} a_n \widehat{K}(c+\lambda_n)$$

for any real number c.

(d) Explain why

$$\Big|\int_{-\infty}^{\infty} K(t)e(ct)S(t)\,dt\Big| \le \int_{-1/2}^{1/2} |S(t)|\,dt\,.$$

(e) Let k be an integer such that $|a_k| = \max_n |a_n|$. Show that

$$\Big|\sum_n a_n \widehat{K}\big(\lambda_n - \lambda_k\big)\Big| \ge |a_k|\bigg(\widehat{K}(0) - \sum_{\substack{n\\ n\ne k}} |\widehat{K}(\lambda_n - \lambda_k)\bigg).$$

(f) Suppose that $\varepsilon > 0$ and that $\lambda_{n+1} - \lambda_n \ge 1 + \varepsilon$ for all n. Show that the expression in large parentheses above is

$$\ge \frac{2}{\pi}\bigg(1 - \frac{1}{2}\sum_{r=1}^{\infty} \frac{1}{((1+\varepsilon)r)^2 - 1/4}\bigg).$$

(g) Show that the expression in large parentheses above is

$$\ge 1 - \frac{1}{2(1+\varepsilon)^2}\sum_{r=1}^{\infty}\frac{1}{r^2 - 1/4}\,.$$

(h) Show that the sum over r above is $= 2$.

(i) By choosing a suitable value of c, show that

$$\int_{-1/2}^{1/2} |S(t)|\,dt \ge C(\varepsilon)\max_n |a_n|$$

where

$$C(\varepsilon) = \frac{2\varepsilon(2+\varepsilon)}{(1+\varepsilon)^2\pi}\,.$$

(j) Show that $C(\varepsilon) > \varepsilon$ if $0 < \varepsilon \le 1/6$.

(k) Suppose that $T > 0$, that $\lambda_{n+1} - \lambda_n \ge \delta > 0$ for all n, and that $\delta T \ge 1 + \varepsilon$. Show that

$$\frac{1}{T}\int_{-T/2}^{T/2} |S(t)|\,dt \ge C(\varepsilon)\max_n |a_n|\,.$$

(l) Let δ and T be as above, and let A be an arbitrary real number. Show that

$$\int_A^{A+T} |S(t)|\,dt \ge C(\varepsilon)\max_n |a_n|\,.$$

The argument above fails to give a positive lower bound when $\delta T = 1$. On might wonder whether a better result could be obtained by the above method, if the kernel K were replaced by a different kernel. Note that by the Poisson summation formula,

$$\sum_n K(n+1/2) = \sum_k (-1)^k \widehat{K}(k)\,.$$

We need $\widehat{K}$ to be absolutely integrable, so it is necessary that K be continuous. Since $K(t) = 0$ for $|t| > 1/2$, it follows that $K(\pm 1/2) = 0$. Thus all terms in the sum on the left above are 0. But the right hand side is

$$\ge \widehat{K}(0) - \sum_{n\ne 0} |\widehat{K}(n)|\,.$$

However, we need this last expression to be positive in order to obtain a positive lower bound, so it seems that this method cannot succeed when $\delta T = 1$. However, see Exercise 13 below.

12. For $0 < \varepsilon \le 1/2$ let $K_\varepsilon(x) = \max(0, 1/\varepsilon - \|x\|)$.
    (a) Show that $\widehat{K}_\varepsilon(0) = 1$.
    (b) By integrating by parts, show that
    $$\widehat{K}_\varepsilon(n) = \Big(\frac{\sin \pi n\varepsilon}{\pi n\varepsilon}\Big)^2$$
    for $n \ne 0$.
    (c) In the notation of the preceding exercise, take $a_n = (-1)^n \widehat{K}_\varepsilon(n)$. Deduce that $S(t) = K_\varepsilon(t+1/2)$.
    (d) Set $T = 1 - 2\varepsilon$. Note that $S(t) = 0$ for $-T/2 \le t \le T/2$, and that $\max_n |a_n| = a_0 = 1$. Here $\delta = 1$, so $\delta T = 1 - 2\varepsilon$, but $\int_{-T/w2}^{T/2} |S(t)|\,dt$ is not bounded below by ca_0 with $c > 0$.

Exer:Ingham 13. Ing50 (Ingham, 1950) The object of this exercise is to show that if

$$f(t) = \sum_{n=1}^{N} a_n e\big(\lambda_n t\big)$$

where $\lambda_{n+1} - \lambda_n \ge 1$ for $1 \le n < N$, then

$$\max_n |a_n| \le 2 \int_{-1/2}^{1/2} |f(t)|\, dt\,.$$

(a) Explain why the following reformulation is equivalent to the above. Let m and n be nonnegative integers, and put

$$f(t) = \sum_{r=-m}^{n} a_r e\big(\lambda_r t\big)$$

where $\lambda_{r+1} - \lambda_r \ge 1$ for $-m \le r < n$. Suppose also that $\lambda_0 = 0$. Then $|a_0| \le 2 \int_{-1/2}^{1/2} |f(t)|\, dt$.

(b) The sum $f(t)$ has $m+n+1$ terms. Let $\mathcal{M}$ be any set of $m+n+1$ integers with $0 \in mathscrM$, and put

$$g(t) = \sum_{\mu \in \mathcal{M}} c(\mu) e(\mu t)$$

where the coefficients $c(\mu)$ are to be determined so $e\big(\lambda_r t\big)$ is orthogonal on the interval $[-1/2, 1/2]$ to $g(t)$ for all $r \ne 0$. This is a system of $m+n$ homogeneous equations in $m+n+1$ variables. To solve this system, put

$$G(u) = \int_{-1/2}^{1/2} g(t) e(-ut)\, dt$$

for real u.

(c) Show that

$$G(u) = \frac{\sin \pi u}{\pi} \sum_{\mu} \frac{(-1)^{\mu} c(\mu)}{u - \mu} = \frac{\sin \pi u}{\pi} F(u),$$

say.

(d) Show that $G(u)$ is an entire function.

(e) We want the rational function $F(u)$ to have poles at the μ and to be zero at the λ_r, so we set $F(u) = P(u)/Q(u)$ where

$$P(u) = \prod_{r \ne 0} \big(u - \lambda_r\big), \qquad Q(u) = \prod_{\mu \in \mathcal{M}} (u - \mu)\,.$$

(f) It is clear that $F(\lambda_r) = 0$. Explain why it follows that $G(\lambda_r) = 0$.

(g) Show that the partial fraction expansion of F is

$$\sum_{\mu} \frac{P(\mu)}{Q'(\mu)(u - \mu)}\,.$$

(h) Show that

$$Q'(\mu) = \prod_{\substack{\mu' \\ \mu' \neq \mu}} (\mu - \mu') .$$

(i) Show that

$$c(\mu) = (-1)^\mu \frac{\prod_{r\neq 0}(u - \lambda_r)}{\prod_{\substack{\mu' \\ \mu' \neq \mu}} (\mu - \mu')} .$$

(j) Show that

$$\int_{-1/2}^{1/2} f(t)g(-t)\,dt = \sum_r G(\lambda_r) = a_0 c(0) .$$

(k) Show that

$$\left| \int_{-1/2}^{1/2} f(t)g(-t)\,dt \right| \le \int_{-1/2}^{1/2} |f(t)|\,dt \sum_\mu |c(\mu)| .$$

(l) Assign indices to the μ, so that $\mu_0 = 0$, $\mu_r = \lceil \lambda_r \rceil$ for $r < 0$, and $\mu_r = \lfloor \lambda_r \rfloor$ for $r > 0$. Show that

$$c(0) = \prod r \neq 0 \frac{\lambda_r}{\mu_r} \ge 1 .$$

(m) Show that $(-1)^{\mu_r} c(\mu_r) < 0$ for all $r \neq 0$.

(n) Show that

$$1 = \lim_{u\to\infty} \frac{uP(u)}{Q(u)} = \sum_\mu (-1)^\mu c(\mu) .$$

(o) Conclude that $|a_0| \le 2 - 1/c(0)$.

(p) Let $\Delta_K(t)$ be the Fejér kernel, and set $f(t) = \Delta_K(\lambda t)$ where λ is slightly less than 2, and K is large. Here the largest coefficient is 1. Show that $\int_{-1/2}^{1/2} |f(t)|\,dt$ is approximately $1/\lambda$. Deduce that the constant 2 in (a) is best possible.

14. Let $K_T(t)$ be defined as in (26.9). By some form of the inversion theorem for Fourier transforms, it follows that

$$\int_{-\infty}^{\infty} \frac{1}{T}\Big(\frac{\sin \pi T \alpha}{\pi \alpha}\Big)^2\,d\alpha = \max(0, 1 - |t|/T) .$$

Give a direct computational proof of the above identity.

15. Suppose that $A_n \ge 0$ for all n, and that $\sum_n A_n < \infty$. Also, let λ_n be arbitrary real numbers, and let $K_T(t)$ be defined as in (26.9).

 (a) Show that

$$\int_{-\infty}^{\infty} K_T(t) \Big| \sum_{j=1}^{\infty} A_j e(\lambda_j t) \Big|^2 \, dt \ge T \sum_{j=1}^{\infty} A_j^2 .$$

 (b) Deduce that

$$\int_{-\infty}^{\infty} \Big| \sum_{j=1}^{\infty} A_j e(\lambda_j t) \Big|^2 \, dt \ge T \sum_{j=1}^{\infty} A_j^2 .$$

Exer:Jutila 16. (a) Let numbers λ_j, a_j, and A_j be as described in Theorem 26.13. Show that

$$\sum_{j,k} \Big| \sum_{n=1}^{\infty} a_n e\big(\lambda_j - \lambda_k\big) \Big|^2 \le \sum_{j,k} \Big| \sum_{n=1}^{\infty} A_n e\big(\lambda_j - \lambda_k\big) \Big|^2 .$$

 (b) Let the Dirichlet series $\alpha(s)$, $A(s)$ be defined as in (26.12), with $|a_n| \le A_n$ for all n, and suppose that σ is chosen so that $\sum A_n n^{-\sigma} < \infty$. Let $t_1, t_2, \ldots, t_R$ be real numbers. Show that

$$\sum_{1 \le j,k \le R} |\alpha\big(\sigma + i(t_j - t_k)\big)|^2 \le \sum_{1 \le j,k \le R} |A\big(\sigma + i(t_j - t_k)\big)|^2 .$$

 (c) Suppose that C is a positive real number such that $|a_n| \le C A_n$ for all n. Show that if Dirichlet characters χ_j and complex numbers s_j are chosen, then

$$\sum_{1 \le j,k \le R} \Big| \sum_{n=1}^{N} a_n \chi_j(n) \overline{\chi_k}(n) n^{-s_j - \overline{s_k}} \Big|^2$$
$$\le C^2 \sum_{1 \le j,k \le R} \Big| \sum_{n=1}^{N} A_n \chi_j(n) \overline{\chi_k}(n) n^{-s_j - \overline{s_k}} \Big|^2 .$$

17. (a) Let $C = [c_{nj}]$ be an arbitrary $N \times J$ matrix, and suppose that $|a_n| \le A_n$ for all n. Show that

$$\sum_{j=1}^{J} \sum_{k=1}^{J} \Big| \sum_{n=1}^{N} a_n c_{nj} \overline{c_{nk}} \Big|^2 \le \sum_{j=1}^{J} \sum_{k=1}^{J} \Big| \sum_{n=1}^{N} A_n c_{nj} \overline{c_{nk}} \Big|^2 .$$

(b) Let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_M$ and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N$ be arbitrary members of an inner product space. Show that

$$\sum_{m=1}^{M}\sum_{n=1}^{N} |\langle \boldsymbol{u}_m, \boldsymbol{v}_n\rangle|^2 \le \Big(\sum_{m=1}^{M}\sum_{\mu=1}^{M} |\langle \boldsymbol{u}_m, \boldsymbol{u}_\mu\rangle|^2\Big)^{1/2} \times \Big(\sum_{n=1}^{N}\sum_{\nu=1}^{N} |\langle \boldsymbol{v}_n, \boldsymbol{v}_\nu\rangle|^2\Big)^{1/2}$$

18. Let numbers a_j, A_j and λ_j be as described in Theorem 26.13. Show that

$$\int_{-T}^{T} \Big|\sum_{j=1}^{\infty} a_j e\big(\lambda_j t\big)\Big|^{2k}\,dt \le 3\int_{-T}^{T} \Big|\sum_{j=1}^{\infty} A_j e\big(\lambda_j t\big)\Big|^{2k}\,dt$$

for any positive integer k.

19. Let a_j, A_j, and the λ_j be as in Theorem 26.13, and consider the possibility of a more general inequality of the form

$$\int_{-cT}^{cT} \Big|\sum_{j=1}^{\infty} a_j e\big(\lambda_j t\big)\Big|^{2}\,dt \le K(c)\int_{-T}^{T} \Big|\sum_{j=1}^{\infty} A_j e\big(\lambda_j t\big)\Big|^{2}\,dt$$

where $c > 0$ and $K(c)$ depends on c.

(a) Show that the above is valid when $K(c) = 1 + \lceil 2c\rceil$.

(b) Show that if the above is valid, then it must be the case that $K(c) \ge \lceil 2c\rceil$.

26.2 Character sums and hybrids

S:hybrids

From the basic orthogonality properties of Dirichlet characters (cf Corollary 4.5) we know that

$$\sum_{\chi \bmod q} \Big|\sum_{n=1}^{q} a_n\chi(n)\Big|^2 = \varphi(q)\sum_{\substack{n=1\\(n,q)=1}}^{q} |a_n|^2 \qquad (26.15)$$

E:chiorthog

for arbitrary complex numbers a_n. Sometimes we want to sum over all χ modulo q, but evaluate our character sum not at χ but rather at the primitive character $\chi^\star$ that induces χ. For this situation we do not have an identity, but at least we have an upper bound.

L:S(chi*) **Lemma 26.14** *Let $\chi^\star$ denote the primitive character that induces χ. Then*

$$\sum_{\chi \bmod q} \Big| \sum_{n=1}^{q} a_n \chi^\star(n) \Big|^2 \le q \sum_{n=1}^{q} |a_n|^2 \tag{26.16}$$ E:S(chi*)

for arbitrary complex numbers a_n.

The bound here is sharp, since equality is attained when all the a_n are equal. (Consider the contribution on the left hand side made by the character χ_0^*.)

Proof Each character χ modulo q is induced by a unique primitive character $\chi^\star$. Let d denote the conductor of $\chi^\star$. Then $d|q$. This is a one-to-one correspondence, so

$$\sum_{\chi \bmod q} \Big| \sum_{n=1}^{q} a_n \chi^\star(n) \Big|^2 = \sum_{d|q} \quad \sideset{}{^\star}\sum_{\chi \bmod d} \Big| \sum_{n=1}^{q} a_n \chi(n) \Big|^2.$$

By Lemma **??** with q replaced by d, we know that

$$\sideset{}{^\star}\sum_{\chi \bmod d} \Big| \sum_{n=1}^{q} a_n \chi(n) \Big|^2 \le \sum_{\substack{b=1 \\ (b,d)=1}}^{d} \Big| \sum_{n=1}^{q} a_n e(bn/d) \Big|^2.$$

On summing this over $d|q$ we deduce that

$$\sum_{\chi \bmod q} \Big| \sum_{n=1}^{q} a_n \chi^\star(n) \Big|^2 \le \sum_{a=1}^{q} \Big| \sum_{n=1}^{q} a_n e(an/q) \Big|^2.$$

By the Parseval identity for the Discrete Fourier Transform (which is equation (4.4) in Chapter 4), the right hand side above is

$$= q \sum_{n=1}^{q} |a_n|^2,$$

which is the desired estimate. □

For similar sums over intervals of arbitrary length we argue less precisely, as follows.

T:charmodq **Theorem 26.15** *Let*

$$S(\chi) = \sum_{M+1}^{M+N} a_n \chi(n)$$

where the a_n are complex numbers. Then

$$\sum_{\chi \bmod q} |S(\chi)|^2 \le \Big(\Big\lfloor \frac{N-1}{q} \Big\rfloor + 1\Big)\varphi(q) \sum_{\substack{n=M+1 \\ (n,q)=1}}^{M+N} |a_n|^2 \qquad (26.17)$$ E:chimodq

and

$$\sum_{\chi \bmod q} |S(\chi^\star)|^2 \le \Big(\Big\lfloor \frac{N-1}{q} \Big\rfloor + 1\Big) q \sum_{n=M+1}^{M+N} |a_n|^2 \qquad (26.18)$$ E:primchimodq

where $\chi^\star$ is the primitive character that induces χ.

Proof Let

$$Z(q,h) = \sum_{\substack{n=M+1 \\ n\equiv h \bmod q}}^{M+N} a_n,$$

as in (**??**). Then

$$\sum_{\chi \bmod q} |S(\chi)|^2 = \sum_{\chi \bmod q} \Big| \sum_{h=1}^{q} Z(q,h)\chi(h)\Big|^2 = \varphi(q) \sum_{\substack{h=1 \\ (h,q)=1}}^{q} |Z(q,h)|^2$$

by (26.15). The sum that defines $Z(q,h)$ has at most $\lfloor \frac{N-1}{q} \rfloor + 1$ terms, so by Cauchy's inequality it follows that

$$|Z(q,h)|^2 \le \Big(\Big\lfloor \frac{N-1}{q} \Big\rfloor + 1\Big) \sum_{\substack{n=M+1 \\ n\equiv h \bmod q}}^{M+N} |a_n|^2,$$

so we have (26.17). The estimate (26.18) is proved in the same way, using Lemma 26.14. □

We now establish a series of fundamental estimates concerning

$$S(s;\chi) = \sum_{n=1}^{N} a_n \chi(n) n^{-s} \qquad (26.19)$$ E:DefPol1

where the a_n are arbitrary complex numbers, χ is a Dirichlet character.

T:smoothhybrid **Theorem 26.16** *Let $S(s;\chi)$ be given by (26.19), and let A be a real number and T be a positive real number. Then*

$$\sum_{\chi \bmod q} \int_A^{A+T} |S(it;\chi)|^2\, dt \ll \frac{\varphi(q)}{q} \sum_{\substack{n=1 \\ (n,q)=1}}^{N} |a_n|^2 (qT+n), \qquad (26.20)$$ E:chit

$$\sum_{\chi \bmod q} \int_A^{A+T} |S(it;\chi^\star)|^2 dt \ll \sum_{n=1}^N |a_n|^2 (qT+n) \qquad (26.21)$$ E:qchi

where $\chi^\star$ *denotes the primitive character that induces* χ*, and*

$$\sum_{q\le Q} \frac{q}{\varphi(q)} \sum_{\chi \bmod q}{}^{\star} \int_A^{A+T} |S(it;\chi)|^2 dt \ll \sum_{n=1}^N |a_n|^2 (Q^2T+n) \qquad (26.22)$$ E:Qchi

where $\sum^\star$ *indicates that the sum is restricted to primitive characters.*

Proof It suffices to prove the theorem with $A = 0$ for then the more general conclusion follows on replacing a_n by $a_n n^{-iA}$. When $T \le 1$ the conclusions follow at once from Theorem 26.15 for the first two inequalities and the large sieve inequality (Theorem **??**) for the third. Thus we may suppose that $T \ge 1$. By Corollary 26.12 we know that

$$\int_0^T |S(it;\chi)|^2 dt \ll T^2 \int_{-\infty}^{\infty} \Big| \sum_{\substack{n=1\\ y<n\le \tau y}}^N a_n\chi(n)\Big|^2 \frac{dy}{y}$$

where $\tau = \exp\left(\frac{1}{T}\right)$. Thus $\tau - 1 \asymp 1/T$. We sum the above over χ (mod q). By (26.17) we see that

$$\sum_\chi \Big| \sum_{y<n\le\tau y} a_n\chi(n)\Big|^2 \ll \frac{\varphi(q)}{q}(q+(\tau-1)y) \sum_{y<n\le\tau y} |a_n|^2.$$

Thus

$$\begin{aligned}\sum_{\chi \bmod q} \int_0^T |S(it;\chi)|^2\, dt &\ll \frac{\varphi(q)}{q} \sum_n |a_n|^2 \int_{n/\tau}^n \Big(\frac{qT^2}{y} + T\Big)\, dy \\ &= \frac{\varphi(q)}{q} \sum_n |a_n|^2 (qT + Tn(1-1/\tau)),\end{aligned}$$

which gives (26.20).

To obtain (26.21) we apply (26.18), from which we see that

$$\sum_\chi \Big| \sum_{y<n\le\tau y} a_n\chi^\star(n)\Big|^2 \ll (q+(\tau-1)y) \sum_{y<n\le\tau y} |a_n|^2.$$

It then suffices to argue as in the first case.

Finally, to obtain (26.22) we apply the large sieve (Theorem **??**), from which we see that

$$\sum_{q\le Q} \frac{q}{\varphi(q)} \sum_\chi{}^{\star} \Big| \sum_{y<n\le\tau y} a_n\chi(n)\Big|^2 \ll (Q^2+(\tau-1)y) \sum_{y<n\le\tau y} |a_n|^2,$$

and then the rest of the argument is the same as before. □

The method of proof of Theorem 26.8 combines well with the estimates of Theorem 26.16 to yield a variant with discrete values of t.

T:discretehybrid1 **Theorem 26.17** *Let $S(s;\chi)$ be given by* (26.19), *let A, $T \ge 2$ and $0 < \delta \le 1$ be a real numbers. Further, for $r = 1, 2, \ldots, R$ let (t_r, χ_r) be a pair in which χ_r is a character* (mod q), $A \le t_r \le A + T$, and $|t_{r_1} - t_{r_2}| \ge \delta$ if $\chi_{r_1} = \chi_{r_2}$. Then

$$\sum_{r=1}^{R} |S(it_r;\chi_r)|^2 \ll \frac{\varphi(q)}{q}(\log N + 1/\delta) \sum_{\substack{n=1\\(n,q)=1}}^{N} |a_n|^2(qT + n), \qquad (26.23)$$ E:chirtr

and

$$\sum_{r=1}^{R} |S(it_r;\chi_r^\star)|^2 \ll (\log N + 1/\delta) \sum_{n=1}^{N} |a_n|^2(qT + n) \qquad (26.24)$$ E:qrchir

where $\chi^\star$ denotes the primitive character that induces χ. Finally, let (t_r, χ_r, q_r) be a triple with χ_r a primitive character (mod q_r) for some $q_r \le Q$, $A \le t_r \le A + T$, and $|t_{r_1} - t_{r_2}| \ge \delta$ if $\chi_{r_1} = \chi_{r_2}$. Then

$$\sum_{r=1}^{R} \frac{q_r}{\varphi(q_r)} |S(it_r;\chi_r)|^2 \ll (\log N + 1/\delta) \sum_{n=1}^{N} |a_n|^2(Q^2T + n). \qquad (26.25)$$ E:Qchirtr

Proof As in the proof of Theorem 26.16, we may assume that $A = 0$. We consider first (26.26). Let $\mathcal{R}(\chi) = \{r : \chi_r = \chi\}$. Let $\mathfrak{M}_r = [t_r - \delta/2, t_r + \delta/2]$ for $r = 1, 2, \ldots, R$. By Lemma **??** it follows that

$$|S(it_r;\chi_r)|^2 \le \frac{1}{\delta}\int_{\mathfrak{M}_r} |S(it;\chi_r)|^2\,dt + \int_{\mathfrak{M}_r} |S(it;\chi_r)S'(it;\chi_r)|^2\,dt.$$

For any χ (mod q), the intervals $\mathfrak{M}_r$ with $r \in \mathcal{R}(\chi)$ are disjoint and lie in the interval $[-\delta, T + \delta]$, so

$$\sum_{r\in\mathcal{R}(\chi)} |S(it_r;\chi)|^2 \le \frac{1}{\delta}\int_{-\delta}^{T+\delta} |S(it;\chi)|^2 + \int_{-\delta}^{T+\delta} |S(it;\chi)S'(it;\chi)|\,dt.$$

We sum this over χ to see that

$$\sum_{r=1}^{R} |S(it_r;\chi_r)|^2 \le \frac{1}{\delta}\sum_{\chi}\int_{-\delta}^{T+\delta} |S(it;\chi)|^2\,dt + \sum_{\chi}\int_{-\delta}^{T+\delta} |S(it;\chi)S'(it,\chi)|\,dt.$$

By the Cauchy–Schwarz inequality, the last term above is

$$\le \Big(\sum_\chi \int_{-\delta}^{T+\delta} |S(it;\chi)|^2\,dt\Big)^{1/2}\Big(\sum_\chi \int_{-\delta}^{T+\delta} |S'(it;\chi)|^2\,dt\Big)^{1/2}.$$

From (26.20) it follows that

$$\sum_{r=1}^R |S(it_r;\chi_r)|^2 \ll \frac{\varphi(q)}{q}\Big(\frac{1}{\delta}\sum_{n=1}^N |a_n|^2(qT+n) + \Big(\sum_{n=1}^N |a_n|^2(qT+n)\Big)^{1/2}\Big(\sum_{n=1}^N |a_n|^2(qT+n)(\log n)^2\Big)^{1/2}\Big),$$

which gives the stated estimate. The estimates (26.27) and (26.28) are derived similarly from (26.21) and (26.22), respectively. □

For points s_r with nonnegative real part we have the following further estimates.

T:discretehybrid2 **Theorem 26.18** *Let $S(s;\chi)$ be given by* (26.19)*, let A, $T \ge 2$ and $0 < \delta \le 1$ be a real numbers. Further, for $r = 1, 2, \ldots, R$ let (s_r, χ_r) be a pair in which χ_r is a character* (mod q), $\sigma_r \ge 0$, $A \le t_r \le A+T$, and $|t_{r_1} - t_{r_2}| \ge \delta$ if $\chi_{r_1} = \chi_{r_2}$. Then

$$\sum_{r=1}^R |S(s_r;\chi_r)|^2 \ll \frac{\varphi(q)}{q}(\log N + 1/\delta) \times \sum_{\substack{n=1\\(n,q)=1}}^N |a_n|^2(qT+n)\Big(1+\log\frac{\log 2N}{\log 2n}\Big), \tag{26.26}$$ E:chirtr

and

$$\sum_{r=1}^R |S\big(s_r;\chi_r^\star\big)|^2 \ll (\log N + 1/\delta) \times \sum_{n=1}^N |a_n|^2(qT+n)\Big(1+\log\frac{\log 2N}{\log 2n}\Big) \tag{26.27}$$ E:qrchir

where $\chi^\star$ denotes the primitive character that induces χ. Finally, let (s_r, χ_r, q_r) be a triple with χ_r a primitive character (mod q_r) for some

$q_r \le Q$, $\sigma_r \ge 0$, $A \le t_r \le A+T$, and $|t_{r_1} - t_{r_2}| \ge \delta$ if $\chi_{r_1} = \chi_{r_2}$. Then

$$\sum_{r=1}^{R} \frac{q_r}{\varphi(q_r)} |S(s_r;\chi_r)|^2 \ll (\log N + 1/\delta) \sum_{n=1}^{N} |a_n|^2 (Q^2T + n)\Big(1 + \log \frac{\log 2N}{\log 2n}\Big). \tag{26.28}$$

E:Qchirtr

Proof We argue in the same way that we did in deriving Theorem 26.9 from Theorem 26.8. In particular, we again use the inequality (26.5). We have $R \ll \varphi(q)T/\delta$ in the case of (26.26) and (26.27), and $R \ll Q^2T/\delta$ for (26.28). Thus the proof is entirely parallel to the former one. □

26.1.1 Exercises

1. As in Lemma 26.14, let $\chi^\star$ denote the primitive character that induces χ.
   (a) Let χ denote a Dirichlet character modulo q. Explain why the assertion that
   $$\sum_\chi \Big|\sum_{n=1}^{q} a_n \chi^\star(n)\Big|^2 \le q \sum_{n=1}^{q} |a_n|^2$$
   for all choices of the a_n is equivalent to the assertion that
   $$\sum_{n=1}^{q} \Big|\sum_\chi b(\chi)\chi^\star(n)\Big|^2 \le q \sum_\chi |b(\chi)|^2$$
   for all choices of the $b(\chi)$.
   (b) Note that the left hand side above is
   $$= \sum_{\chi_1,\chi_2} b(\chi_1)\overline{b(\chi_2)} \sum_{n=1}^{q} \chi_1^*(n)\overline{\chi_2^*}(n)\,.$$
   (c) Show that the sum over n above is $= q\varphi(d)/d$ if $\chi_1 = \chi_2$ and their conductor is d, and is $= 0$ otherwise.
   (d) Deduce that
   $$\sum_{n=1}^{q} \Big|\sum_\chi b(\chi)\chi^\star(n)\Big|^2 = q \sum_{d|q} \frac{\varphi(d)}{d} \sideset{}{^\star}\sum_{\chi \bmod d} |b(\chi)|^2\,.$$
   (e) Explain why this gives a second proof of Lemma 26.14.

26.3 Notes

S: MeanVals Notes

Section 21.1. Concerning majorant inequalities, antecedents of Theorem 26.13 are found in the work of Wiener (unpubished — see Theorem 12.6.12 of Boas (1954), Erdős and Fuchs (1956), Wiener and Wintner (1956), and Halász (1968). Logan (1988) showed that the constant 3 is best possible. For a general discussion of majorant principles, see Shapiro (1975).

26.4 References

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26.5 Mean values of L-functions

S:MVLFcns

Suppose that $\sigma > 1$. Then the Dirichlet series for $\zeta(s)$ is absolutely convergent, and hence the Dirichlet series for $\zeta(s)^k$ is also absolutely convergent:

$$\zeta(s)^k = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} = \prod_p \Big(1 - \frac{1}{p^s}\Big)^{-k} = \prod_p \Big(\sum_{r=0}^{\infty} \binom{k+r-1}{r} p^{-rs} \Big) .$$

Here $d_k(n)$ is known as the k^{th} divisor function. It is the unique multiplicative function with the property that $d_k\big(p^r\big) = \binom{k+r-1}{r}$. Thus (ignoring for the moment the sharp estimates we derived in §21.1),

$$\begin{aligned} \int_0^T |\zeta(\sigma+it)|^{2k}\,dt &= \int_0^T \Big| \sum_{n=1}^{\infty} \frac{d_k(n)}{n^\sigma} n^{-it} \Big|^2 \,dt \\ &= \sum_{m,n} \frac{d_k(m)d_k(n)}{(mn)^\sigma} \int_0^T \Big(\frac{m}{n}\Big)^{it}\,dt\,. \end{aligned}$$

The functions m^{-it} and n^{-it} are asymptotically orthogonal to the extent that

$$\frac{1}{T}\int_0^T \Big(\frac{m}{n}\Big)^{it}\,dt = \begin{cases} 1 & (m=n), \\ O_{m,n}(1/T) & (m \ne n)\,. \end{cases}$$

The double sum of the coefficients,

$$\sum_{m,n} \frac{d_k(m)d_k(n)}{(mn)^\sigma},$$

is absolutely convergent $\big($with value $\zeta(\sigma)^{2k}\big)$, so by the principle of dominated convergence it follows that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |\zeta(\sigma+it)|^{2k}\,dt = \sum_{n=1}^{\infty} \frac{d_k(n)^2}{n^{2\sigma}} \qquad (26.29)$$ E:zeta2kthmv

for any fixed $\sigma > 1$. The question before us is to try to determine what combination of values of k and $\sigma \le 1$ (if any) the above relation continues to be valid. In this connection, the following principle is often useful.

T:alpha2sigma **Theorem 26.19** *Let k be a positive integer, and suppose that $\alpha \ge 1/2$ is a number such that*

$$\int_0^T |\zeta(\alpha+it)|^{2k}\,dt \ll_\varepsilon T^{1+\varepsilon} \qquad (26.30)$$ E:zetamvEst

for every $\varepsilon > 0$. Then the relation (26.29) *holds for all $\sigma > \alpha$.*

Proof We may suppose that $\sigma \le 1$. We suppose also that $2 \le x \le T^A$ for some constant A. Then by the formula (5.25) for the inverse Mellin transform with abelian weights we know that

$$\sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} e^{-n/x} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta(s+w)^k \Gamma(w) x^w \, dw$$

for $c > 1 - \sigma$. Write $w = u + iv$. When we move the contour to the abscissa $u = \alpha - \sigma$, we pass two poles: One at $w = 0$ with residue $\zeta(s)^k$, and the other at $w = 1 - s$. To estimate the residue at this second pole, let C be a circle of radius $1/\log x$ centered at $1 - s$. For w on this circle, $|\zeta(s+w)^k| \asymp (\log x)^k$, $|x^w| \asymp x^{1-\sigma}$, and $|\Gamma(w)| \asymp e^{-\pi\tau/2}\tau^{1/2-\sigma}$ by (25.3). Hence the residue at $1 - s$ is $\ll e^{-\pi\tau/2}\tau^{1/2-\sigma}x^{1-\sigma}(\log x)^{k-1}$. Thus the above is

$$= \zeta(s)^k + \frac{1}{2\pi} \int_{-\infty}^{\infty} \zeta(\alpha + it + iv)^k \Gamma(\alpha - \sigma + iv) x^{\alpha - \sigma + iv} \, dv$$
$$+ O\big(e^{-\pi\tau/2}\tau^{1/2-\sigma}x^{1-\sigma}(\log x)^{k-1}\big),$$

and consequently

$$\int_T^{2T} \Big| \zeta(s)^k - \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} e^{-n/x} \Big|^2 dt$$
$$\ll x^{2(\alpha-\sigma)} \int_T^{2T} \Big| \int_{-\infty}^{\infty} |\zeta(\alpha + it + iv)^k \Gamma(\alpha - \sigma + iv)| \, dv \Big|^2 dt + e^{-T}.$$

By the Cauchy–Schwarz inequality we see that the square of the modulus of the above integral over v is

$$\le \int_{-\infty}^{\infty} |\Gamma(\alpha - \sigma + iv)| \, dv \int_{-\infty}^{\infty} |\zeta(\alpha + it + iv)|^{2k} |\Gamma(\alpha - \sigma + iv)| \, dv \, .$$

The first integral above is bounded, so

$$\int_T^{2T} \Big| \zeta(s)^k - \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} e^{-n/x} \Big|^2 dt$$
$$\ll x^{2(\alpha-\sigma)} \int_{-\infty}^{\infty} \int_T^{2T} |\zeta(\alpha + it + iv)|^{2k} \, dt \, |\Gamma(\alpha - \sigma + iv)| \, dv + e^{-T}.$$

To estimate the double integral we consider two ranges of v. First, if $-4T \le v \le 4T$, then the integral over t is $\ll T^{1+\varepsilon}$, so the contribution of such v is also $\ll T^{1+\varepsilon}$. Secondly, if $|v| > 4T$, then by the trivial bound $\zeta(\alpha+it) \ll \tau^{1/4}$ we see that the integral over t is $\ll Tv^{k/2}$, but by (25.3) the resulting contribution is $\ll e^{-T}$. We conclude that the above is

$$\ll x^{2(\alpha-\sigma)}T^{1+\varepsilon}\,. \qquad (26.31)$$ E:ApproxEst

By Corollary 26.4 we see that

$$\int_T^{2T} \Big|\sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} e^{-n/x}\Big|^2\,dt = \sum_{n=1}^{\infty} \frac{d_k(n)^2}{n^{2\sigma}} e^{-2n/x}\big(T + O(n)\big)\,.$$

Since $e^{-2n/x} = 1 + O(n/x)$ for $n \le x$, we find that the right hand side above is

$$= T\sum_{n=1}^{\infty}\frac{d_k(n)^2}{n^{2\sigma}} + O\Big(T\sum_{n>x}\frac{d_k(n)^2}{n^{2\sigma}}\Big) + O\Big(\frac{T}{x}\sum_{n\le x}\frac{d_k(n)^2}{n^{2\sigma-1}}\Big) + O\Big(\sum_{n=1}^{\infty}\frac{d_k(n)^2}{n^{2\sigma-1}}e^{-2n/x}\Big).$$

From Corollary 2.15 we deduce that

$$\sum_{U<n\le 2U} d_k(n)^2 \ll U(\log 2U)^{k^2-1},$$

and from this it follows that

$$\sum_{n\le x}\frac{d_k(n)^2}{n^{2\sigma-1}} \ll x^{2-2\sigma}(\log x)^{k^2}$$

uniformly for $1/2 \le \sigma \le 1$. Consequently,

$$\int_T^{2T} \Big|\sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} e^{-n/x}\Big|^2\,dt = T\sum_{n=1}^{\infty}\frac{d_k(n)^2}{n^{2\sigma}} \ + O\big((T+x)x^{1-2\sigma}(\log x)^{k^2}\big).$$

We take $x = T$ and combine this with (26.31) to obtain the stated result. □

We argued qualitatively above. If we wanted a quantitative estimate, we would need to employ the triangle inequality for L^2 norms; see Exercise 26.5.1.2.

We recall that the Lindelöf Hypothesis (LH) asserts that

$$\zeta(1/2+it) \ll_\varepsilon \tau^\varepsilon \qquad (26.32)$$ E:DefLH

for every $\varepsilon > 0$. As we discussed already in §§10.1,13.2, RH implies LH.

Cor:LH2mvt **Corollary 26.20** *Assume the Lindelöf Hypothesis. Then the relation* (26.29) *holds for all positive integers k, and all fixed $\sigma > 1/2$.*

We have some nontrivial bounds for the zeta function, but they fall short of LH, and so it is to be expected that our results concerning mean values are a bit fragmentary and unsatisfactory. We now establish a first mean value theorem for the zeta function on the 1/2-line.

T:zeta(1/2+it)^2 **Theorem 26.21** *For $T \ge 2$,*

$$\int_0^T |\zeta(1/2+it)|^2\,dt = T\log T + O(T)\,.$$

Proof It suffices to show that

$$\int_T^{2T} |\zeta(1/2+it)|^2\,dt = T\log T + O(T),$$

since we can replace T by $T/2^k$ in the above, and then sum over k to obtain the stated result. We write the formula (**??**) as

$$\zeta(s) = \sum_{n\le x} n^{-s} \;+\; R(s),$$

and take $x = 4T$. Thus for $s = 1/2 + it$ with $T \le t \le 2T$ we have $R(s) \ll T^{-1/2}$. By Corollary 26.4 we find that

$$\int_T^{2T} \Big|\sum_{n\le 4T} n^{-1/2-it}\Big|^2\,dt = T\sum_{n\le 4T}\frac{1}{n} + O\Big(\sum_{n\le 4T} 1\Big) = T\log T + O(T)\,.$$

Hence, as in Exercise 26.5.1.2(b), we have

$$\begin{aligned}\int_T^{2T} |\zeta(1/2+it)|^2\,dt &= \big(T\log T + O(T)\big)\Big(1 + O\big((T\log T)^{-1/2}\big)\Big)\\ &= T\log T + O(T)\,.\end{aligned}$$

□

By combining the above with Theorem 26.19 we obtain the following further result.

Cor:zetamv2 **Corollary 26.22** *Let $\sigma > 1/2$ be fixed. Then*

$$\int_0^T |\zeta(\sigma+it)|^2\,dt \sim \zeta(2\sigma)T$$

as $T \to \infty$.

By using our averaged form of the approximate functional equation for the zeta function, we obtain the following useful estimate.

T:zeta4mean **Theorem 26.23** *Suppose that* $T \ge 60$. *Then*

$$\int_0^T |\zeta(\sigma+it)|^4\,dt \ll T(\log T)^4$$

uniformly for $|\sigma - 1/2| \le 2/\log T$.

Proof From (25.22) we see that

$$|\zeta(s)|^4 \ll \Big|\sum_n w(n/\sqrt{\tau})n^{-s}\Big|^4 + \Big|\sum_n w(n/\sqrt{\tau})n^{s-1}\Big|^4 + \tau^{-1}(\log\tau)^4$$

for $|\sigma - 1/2| \le 2/\log\tau$. Here $w(u)$ is defined as in (25.23). By taking $k = 2$ in (25.26) we find that

$$\Big|\sum_n w(n/\sqrt{\tau})n^{-s}\Big|^4 \le \int_{T^{1/2}}^{8T^{1/2}} |A(s,x)|^4\,\frac{dx}{x} \qquad (26.33)$$ E:MainStep

uniformly for $T \le t \le 2T$. Here $A(s,x)$ is defined as in (25.24). Note that

$$A(s,x)^2 = \sum_{n\le x^2} c_n n^{-s}$$

where $c_n = c_{n,x}$ counts only some of the divisors of n, so that $0 \le c_n \le d(n)$ for all x. Hence when we integrate both sides of (26.33) with respect to t and apply Corollary 26.2, we find that

$$\int_T^{2T}\Big|\sum_n w(n/\sqrt{\tau})n^{-s}\Big|^4\,dt \ll \int_{T^{1/2}}^{8T^{1/2}} \sum_{n\le x^2}\frac{d(n)^2}{n^{2\sigma}}\big(T+x^2\big)\,\frac{dx}{x}\,.$$

From Corollary 2.15 we know that $\sum_{n\le U} d(n)^2 \ll U(\log U)^3$, and hence that $\sum_{n\le U} d(n)^2/n \ll (\log U)^4$. Thus the above is $\ll T(\log T)^4$. The same bound applies with s replaced by $1-s$, so we have the desired result. □

Cor:4thmom **Corollary 26.24** *Suppose that* $\sigma > 1/2$. *Then*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T |\zeta(\sigma+it)|^4\,dt = \frac{\zeta(2\sigma)^4}{\zeta(4\sigma)}\,.$$

Here we have used the identity

$$\sum_{n=1}^\infty d(n)^2/n^s = \zeta(s)^4/\zeta(2s), \qquad (26.34)$$ E:RamId

which is a special case of the identity discussed in Exercise 4.3.1.4.

Of the many consequences of Theorem 26.23, we note one particularly useful example.

Cor:2mzeta*zeta'

Corollary 26.25 *Suppose that $T \ge 60$. Then*

$$\int_0^T |\zeta(\sigma+it)\zeta'(\sigma+it)|^2\,dt \ll T(\log T)^6$$

uniformly for $|\sigma - 1/2| \le 1/\log T$.

Proof If $f(z)$ is analytic in a neighborhood of $z=0$, then by Cauchy's formula we know that

$$f'(0) = \frac{1}{2\pi i}\oint \frac{f(z)}{z^2}\,dz\,.$$

We take $f(z) = \zeta(s+z)^2$ and set $z = re^{i\theta}$ where $r = 1/\log T$. By the Cauchy–Schwarz inequality we find that

$$|\zeta(s)\zeta'(s)|^2 \ll (\log T)^2 \int_0^{2\pi} \left|\zeta\big(s+re^{i\theta}\big)\right|^4.$$

We integrate this over $0 \le t \le T$ to obtain the stated result. □

Cor:disczeta4mv

Corollary 26.26 *Suppose that $T \ge 60$, that $\delta > 0$, and that $\delta/2 \le t_1 < t_2 < \cdots < t_R < T - \delta/2$ where $t_{r+1} - t_r \ge \delta$ for $r = 1, 2, \dots, R-1$. Then*

$$\sum_{r=1}^R |\zeta(1/2+it_r)|^4 \ll (1/\delta + \log T)T(\log T)^4\,.$$

Proof By the Sobolev inequality (**??**) of Lemma **??** we see that

$$|\zeta(1/2+it_r)|^4 \le \delta^{-1}\int_{t_r-\frac12\delta}^{t_r+\frac12\delta} |\zeta(1/2+it)|^4\,dt + 2\int_{t_r-\frac12\delta}^{t_r+\frac12\delta} |\zeta(1/2+it)^3\zeta'(1/2+it)|\,dt\,.$$

The intervals of integration are disjoint, so on summing over r we find that

$$\sum_{r=1}^R |\zeta(1/2+it_r)|^4 \ll \delta^{-1}\int_0^T |\zeta(1/2+it)|^4\,dt + \int_0^T |\zeta(1/2+it)\zeta'(1/2+it)|\,dt\,.$$

By the Cauchy–Schwarz inequality the second integral above is

$$\le \Big(\int_0^T |\zeta(1/2+it)|^4\,dt\Big)^{1/2}\Big(\int_0^T |\zeta(1/2+it)\zeta'(1/2+it)|^2\,dt\Big)^{1/2}.$$

The desired result now follows from the Theorem and the preceding Corollary. □

When we form a weighted average a Dirichlet series $\alpha(s) = \sum a_n n^{-s}$ over a vertical line, its coefficients are diminished. For example, in Exercise 5.1.1.5 we saw that if $\sigma_c < 0$, then

$$\frac{1}{2\pi}(\log N)\int_{-\infty}^{\infty} \alpha(it)\Big(\frac{\sin \frac{1}{2}t\log N}{\frac{1}{2}t\log N}\Big)^2 dt = \sum_{n\le N} a_n\Big(1-\frac{\log n}{\log N}\Big). \quad (26.35)$$ E:Riesz1

We also know that averaging a function causes its norm to decrease. For example, suppose that $w(x)$ is a weight function such that $w(x) \ge 0$ for all x and $\int_{-\infty}^{\infty} w(x)\,dx = 1$. If $f \in L^p(\mathbb{R})$ for some real number $p \ge 1$, and we define F to be the convolution

$$F(x) = (w * f)(x) = \int_{-\infty}^{\infty} w(u)f(w-u)\,du,$$

then not only is $F \in L^p(\mathbb{R})$, but also $\|F\|_p \le \|f\|_p$. (See Exercise 5.) We use these ideas to derive the following useful complement to our upper bounds.

T:zetakmvlb **Theorem 26.27** *Suppose that the real number σ and a positive integer k are fixed, with $1/2 < \sigma < 1$. Then*

$$\liminf_{T\to\infty} \frac{1}{T}\int_0^T |\zeta(\sigma+it)|^{2k}\,dt \ge \sum_{n=1}^{\infty} \frac{d_k(n)^2}{n^{2\sigma}}. \quad (26.36)$$ E:zetakmvlb

Proof Since σ and k are fixed, implicit constants in this proof may depend on these quantities. For our present purposes we need a formula similar to (26.35), but in which the kernel decay more quickly. To this end we start by putting

$$K(w) = c_k\Big(\frac{\sinh \frac{w}{4k}}{w}\Big)^{4k} = c_k\Big(\frac{e^{w/(4k)} - e^{-w/(4k)}}{2w}\Big)^{4k} \quad (26.37)$$ E:DefK

where $w = u + iv$ is a complex variable, and c_k is chosen so that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} K(iv)\,dv = 1. \quad (26.38)$$ E:IntKiv

We note that $K(w)$ is an entire function. Since $|\sinh w| \le e^{|u|}$, it follows that

$$K(w) \ll \frac{e^{|u|}}{|w|^{4k}}. \quad (26.39)$$ E:KEst

The implicit constant in the above may depend on k, but we suppress this, here and elsewhere, since k is considered to be fixed. We note that $K(-w) = K(w)$ for all complex w, and that

$$K(iv) = c_k\Big(\frac{\sin\frac{v}{4k}}{v}\Big)^{4k} \ge 0$$

for all real v. To create a kernel that yields a weighted partial sum of a Dirichlet series, we rescale $K(w)$ and put

$$\begin{aligned} K_N(w) &= K(w\log N)\log N \\ &= c_k(\log N)\Big(\frac{N^{w/(4k)} - N^{-w/(4k)}}{2w\log N}\Big)^{4k}. \end{aligned} \tag{26.40}$$ E:DefKN

For real α let

$$W(\alpha) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} K(w)e^{\alpha w}\,dw = \frac{1}{2\pi}\int_{-\infty}^{\infty} K(iv)e^{i\alpha v}\,dv\,. \tag{26.41}$$ E:DefW

Thus W is the inverse Laplace transform of K. We note that the value of the above integrals are independent of the value of the real number a. Suppose that $a > 0$. From (26.39) it follows that

$$|W(\alpha)| \le \frac{e^{a(1+\alpha)}}{2\pi}\int_{-\infty}^{\infty}\frac{dv}{|a+iv|^{4k}} \ll \frac{e^{a(1+\alpha)}}{a^{4k-1}}\,.$$

If $\alpha \le -1$, then this upper bound tends to 0 as $a \to \infty$. Hence $W(\alpha) = 0$ if $\alpha \le -1$. Since K is even it follows that W is also even, and hence that $W(\alpha) = 0$ if $\alpha \ge 1$. We now take $a = 0$ to see that

$$W(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} K(iv)e^{i\alpha v}\,dv\,.$$

From this it follows that $W(0) = 1$, that $|W(\alpha)| \le 1$ for all α, and that W is continuous. We now set

$$W_N(x) = W\Big(-\frac{\log x}{\log N}\Big) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} K_N(s)x^{-s}\,ds\,. \tag{26.42}$$ E:DefWN

Thus W_N is the inverse Mellin transform of K_N. On expanding the binomial on the right hand side of (27.36) we see that W_N is a linear combination of Riesz typical means of order $4k-1$ with truncations at $N^{j/(2k)}$ for $j = 1, 2, \ldots, 2k$.

If $\alpha(s) = \sum a_n n^{-s}$ is a Dirichlet series with abscissa of convergence σ_c, then

$$\sum_{n\le N}\frac{a_n}{n^s}W_N(n) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\alpha(s+w)K_N(w)\,dw$$

for $a > \sigma_c$. In particular, if $\alpha(s) = \zeta(s)^k$ and $1/2 < \sigma \le 1$, then

$$\sum_{n\le N} \frac{d_k(n)}{n^s} W_N(n) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \zeta(s+w)^k K_N(w)\,dw\,.$$

On moving the contour to $a = 0$ we see that this is

$$\begin{aligned} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \zeta(s+iv)^k K_N(iv)\,dv \\ &\quad + \operatorname{Res}\big[\zeta(s+w)^k K_N(w)\big|_{w=1-s}\,. \end{aligned} \tag{26.43}$$

E:MainId

Here the first term above is a weighted average of the numbers $\zeta(s+iv)^k$, with the bulk of the weight attached to points for which $v \ll 1/\log N$.

Let $A \ge 1$ and $N \ge 1$ be parameters that will eventually be chosen to be functions of T, with $A = o(T)$ and $N = o(T)$. Then by Corollary 26.2 we see that

$$\int_A^{T-A} \Big|\sum_{n\le N} \frac{d_k(n)}{n^{\sigma+it}} W_N(n)\Big|^2 dt = (T - 2A + O(N))\sum_{n\le N} \frac{d_k(n)^2}{n^{2\sigma}} W_N(n)^2\,.$$

Since W is continuous at 0 and $W(0) = 1$, it follows that for any $\varepsilon > 0$ there is a $\delta > 0$ such that $|W_N(n) - 1| < \varepsilon$ for all $n < N^\delta$. Thus if $N = T^{1/2}$, then

$$\int_A^{T-A} \Big|\sum_{n\le N} \frac{d_k(n)}{n^{\sigma+it}} W_N(n)\Big|^2 dt \sim T\sum_{n=1}^{\infty} \frac{d_k(n)^2}{n^{2\sigma}}\,.$$

To assess the residue in (26.43) let $\mathcal{C}$ be a circle of radius $1/\log N$ centered at $1 - s$. Thus the residue is

$$\begin{aligned} \frac{1}{2\pi i}\int_{\mathcal{C}} \zeta(s+w)^k K_N(w)\,dw &\ll (\log N)^{k-1} \max_{w\in\mathcal{C}} |K_N(w)| \\ &= (\log N)^k \max_{w\in\mathcal{C}} |W(w\log N)| \\ &\ll \frac{N^{1-\sigma}}{\tau^{4k}(\log N)^{3k}}\,. \end{aligned}$$

The mean square of this for $A \le t \le T - A$ is $\ll N^{2-2\sigma}A^{-8k+1} \ll N$ $= o(T)$. Thus

$$\int_A^{T-A} \Big|\frac{1}{2\pi}\int_{-\infty}^{\infty} \zeta(s+iv)^k K_N(iv)\,dv\Big|^2 dt \sim T\sum_{n=1}^{\infty} \frac{d_k(n)^2}{n^{2\sigma}}\,.$$

By the Cauchy–Schwarz inequality,

$$\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}\zeta(s+iv)^k K_N(iv)\,dv\right|^2 \le \frac{1}{2\pi}\int_{-\infty}^{\infty} K_N(iv)\,dv \times \frac{1}{2\pi}\int_{-\infty}^{\infty}|\zeta(\sigma+i(t+v))|^{2k}K_N(iv)\,dv\,.$$

Here the first integral on the right hand side is equal to 1. We integrate the above with respect to t to see that

$$T\sum_{n=1}^{\infty}\frac{d_k(n)^2}{n^{2\sigma}} \lesssim \frac{1}{2\pi}\int_A^{T-A}\int_{-\infty}^{\infty}|\zeta(\sigma+i(t+v))|^{2k}K_N(iv)\,dv\,dt\,.$$

With the change of variable $y = t+v$ we see that the right hand side above is

$$\begin{aligned}&= \int_{-\infty}^{\infty}|\zeta(\sigma+iy)|^{2k}\left(\frac{1}{2\pi}\int_{y-T+A}^{y-A}K_N(iv)\,dv\right)dy\\ &= \int_{-\infty}^{0}dy + \int_0^T dy + \int_T^{\infty}dy = I_1+I_2+I_3,\end{aligned}$$

say. Since $K_N(iv) \ge 0$ for all v, it follows from (26.38) that

$$\frac{1}{2\pi}\int_{y-T+A}^{y-A}K_N(iv)\,dv \le \frac{1}{2\pi}\int_{-\infty}^{\infty}K_N(iv)\,dv = \frac{1}{2\pi}\int_{-\infty}^{\infty}K(iv)\,dv = 1$$

for all y. Thus

$$I_2 \le \int_0^T|\zeta(\sigma+it)|^{2k}\,dt\,.$$

From Corollary 1.17 we know that $\zeta(\sigma+it) \ll \tau^{1/2}$. Hence

$$\begin{aligned}I_1 &\ll \int_{-\infty}^{0}(|y|+4)^k\int_{-\infty}^{y-A}|v|^{-4k}\,dv\,dy(\log N)^{-4k+1}\\ &\ll \int_{-\infty}^{0}(|y|+4)^k(|y|+A)^{-4k+1}\,dy \ll A^{-3k+2} \ll A^{-1} = o(T)\,.\end{aligned}$$

Similarly,

$$\begin{aligned}I_3 &\ll \int_T^{\infty}y^k\int_{y-T+A}^{\infty}v^{-4k}\,dv\,dy(\log N)^{-4k+1}\\ &\ll \int_T^{\infty}y^k(y-T+A)^{1-4k}\,dy \ll T^kA^{2-4k}\,.\end{aligned}$$

We take $A = T^{1/2}$, with the result that the above is $\ll T^{1-k} \ll 1 = o(T)$. Thus we have the result. □

For many purposes the upper bound of Theorem 26.23 for the fourth moment of the zeta function suffices. By taking more care, we now show that on the 1/2-line we can derive a more precise estimate.

T:zeta4mean2 **Theorem 26.28** *For* $T \ge 2$,

$$\int_0^T |\zeta(1/2+it)|^4\,dt = \Big(\frac{1}{2\pi^2} + O\big((\log x)^{-1/2}\big)\Big)T(\log T)^4.$$

Proof We first derive a useful formula for $\zeta(1/2+it)^2$. Put

$$M_1(t) = \sum_{n\le x} d(n)n^{-1/2-it} \qquad (26.44)$$ E:DefM1

where x is a parameter whose value will be chosen later. Weighted partial sums are more easily manipulated than unweighted ones, so we set

$$S_1(t) = \sum_{n\le x} d(n)n^{-1/2-it}\Big(1-\frac{n}{x}\Big)^2$$

with the result that $M_1(t) = S_1(t) + R_1(t)$ where

$$R_1(t) = \sum_{n\le x} d(n)n^{-1/2-it}\Big(1-\Big(1-\frac{n}{x}\Big)^2\Big). \qquad (26.45)$$ E:DefR1

Let $K(x,w) = 2x^w/\big(w(w+1)(w+2)\big)$ By the formulæ (5.17), (5.19) concerning the Mellin and inverse Mellin transforms relating to Cesàro partial sums, we see that

$$S_1(t) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \zeta\big(1/2+it+w\big)^2 K(x,w)\,dw\,.$$

From Corollary 10.5 we know that if α is fixed, $\alpha < 0$, then $\zeta(s) \ll \tau^{1/2-\alpha}$ uniformly for $\sigma \ge \alpha$, $|t| \ge 1$. Let ϕ be fixed, $-1 < \phi < -1/2$, and write $w = u+iv$. If $u \ge \phi$, then

$$\frac{\zeta(1/2+it+u+iv)^2}{w(w+1)(w+2)} \ll \frac{(|t+v|+4)^{-2\phi}}{(|v|+4)^3}\,.$$

Thus in the integral above we may move the path of integration from the 1-line to the abscissa ϕ. In doing so, we pass poles at $w=0$ and at $w = 1/2-it$. Hence $S_1(t) = \zeta(1/2+it)^2 + S_2(t) + R_2(t)$ where

$$S_2(t) = \frac{1}{2\pi i}\int_{\phi-i\infty}^{\phi+i\infty} \zeta(1/2+it+w)^2 K(x,w)\,dw + \zeta(1/2+it)^2$$

and

$$R_2(t) = \operatorname{Res}\Big[\zeta(1/2+it+w)^2 K(x,w)\Big|_{w=1/2-it}\,. \qquad (26.46)$$ E:DefR2

We write the functional equation of the zeta function in the asymmetric form as $\zeta(s) = \Delta(s)\zeta(1-s)$ where $\Delta(s) = 2^s\pi^{s-1}\Gamma(1-s)\sin \pi s/2$ by Corollary 10.4. Hence

$$\zeta(1/2+\phi+it+iv)^2 = \Delta(1/2+\phi+it+iv)^2\zeta(1/2-\phi-it-iv)^2 \,.$$

We write

$$\zeta(1/2-\phi-it-iv)^2 = \sum_{n\le y} d(n)n^{-1/2+\phi+it+iv} + \sum_{n>y} d(n)n^{-1/2+\phi+i(t+v)} \,.$$

Thus $S_2(t) = S_3(t) + R_3(t)$ where

$$S_3(t) = \frac{1}{2\pi i}\int_{\phi-i\infty}^{\phi+i\infty} \Delta(1/2+it+w)^2\Big(\sum_{n\le y} d(n)n^{-1/2+it+w}\Big)K(x,w)\,dw \tag{26.47}$$ E:DefS3

and

$$R_3(t) = \frac{1}{2\pi i}\int_{\phi-i\infty}^{\phi+i\infty} \Delta(1/2+it+w)^2\Big(\sum_{n>y} d(n)n^{-1/2+it+w}\Big)K(x,w)\,dw \,. \tag{26.48}$$ E:DefR3

Let θ be fixed with $0<\theta<1/2$. We move the contour in (26.47) from the abscissa ϕ to the abscissa θ. In doing so we pass a pole at $w=0$. Thus $S_3(t) = R_4(t) - M_2(t)$ where

$$R_4(t) = \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty} \Delta(1/2+it+w)^2\Big(\sum_{n\le y} d(n)n^{-1/2+it+w}\Big)K(x,w)\,dw \tag{26.49}$$ E:DefR4

and

$$M_2(t) = \Delta(1/2+it)^2\sum_{n\le y} d(n)n^{-1/2+it} \,. \tag{26.50}$$ E:DefM2

Thus we have shown that

$$\zeta(1/2+it)^2 = M_1(t) + M_2(t) - R_1(t) - R_2(t) - R_3(t) - R_4(t) \,.$$

It suffices to show that

$$\int_T^{2T} |\zeta(1/2+it)|^4\,dt = \Big(\frac{1}{2\pi^2} + O\big((\log T)^{-1/2}\big)\Big)T(\log T)^4, \tag{26.51}$$ E:zeta4meanEst2

so we consider the mean square size of the M_i and R_j over the interval $[T,2T]$.

Since $1-(1-n/x)^2 \ll n/x$ for $1\le n\le x$, it follows by Corollary 26.2

that

$$\int_T^{2T} |R_1(t)|^2 \ll \sum_{n\le x} d(n)^2 n x^{-2}(T+x)\Big(\frac{T}{x}+1\Big)\sum_{n\le x} d(n)^2 \ll (T+x)(\log x)^3 \tag{26.52}$$ E:R1Est

since $\sum_{n\le x} d(n)^2 \ll x(\log x)^3$ by (2.31).

Let $\mathcal{C}$ be a circle centered at $1/2 - it$ and radius $1/\log x$. Then

$$R_2(t) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{2\zeta(1/2+it+w)^2 x^w}{w(w+1)(w+2)}\,dw\,.$$

For $w \in \mathcal{C}$, we see that $|\zeta(1/2+it+w)| \asymp (\log x)^2$, $|x^w| \asymp x^{1/2}|$, and $|w(w+1)(w+2)| \asymp \tau^3$. Hence $R_2(t) \ll x^{1/2}\tau^{-3}(\log x)^3$, and so

$$\int_T^{2T} |R_2(t)|^2\,dt \ll \frac{x(\log x)^6}{T^5}\,. \tag{26.53}$$ E:R2Est

Let $W_\phi(x;t,v) = |\Delta(1/2+\phi+i(t+v))^2 K(x,\phi+iv)|$. By the Cauchy–Schwarz inequality,

$$\begin{aligned}|R_3(t)|^2 \le &\int_{-\infty}^{\infty} W_\phi(x;t,v)\,dv\\ &\times\int_{-\infty}^{\infty}\Big|\sum_{n>y} d(n)n^{-1/2+\phi+i(t+v)}\Big|^2 W_\phi(x;t,v)\,dv\,.\end{aligned}$$

From Corollary 10.5 we deduce that $\Delta(1/2+\phi+i(t+v))^2 \ll (|t+v|+4)^{-2\phi}$. It is also clear that $K(x;\phi+iv) \ll x^\phi(|v|+4)^{-3}$. Suppose that $T \le t \le 2T$. Then the first integral above is $\ll x^\phi T^{-2\phi}$. By Corollary 26.4 we see that

$$\begin{aligned}\int_A^{A+T}\Big|\sum_{n>y} d(n)n^{-1/2+\phi+it}\Big|^2\,dt &= \sum_{n>y} d(n)^2 n^{-1+2\phi}\big(T+O(n)\big)\\ &\ll Ty^{2\phi}(\log y)^3 + y^{1+2\phi}(\log y)^3.\end{aligned}$$

uniformly for any real number A. Hence

$$\begin{aligned}&\int_T^{2T} |R_3(t)|^2\,dt\\ &\quad\ll x^\phi T^{-2\phi}\int_{-\infty}^{\infty}\int_T^{2T}\Big|\sum_{n>y} d(n)n^{-1/2+\phi+i(t+v)}\Big|^2\,dt\,W_\phi(x;t,v)\,dv\\ &\quad\ll (T+y)\Big(\frac{T^2}{xy}\Big)^{-2\phi}(\log y)^3\,.\end{aligned} \tag{26.54}$$ E:R3Est

Similarly, by the Cauchy–Schwarz inequality,

$$|R_4(t)|^2 \le \int_{-\infty}^{\infty} W_\theta(x;t,v)\,dv \times \int_{-\infty}^{\infty} \Big| \sum_{n\le y} d(n) n^{-1/2+\theta+i(t+v)} \Big|^2 W_\theta(x;t,v)\,dv\,.$$

The estimates for $\Delta(1/2+\phi+i(t+v))$ and for $K(x;\phi+iv)$ derived above are still valid when ϕ is replaced by θ, so $W_\theta(x;t,v) \ll x^\theta(|t+v|+4)^{-2\theta}(|v|+4)^{-3}$, and hence the first integaral above is $\ll x^\theta T^{-2\theta}$ uniformly for $T \le t \le 2T$. By Corollary 26.2 we see that

$$\int_A^{A+T} \Big| \sum_{n\le y} d(n) n^{-1/2+\theta+it} \Big|^2 = (T+O(y)) \sum_{n\le y} d(n)^2 n^{-1+2\theta} \ll (T+y) y^{2\theta} (\log y)^3$$

uniformly in A. Hence

$$\begin{aligned}\int_T^{2T} &|R_4(t)|^2\,dt \\ &\ll x^\theta T^{-2\theta} \int_{-\infty}^{\infty} \int_T^{2T} \Big| \sum_{n\le y} d(n) n^{-1/2+\theta+i(t+v)} \Big|^2 dt\, W_\theta(x;t,v)\,dv \\ &\ll (T+y)\Big(\frac{xy}{T^2}\Big)^{2\theta} (\log y)^3\,. \end{aligned} \tag{26.55}$$ E:R4Est

To treat the main terms we first observe that

$$|M_1(t)+M_2(t)|^2 = |M_1(t)|^2 = 2\,\mathrm{Re}\, M_1(t)\overline{M_2(t)} + |M_2(t)|^2\,.$$

By Corollary 26.4 we see that

$$\int_T^{2T} |M_1(t)|^2\,dt = \sum_{n\le x} \frac{d(n)^2}{n}(T+O(n)) = T \sum_{n\le x} \frac{d(n)^2}{n} \ + \ O\big(x(\log x)^3\big).$$

We now show that

$$\sum_{n\le x} \frac{d(n)^2}{n} = \frac{1}{4\pi^2}(\log x)^4 + O\big((\log x)^3\big)\,. \tag{26.56}$$ E:MainSumEst

To see why this is so, we first observe that by computing Euler products it is clear that

$$\sum_{n=1}^{\infty} d(n)^2 n^{-s} = \frac{\zeta(s)^4}{\zeta(2s)}$$

for $\sigma > 1$. Hence by Perron's formula (Theorem 5.1) we see that

$$\sum_{n \le x} \frac{d(n)^2}{n} = \frac{1}{2\pi i} \int_{\varepsilon - i\infty}^{\varepsilon + i\infty} \frac{\zeta(s+1)^4 x^s}{\zeta(2s+2)s} \, ds \, .$$

From the trivial estimate of Corollary 1.17 we know that $\zeta(s) \ll \tau^{1/4}$ for $\sigma \ge 3/4$, $|t| \ge 1$. Hence by moving the contour to the rectilinear path with vertices $\varepsilon - i\infty$, $\varepsilon - iT$, $3/4 - iT$, $3/4 + it$, $\varepsilon + iT$, $\varepsilon + i\infty$ with $T = x^{1/8}$ we see that

$$\sum_{n \le x} \frac{d(n)^2}{n} = \operatorname{Res} \left[\frac{\zeta(s+1)^4 x^s}{2\pi i \zeta(2s+2)s} \right|_{s=0} + O\big(x^{-1/8+2\varepsilon}\big) \, .$$

This gives (26.56), and so we see that

$$\int_T^{2T} |M_1(t)|^2 \, dt = \frac{1}{4\pi^2} T (\log x)^4 + O\big((T + x)(\log x)^3\big) \, . \tag{26.57}$$

E:M1Est

Since $|\Delta(1/2 + it)| = 1$, it follows similarly that

$$\int_T^{2T} |M_2(t)|^2 \, dt = \frac{1}{4\pi^2} T (\log y)^4 + O\big((T + y)(\log y)^3\big) \, . \tag{26.58}$$

E:M2Est

In (26.54) we see that we want $xy \ll T^2$, while in (26.55) we see that we want $xy \gg T^2$. Also, in our estimates we see that it would be useful to have $x \ll T$ and $y \ll T$. Hence we now take $x = y = cT$ where c is a positive constant. To complete the final estimate, below, it is convenient to take c to be rather small, say $c = 1/8$. With our parameters chosen in this way, we now show that

$$\int_T^{2T} M_1(t) \overline{M_2(t)} \, dt \ll T (\log T)^2 \, . \tag{26.59}$$

E:M1M2Est

To this end we first note that $\overline{M_2(t)} = M_1(t) \overline{\Delta(1/2 + it)^2}$. As for the second factor, we recall that the functional equation for the zeta function in the symmetric form asserts that

$$\zeta(1/2 + it) \Gamma(1/4 + it/2) \pi^{-it/2} = \zeta(1/2 - it) \Gamma(1/4 - it/2) \pi^{it/2} \, .$$

We divide both sides by $|\Gamma(1/4 + it/2)| = |\Gamma(1/4 - it/2)|$, and let $Z(t)$ denote the result:

$$\begin{aligned} Z(t) &= \zeta(1/2 + it) \frac{\Gamma(1/4 + it/2) \pi^{-it/2}}{|\Gamma(1/4 + it/2)|} \qquad (26.60) \\ &= \zeta(1/2 - it) \frac{\Gamma(1/4 - it/2) \pi^{it/2}}{|\Gamma(1/4 - it/2)|} \, . \end{aligned}$$

E:DefZ

This is *Hardy's Z-function*, as we defined it in §14.2. Since the second formula above is the complex conjugate of the first, it is clear that $Z(t)$ is a real-valued function of the real variable t. Define $\theta(t)$ so that the cofactor of $\zeta(1/2+it)$ above is $e^{i\theta(t)}$. Thus $\theta(t)$ is a smooth function of t and $\zeta(1/2+it) = e^{-2i\theta(t)}\zeta(1/2-it)$. Hence $\Delta(1/2+it) = e^{-2i\theta(t)}$, and so $\overline{\Delta(1/2+it)^2} = e^{4i\theta(t)}$. From our discussion of Stirling's formula in Appendix C we see that

$$\log\Gamma(s) = s\log s - s - \frac{1}{2}\log s + \frac{1}{2}\log(2\pi) + O(1/|s|)$$

uniformly for $|s| \ge 1$ and $|\arg s| \le \pi - \delta$. With a little calculation, we deduce from this that

$$\theta(t) = \frac{t}{2}\log\frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + O(1/\tau)\,. \tag{26.61}$$

E:thetaEst

Let $F(t)$ denote the main term above. Then the left hand side of (26.59) is

$$\begin{aligned} &= \int_T^{2T} M_1(t)^2\big(e^{4iF(t)} + O(1/t)\big)\,dt \\ &= \int_T^{2T} M_1(t)^2 e^{4iF(t)}\,dt + O\Big(\Big(\sum_{n\le T/8} d(n)n^{-1/2}\Big)^2\Big)\,. \end{aligned} \tag{26.62}$$

E:M1M2Est2

Here the error term is $\ll T(\log T)^2$. The integral above is

$$\sum_{m\le T/8}\sum_{n\le T/8}\frac{d(m)d(n)}{\sqrt{mn}}\int_T^{2T} e^{i(4F(t)-t\log mn)}\,dt\,.$$

For a given pair m, n let $f(t) = 4F(t) - t\log mn$. Then

$$f'(t) = \log\frac{t^2}{4\pi^2 mn}\,.$$

This is an increasing function, and on the interval $[T, 2T]$ it is bounded below by

$$f'(T) = \log\frac{T^2}{4\pi^2 mn} \ge \log\frac{16}{\pi^2} \gg 1\,.$$

By Corollary **??** it follows that the integrals above are uniformly bounded, and hence that the integral in (26.62) is

$$\ll \Big(\sum_{n\le T/8}\frac{d(n)}{n^{1/2}}\Big)^2 \ll T(\log T)^2\,.$$

This completes the proof of (26.59). To obtain the Theorem, it remains only to combine this with the estimates (26.52), (26.53), (26.54), (26.55), (29.14), and (29.15). □

We now establish q-analogues of Theorem 26.23.

T:LqT4mean

Theorem 26.29 *Let χ denote a character modulo q, $q \ge 2$, and $\chi^\star$ the primitive character that induces χ. If $T \ge 2$ and $|\sigma - 1/2| \le 1/(4\log qT)$, then*

$$\sum_\chi \int_0^T |L(\sigma + it, \chi^\star)|^4 \, dt \ll qT(\log qT)^4 \, .$$

Proof We apply Theorem 25.4 with $\delta = 1/4$. Let

$$W(u, r) = \int_0^\infty \exp\big(-u^2/r\big)(u^2/r)^{1/8} u^{-1} \, .$$

From Theorem 25.4 *via* Hölder's inequality, we see that

$$\begin{aligned} |L(s,\chi^\star)|^4 \ll & \int_0^\infty \big(|A(s,\chi^\star;u)|^4 + |A(1-s,\overline{\chi^\star};u)|^4\big) W(u,q\tau)\,du \\ & \times \Big(\int_0^\infty W(u,q\tau)\,du\Big)^3 . \end{aligned}$$

We have already remarked that $\int_0^\infty W(u,r)\,du \ll 1$ uniformly in r. We note further that if $0 < r_1 \le r_2 \le 4r_1$, then $W(r_1,u) \ll W(r_2,u)$ uniformly for $u > 0$. Since $\tau = |t| + 4$, it follows that if $2 \le T \le t \le 2T$, then $\tau \le 4T$, so the above is

$$\ll \int_0^\infty \big(|A(s,\chi^\star;u)|^4 + |A(1-s,\overline{\chi^\star};u)|^4\big) W(u,4qT)\,du \, .$$

Hence

$$\begin{aligned} & \sum_\chi \int_T^{2T} |L(\sigma+it,\chi^\star)|^4 \, dt \\ & \quad \ll \int_0^\infty \sum_\chi \int_T^{2T} \big(|A(s,\chi^\star)|^4 + |A(1-s,\overline{\chi^\star})|^4\big)\, dt \, W(u,4qT)\,du \, . \end{aligned} \tag{26.63}$$

E:L4momEst1

We note that

$$A(s,\chi^\star;u)^2 = \sum_{k \le u^2} \frac{\chi^\star(k)}{k^s} \sum_{\substack{m,n\le u \\ mn=k}} 1 = \sum_{k\le u^2} \frac{\chi^\star(k) d(k,u)}{k^s},$$

say. Thus by Theorem 26.16 we see that the expression (26.63) is

$$\ll \int_0^\infty \sum_{k \le u^2} d(k,u)^2 \Big(\frac{1}{k^{2\sigma}} + \frac{1}{k^{2-2\sigma}}\Big)(qT+u^2)W(u,4qT)\,du\,.$$

Put $\kappa = 1/\log qT$. Since $0 \le d(k,u) \le d(k)$ for all k, and $\sum_{k\le y} d(k)^2/k \ll (\log 2y)^4$ for $y \ge 1$, the above is

$$\ll \int_1^\infty (\log 2u)^4 u^{4\kappa}(qT+u^2)W(u,4qT)\,du\,.$$

By means of the change of variable $v = u^2/r$ we see that if $r \ge 1$ and $\alpha \ge 0$, then

$$\begin{aligned}
&\int_0^\infty (\log 2u)^4 u^\alpha W(u,r)\,du \ll \int_0^\infty (\log 2rv)^4 (rv)^{\alpha/2} v^{1/8} e^{-v}\,\frac{dv}{v} \\
&\ll (\log 2r)^4 r^{\alpha/2}\int_0^\infty v^{\alpha/2+1/8}e^{-v}\,\frac{dv}{v} + r^{\alpha/2}\int_0^\infty (\log v)^4 v^{\alpha/2+1/8}e^{-v}\,\frac{dv}{v} \\
&= (\log 2r)^4 r^{\alpha/2}\Gamma(\alpha/2+1/8) + r^{\alpha/2}\Gamma^{(4)}(\alpha/2+1/8) \ll r^{\alpha/2}(\log 2r)^4
\end{aligned}$$

for bounded values of α. We apply this with $\alpha = 4\kappa$ and with $\alpha = 2+4\kappa$, and conclude that

$$\sum_\chi \int_T^{2T} |L(\sigma+it,\chi^\star)|^4\,dt \ll qT(\log 2qT)^4$$

for $T \ge 2$. We take $T = 2, 4, 8, \ldots$ and sum to deduce that

$$\int_2^T \sum_\chi L(\sigma+it,\chi^\star)|^4\,dt \ll qT(\log 2qT)^4\,.$$

It remains to treat the interval $0 \le t \le 2$. Our method applies to any interval in which τ changes by at most a bounded factor. When $t = 0$ we have $\tau = 4$, and when $t = 2$ we have $\tau = 6$. Thus our method, when applied to the interval $[0,2]$ yields the estimate

$$\int_0^2 \sum_\chi |L(\sigma+it,\chi^\star)|^4\,dt \ll q(\log 2q)^4\,.$$

This completes the proof. □

Cor:2mqTL*L'

Corollary 26.30 *Let χ denote a character modulo q, $q \ge 2$, and $\chi^\star$ the primitive character that induces χ. If $T \ge 2$, then*

$$\int_0^T \sum_\chi |L(1/2+it,\chi^\star)L'(1/2+it,\chi^\star)|^2 \ll qT(\log qT)^6\,.$$

Proof This follows from Theorem 26.29 by the same method that we used to derive Corollary 26.25 from Theorem 26.23. □

Cor:discqT4mv **Corollary 26.31** *Suppose that $\delta > 0$, that $T \ge 2$, and that for each character χ modulo q we have numbers $t_{j,\chi}$, $j = 1, 2, \ldots, J_\chi$ in the interval $[\delta/2, T - \delta/2]$ such that $|t_{j,\chi} - t_{k,\chi}| \ge \delta$ when $j \ne k$. Then*

$$\sum_{\chi} \sum_{j=1}^{J_\chi} |L(1/2 + it_{j,\chi}, \chi^\star)|^4 \ll (1/\delta + \log qT) qT (\log qT)^4 .$$

Proof This follows from Theorem 26.29 and Corollary 26.31 by the same method that we used to derive Corollary 26.26 from Theorem 26.23 and Corollary 26.25. □

T:LQ2T4mean **Theorem 26.32** *Suppose that $Q \ge 2$, $T \ge 2$, and that $|\sigma - 1/2| \le 1/(4 \log QT)$. Then*

$$\sum_{q \le Q} \sideset{}{^\star}\sum_{\chi} \int_0^T |L(\sigma + it, \chi)|^4 \ll Q^2 T (\log QT)^4 .$$

Proof We show first that

$$\sum_{Q/2 < q \le Q} \sideset{}{^\star}\sum_{\chi} \int_{T/2}^T |L(\sigma + it, \chi)|^4 \ll Q^2 T (\log QT)^4,$$

and then sum over diadic blocks. The above is proved by the same method used to prove Theorem 26.29, but with an appeal to (26.22) instead of (26.21). □

Cor:2mQ2TL*L' **Corollary 26.33** *Suppose that $Q \ge 2$ and that $T \ge 2$. Then*

$$\sum_{q \le Q} \sideset{}{^\star}\sum_{\chi} \int_0^T |L(1/2 + it, \chi) L'(1/2 + it, \chi)|^2 \ll Q^2 T (\log QT)^6 .$$

discQ2T4mv **Corollary 26.34** *Suppose that $\delta > 0$, that $Q \ge 2$, $T \ge 2$, and that for each primitive character χ modulo q with $q \le Q$ we have numbers $t_{j,\chi}$, $j = 1, 2, \ldots, J_\chi$ in the interval $[\delta/2, T - \delta/2]$ such that $|t_{j,\chi} - t_{k,\chi}| \ge \delta$ when $j \ne k$. Then*

$$\sum_{q \le Q} \sideset{}{^\star}\sum_{\chi} \sum_{j=1}^{J_\chi} |L(1/2 + it_{j,\chi}, \chi)|^4 \ll (1/\delta + \log QT) Q^2 T (\log QT)^4 .$$

26.5.1 Exercises

1. By arguing in the same style that we used to derive (26.29), show that if $\alpha > 1$, $\beta > 1$, and μ and ν are nonnegative integers, then

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \zeta^{(\mu)}(\alpha+it)\zeta^{(\nu)}(\beta-it) = \zeta^{(\mu+\nu)}(\alpha+\beta)\,.$$

Exer:Lptriineq

2. Let p be a real number, $1 \le p < \infty$, and let

$$\|f\|_p = \Big(\int_a^b |f(x)|^p\,dx\Big)^{1/p}$$

denote the L^p norm of a function f on the interval $[a,b]$. We recall that such norms have a triangle inequality: $\|f+g\|_p \le \|f\|_p + \|g\|_p$.

(a) Suppose that $f = f_1 + f_2$, and that $\|f_2\|_p < \|f_1\|_p$. Show that

$$\|f_1\|_p\|\Big(1-\frac{\|f_2\|_p}{\|f_1\|_p}\Big) \le \|f\|_p \le \|f_1\|_p\Big(1+\frac{\|f_2\|_p}{\|f_1\|_p}\Big)\,.$$

(b) Deduce in particular that

$$\int_a^b |f(x)|^2\,dx = \int_a^b |f_1(x)|^2\,dx\Big(1+O\Big(\frac{\|f_2\|_2}{\|f_1\|_2}\Big)\Big).$$

(c) In the context of Theorem 26.19, by choosing the parameter x appropriately, show that if $1/2 \le \alpha < \sigma \le 1$, then

$$\int_0^T |\zeta(\sigma+it)|^{2k} = T\Big(\sum_{n=1}^\infty \frac{d_k(n)^2}{n^{2\sigma}}\Big)\Big(1+O\big(T^{-\frac{\sigma-\alpha}{2-\alpha-\sigma}+\varepsilon}\big)\Big).$$

3. Suppose that $-1/\log T \le \alpha \le 1/\log T$, that $-1/\log T \le \beta \le 1/\log T$, and that $-1 \le \delta \le 1$.

(a) Show that

$$\int_0^T \zeta(1/2+\alpha+it)\overline{\zeta(1/2+\beta+i\delta+it)}\,dt = T\int_1^T u^{-1-\alpha-\beta+i\delta}\,du + O(T)\,.$$

(b) Show that

$$\int_0^T \zeta'(1/2+\alpha+it)\overline{\zeta(1/2+\beta+i\delta+it)}\,dt = -T\int_1^T u^{-1-\alpha-\beta+i\delta}\log u\,du + O(T\log T)\,.$$

4. (a) By mimicking the proof of Theorem 26.23, show that if $k \ge 2$ is an integer and $1/2 \le \sigma \le 1$, then

$$\int_0^T |\zeta(\sigma + it)|^{2k}\, dt \ll T + T^{k(1-\sigma)+\varepsilon}.$$

(b) Deduce that the relation (26.29) holds for $\sigma > 1 - 1/k$.

(c) Show that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |\zeta(1 + it)|^{2k}\, dt = \sum_{n=1}^{\infty} \frac{d_k(n)^2}{n^2}$$

for all positive integers k.

Exer:pNormIneq

5. Let p be a real number, $1 \le p < \infty$.

(a) Suppose that $w \in L^1(\mathbb{R})$, and that $f \in L^p(\mathbb{R})$. Put $F(x) = (w * f)(x) = \int_{-\infty}^{\infty} w(u)f(x - u)\, du$. Show that $\|F\|_p \le \|w\|_1 \|f\|_p$.

(b) Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ denote the circle group. Suppose that $w \in L^1(\mathbb{T})$, and that $f \in L^p(\mathbb{T})$. Put $F(x) = (w * f)(x) = \int_0^1 w(u)f(x-u)\, du$. Show that $\|F\|_p \le \|w\|_1 \|f\|_p$.

6. (Balasubramanian & Ramachandra 1990) Show that if k is a positive integer, then

$$\int_0^T |\zeta(1/2 + it)|^{2k}\, dt \gg_k T(\log T)^{k^2}$$

for $T \ge 2$.

7. Suppose that k is a positive integer, that $1/2 < \sigma < 1$, and that $\delta > 0$. Show that if $H \ge T^\delta$, then

$$\int_T^{T+H} |\zeta(\sigma + it)|^{2k}\, dt \ge (1 + o(1))H \sum_{n=1}^{\infty} \frac{d_k(n)^2}{n^{2\sigma}}\,.$$

8. Show that

$$\int_0^T |\zeta(1/2 + it)\zeta'(1/2 + it)|^2\, dt \gg T(\log T)^6$$

for $T \ge 2$.

27
Large Values of Dirichlet Polynomials

C:MeanLargeVals

Let $m(V) = \text{meas}\{x \in [a,b] : |f(x)| \ge V\}$ where f is a measurable function defined on $[a,b]$ and $V > 0$. Then

$$V^p m(V) = \int_{\substack{a \le x \le b \\ |f(x)| \ge V}} V^p \, dx \le \int_a^b |f(x)|^p \, dx = \|f\|_p^p \, .$$

Hence

$$m(V) \le \frac{\|f\|_p^p}{V^p} \, .$$

For a general function this is all that we can say about the measure of the set on which it is large, based on its L^p norm. If we have bounds for $\|f\|_p$ and for $\|f\|_q$, then

$$m(V) \le \min\Big(\frac{\|f\|_p^p}{V^p}, \frac{\|f\|_q^q}{V^q}\Big),$$

and there is still not much more that we can say, even if f is analytic. However, if f is a Dirichlet polynomial, then there is much more that we can say about its large values beyond what follows from mean value estimates.

Suppose that

$$D(s) = \sum_{n=1}^{N} a_n n^{-s}, \tag{27.1}$$ E:DefD(s)

that $t_r \in [0,T]$ for $r = 1, 2, \ldots, R$, that $\left|t_{r_1} - t_{r_2}\right| \ge 1$ for $r_1 \ne r_2$. Let Δ have the property that

$$\sum_{r=1}^{R} \Big| \sum_{n=1}^{N} a_n n^{-it_r} \Big|^2 \le \Delta^2 \sum_{n=1}^{N} |a_n|^2 \tag{27.2}$$ E:HM-1

for all choices of the a_n. This is a bilinear form inequality, so by the duality theorem (Theorem **??**), Δ has the equivalent property that

$$\sum_{n=1}^{N}\Big|\sum_{r=1}^{R} y_r n^{-it_r}\Big|^2 \le \Delta^2 \sum_{r=1}^{R}|y_r|^2 \qquad (27.3) \quad \boxed{\texttt{E:HM0}}$$

for all choices of the y_r. We expand the square on the left hand side above and take the sum over n inside, to see that it is

$$= \sum_{1\le r_1,r_2\le R} y_{r_1}\overline{y_{r_2}} \sum_{n=1}^{N} n^{i(t_{r_2}-t_{r_1})}\,. \qquad (27.4) \quad \boxed{\texttt{E:HMEst0}}$$

Now $\big|y_{r_1}\overline{y_{r_2}}\big| \le \frac{1}{2}\big|y_{r_1}\big|^2 + \frac{1}{2}\big|y_{r_2}\big|^2$ by the geometric–arithmetic mean inequality, so the above is

$$\le \sum_{r_1=1}^{R}|y_{r_1}|^2 \sum_{r_2=1}^{R}\Big|\sum_{n=1}^{N} n^{i(t_{r_2}-t_{r_1})}\Big|. \qquad (27.5) \quad \boxed{\texttt{E:HMEst1}}$$

Thus

$$\Delta^2 \le \max_{1\le r_1\le R}\sum_{r_2=1}^{R}\Big|\sum_{n=1}^{N} n^{i(t_{r_2}-t_{r_1})}\Big|.$$

Our argument here is reminiscent of one of our approaches to the large sieve, where we found that it was fruitful to introduce weights. We can introduce weights here, also. Suppose that $w_n \ge 0$ for all n, that $w_n \ge 1$ for $1 \le n \le N$, and that $\sum_{n=1}^{\infty} w_n < \infty$. Then the left hand side of (27.3) is

$$\le \sum_{n=1}^{\infty} w_n\Big|\sum_{r=1}^{R} y_r n^{-it_r}\Big|^2.$$

On continuing as above, we find that

$$\Delta^2 \le \max_{1\le r_1\le R}\sum_{r_2=1}^{R}\big|W\big(i\big(t_{r_1}-t_{r_2}\big)\big)\big| \qquad (27.6) \quad \boxed{\texttt{E:HM1}}$$

where $W(s) = \sum_{n=1}^{\infty} w_n n^{-s}$ for $\sigma \ge 0$. Suppose that we take $w_n = 2 - n/N$ for $1 \le n \le 2N$, and $w_n = 0$ for $n > 2N$. Then by the formula (5.19) for the inverse Mellin transform with Cesàro weights we see that

$$W(it) = \frac{1}{\pi i}\int_{2-i\infty}^{2+i\infty}\zeta(z+it)\frac{(2N)^z}{z(z+1)}\,dz\,.$$

On moving the path of integration to the abscissa $\operatorname{Re} z = 1/2$, we see that the above is

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \zeta(1/2 + i(t+y))\frac{(2N)^{1/2+iy}}{(1/2+iy)(3/2+iy)}\,dy + \frac{2(2N)^{1-it}}{(1-it)(2-it)}\,.$$

If the Lindelöf Hypothesis (LH) is true, then the above is

$$\ll C(\varepsilon)N^{1/2}\tau^{\varepsilon} + \frac{N}{\tau^2}\,.$$

Hence

$$\sum_{r_2=1}^{R} \big|W\big(i\big(t_{r_1} - t_{r_2}\big)\big)\big| \ll N + C(\varepsilon)N^{1/2}T^{\varepsilon}R,$$

for all r_1, so

$$\sum_{r=1}^{R} |D(it_r)|^2 \ll \big(N + C_1(\varepsilon)N^{1/2}T^{\varepsilon}R\big)\sum_{n=1}^{N} |a_n|^2\,. \tag{27.7}$$ E:HalaszLH

Suppose that V is a number such that $|D(it_r)| \ge V$ for all r. Then

$$V^2R \ll N\sum_{n=1}^{N} |a_n|^2 \ + \ C_1(\varepsilon)N^{1/2}T^{\varepsilon}R\sum_{n=1}^{N} |a_n|^2\,. \tag{27.8}$$ E:V^2JEst

Hence there is a constant $C_2(\varepsilon)$ with the property that if

$$V^2 \ge C_2(\varepsilon)N^{1/2}T^{\varepsilon}\sum_{n=1}^{N} |a_n|^2, \tag{27.9}$$ E:Vlb

then the second term on the right hand side of (27.8) does not majorize the left hand side, and hence $V^2R \ll N\sum_{n=1}^{N}|a_n|^2$, which is to say that we have proved

T:HalaszLH **Theorem 27.1** *Assume the Lindelöf Hypothesis. Let $D(s)$ be a Dirichlet polynomial as in* (27.1)*, and $\mathcal{T}$ be a set of R real numbers in the interval $[0,T]$ such that $|t-t'| \ge 1$ whenever $t,t' \in \mathcal{T}$ and $t \ne t'$. There is a constant $C_2(\varepsilon)$ such that if V satisfies* (27.9)*, then*

$$R \ll \frac{NG}{V^2} \tag{27.10}$$ E:HEst

where

$$G = \sum_{n=1}^{N} |a_n|^2\,. \tag{27.11}$$ E:DefG

Suppose that $|a_n| \asymp 1$ for all n. Then $\sup|D(it)| \ll N$, the asymptotic root-mean-value of $|D(it)|$ is $\asymp N^{1/2}$, and from the above we see that if V is a little larger than $T^\varepsilon N^{3/4}$, then

$$R \ll \frac{N^2}{V^2}\,. \tag{27.12}$$ E:HMEst

From Theorem 26.9 we also know that

$$R \ll \frac{(T+N)N\log N}{V^2}\,.$$

If N is small compared with T, say of the size of a small fractional power of T, then the first upper bound above is enormously better than the second. Of course the first depends on LH and applies only to large values of V, while the second is unconditional and is valid for all $V > 0$.

In passing from (27.4) to (27.5) we used the triangle inequality, and we would expect that in doing so a great deal of cancellation has been lost. This would suggest that something substantially stronger than (27.7) should hold. But this is false! The surprise here is that (27.7) is within a factor T^ε of being best possible. We see this from the following

Exam:Bourgain **Example 27.1** (Bourgain) Suppose that $\delta > 0$ is a small positive absolute constant, that $H \le \delta N^{1/2}$, that

$$D(s) = \sum_{N-H<n\le N} n^{-s},$$

and that $\mathcal{T} = \mathcal{T}_\delta\{a + 2\pi Nb : a,b\in\mathbb{Z}, -A \le A, -B \le b \le B\}$ where $A = \delta N/H$, $B = \delta N/H^2$. If $N - H < n \le N$, then

$$n^{-it} = N^{-it}\exp\Big(-it\log\frac{n}{N}\Big) = N^{-it}\exp\Big(-it\log\Big(1-\frac{N-n}{N}\Big)\Big)\,.$$

For $0 \le u \le 1/2$ let r be defined by the equation $-\log(1-u) = u + r$. Then r is real and $r \ll u^2$, so that if $u = (N-n)/N$ we see that the above is

$$= N^{-it}e^{itu}e^{itr} = N^{-it}\exp\Big(it\frac{N-n}{N}\Big)\Big(1 + O\big(|t|(N-n)^2/N^2\big)\Big).$$

If $t = a + 2\pi Nb \in \mathcal{T}_\delta$, then the above is

$$\begin{aligned} &= N^{-it}\exp(ia(N-n)/N)\exp(2\pi ib(N-n))\big(1+O\big((A+NB)H^2/N^2\big)\big)\\ &= N^{-it}\big(1+O(AH/N)\big)\big(1+O\big((A+NB)H^2/N^2\big)\big)\,. \end{aligned}$$

In view of the definitions of A and of B, the above is

$$= N^{-it}\big(1+O(\delta)\big)\,.$$

We sum over n to see that if $t \in \mathcal{T}$, then $D(it) = HN^{-it}\big(1 + O(\delta)\big)$. If we apply (27.7) to the points $\mathcal{T}$, then $R = \operatorname{card}\mathcal{T}$, the left hand side is of the order RH^2, while the right hand side is comparable to $N^{1/2}T^\varepsilon RH$. If $H \sim \delta N^{1/2}$, then the right hand side is larger than the left by T^ε. To prove Theorem 27.1 we introduced weights, but the two sides of the resulting relation are essentially unchanged.

Before Bourgain invented his Example, it was conjectured that $\Delta^2 \ll N + RT^\varepsilon$, but this is false because it is much stronger than (27.7), which we now recognize is essentially best possible.

The Dirichlet polynomial in Bourgain's Example is just a short sum, and is therefore not typical of the Dirichlet polynomials that we most often encounter. We avoid Bourgain's Example by proposing a bound in terms of the maximum of the coefficients instead of their mean square size.

Conj:Halaszlinfty **Conjecture 27.2** *Let $D(s)$ be a Dirichlet polynomial as in* (27.1)*, and suppose that $|a_n| \le 1$ for all n. Let $\mathcal{T}$ be a set of R real numbers in the interval $[0,T]$ such that $|t - t'| \ge 1$ whenever $t, t' \in \mathcal{T}$ and $t \ne t'$. Then*

$$\sum_{t\in\mathcal{T}} |D(it)|^2 \ll_\varepsilon (N + R)N^{1+\varepsilon}.$$

If $|D(it)| \ge V$ for all $t \in \mathcal{T}$, and $V > N^{1/2+\varepsilon}$, then it follows from the above that

$$R \ll_\varepsilon \frac{N^{2+\varepsilon}}{V^2}\,. \tag{27.13}$$ E:HMConj

To obtain unconditional results using our new ideas, we could simply replace the appeal to LH by one to a known bound for $|\zeta(1/2+it)|$, such as the one found in Theorem **??**. One could also proceed along these lines using a different abscissa instead of $1/2$, as discussed in Exercise 2. For larger values of N, say $T^{1/2} \le N \le T$, a further approach works well: From Theorem **??** we see that

$$\sum_{n\le x} n^{-it} = \frac{x^{1-it}}{1-it} + O\big(\tau^{1/2}\log\tau\big)\,.$$

We integrate this over $0 \le x \le N$, and then divide by N to see that

$$\begin{aligned}\sum_{n=1}^{N}(1-n/N)n^{-it} &= \frac{N^{1-it}}{(1-it)(2-it)} + O\big(\tau^{1/2}\log\tau\big)\\ &\ll \tau^{1/2}\log\tau + \frac{N}{\tau^2}\,.\end{aligned} \tag{27.14}$$ E:whtedn-it

By replacing N by $2N$ we obtain the same weights that we already used, and we see that if the numbers t_r are well-spaced in the interval $[0, T]$, then

$$\sum_{r_2=1}^{R} \left|W\big(i(t_{r_1} - t_{r_2})\big)\right| \ll N + RT^{1/2}\log T\,.$$

In view of (27.6), this gives

T:HM2 **Theorem 27.3** *Let $D(s)$ be a Dirichlet polynomial as in* (27.1)*, let $\mathcal{T}$ be a set of R real numbers in the interval $[0, T]$ such that $|t - t'| \geq 1$ whenever $t, t' \in \mathcal{T}$ and $t \neq t'$, and let G be defined as in* (27.11)*. Then*

$$\sum_{t\in\mathcal{T}} |D(it|^2 \ll (N + RT^{1/2}\log T)G\,. \tag{27.15}$$ E:HM2

From the above we see that there is an absolute constant C such that if $|D(it)| \geq V$ for all $t \in \mathcal{T}$ and

$$V^2 \geq CT^{1/2}(\log T)G, \tag{27.16}$$ E:Vlb2

then we have (27.10). At first sight it would seem that for smaller V we have no bound, but by exercising a little care we obtain the following comprehensive result.

T:HM3 **Theorem 27.4** *Let $D(s)$ be a Dirichlet polynomial as in* (27.1)*, let $t_1, t_2, \ldots, t_R$ be real numbers in the interval $[0, T]$ such that $\left|t_{r_1} - t_{r_2}\right| \geq 1$ whenever $r_1 \neq r_2$, and let G be defined by* (27.11)*. If $|D(it_r)| \geq V > 0$ for all r, then*

$$R \ll \frac{NG}{V^2} + \frac{NTG^3(\log T)^2}{V^6}\,. \tag{27.17}$$ E:HM3

Proof If the condition (27.16) holds, then there is nothing to prove. Thus we may assume that (27.16) fails. Let T_1 be the number for which $V^2 = CT_1^{1/2}(\log T_1)G$ where C is the constant in (27.16). Thus $T_1 < T$. We divide the interval $[0, T]$ into $\lceil T/T_1 \rceil$ intervals, each one of length $T/\lceil T/T_1 \rceil \leq T_1$. Thus Theorem T:HM2 ? applies to each subinterval. Since $\lceil T/T_1 \rceil \asymp T/T_1$, it follows that the total number of poiints t, summed over all subintervals, is

$$R \ll \Big(1 + \frac{T}{T_1}\Big)\frac{NG}{V^2} = \frac{NG}{V^2} + \frac{TNG}{T_1V^2}\,.$$

Since $T_1(\log T_1)^2 \asymp V^4/G^2$, it follows that the second term on the right above is

$$\frac{TNG(\log T_1)^2}{T_1(\log T_1)^2V^2} \asymp \frac{TNG^3(\log T_1)^2}{V^6} \leq \frac{TNG^3(\log T)^2}{V^6}$$

since $T_1 \le T$. This gives the result. □

The bound $R \le T$ is trivial, and we expect that $|D(it)| \ll G^{1/2}$ a positive proportion of t. Thus we expect to have a nontrivial bound for R only when V^2 is somewhat larger than G. On combining the estimates derived from Theorems 26.8 and 27.4, we find the following:

$$R \ll \begin{cases} T & (V^2 \le G\log N), \\ \frac{TG\log N}{V^2} & (G\log N \le V^2 \le N^{1/2}G(\log T)^{1/2}), \\ \frac{TNG^3(\log T)^2}{V^6} & (N^{1/2}G(\log T)^{1/2} \le V^2 \le T^{1/2}\log T), \\ \frac{NG}{V^2} & (V^2 \ge T^{1/2}G\log T). \end{cases}$$

We now extend our discussion in two directions. First, we replace n^{-it} by $\chi(n)$, or by $\chi(n)n^{-it}$. Secondly, in applications it frequently happens that we want to sample not just at points it on the imaginary axis, but at points $s = \sigma + it$ with well-spaced t and $\sigma \ge 0$. These extensions are easily obtained from the following useful

L:weightedhybridPV **Lemma 27.5** *Suppose that $s = \sigma + it$ with $\sigma \ge 0$, and that χ is a character modulo q. Then*

$$\sum_{n=1}^{N}\Big(1-\frac{n}{N}\Big)\chi(n)n^{-s} \ll (q\tau)^{1/2}\log q\tau + E_0(\chi)\frac{N}{\tau^2} \qquad (27.18)$$ E:weightedhybridPV

where $E_0(\chi) = 1$ if $\chi = \chi_0$, and $E_0(\chi) = 0$ otherwise.

Proof We recall that Theorem **??** asserts that

$$\sum_{n\le u}\chi(n)n^{-it} = E_0(\chi)\frac{\varphi(q)}{q}\cdot\frac{u^{1-it}}{1-it} + O\big((q\tau)^{1/2}\log q\tau\big)\,.$$

We integrate both sides of this over the interval $0 \le u \le x$, and then divide both sides by x to see that

$$\begin{aligned}\sum_{n\le x}\Big(1-\frac{n}{x}\Big)\chi(n)n^{-it} &= E_0(\chi)\frac{\varphi(q)x^{1-it}}{q(1-it)(2-it)} + O\big((q\tau)^{1/2}\log q\tau\big) \\ &\ll E_0(\chi)\frac{x}{\tau^2} + (q\tau)^{1/2}\log q\tau\,. \qquad (27.19)\end{aligned}$$ E:qthybridEst

By restricting the real parameter x to the integer value N, we obtain the desired result when $\sigma = 0$. To extend the above to allow $\sigma > 0$, we express the weight $\max(0, 1-n/N)n^{-\sigma}$ as a nonnegative linear combination of the weights $\max(0, 1-n/x)$ with $0 \le x \le N$. Specifically, we show that if $a_1, a_2, \dots$ is a sequence of real or complex numbers and

$$A(x) = \sum_{1\le n\le x} a_n, \qquad B(x) = \sum_{1\le n\le x} (x-n)a_n,$$

then

$$\sum_{1\le n\le x} (x-n)a_n n^{-\sigma} = \frac{B(x)}{x^\sigma} + 2\sigma \int_0^x B(v)v^{-\sigma-1}\,dv + \sigma(\sigma+1)\int_0^x B(v)(x-v)v^{-\sigma-2}\,dv \tag{27.20}$$ E:lcwhts

for $\sigma \ge 0$. To this end we first note that

$$\sum_{1\le n\le x} (x-n)a_n n^{-\sigma} = \int_0^x \sum_{1\le n\le u} a_n n^{-\sigma}\,du\,. \tag{27.21}$$ E:IntA=B

By Riemann–Stieltjes integration by parts (as in the proof of Theorem 1.3) we see that the integrand above is

$$= \frac{A(u)}{u^\sigma} + \sigma \int_1^v A(v)v^{-\sigma-1}\,dv\,.$$

Hence the right hand side of (27.21) is

$$= \int_1^x A(u)u^{-\sigma}\,du + \sigma \int_1^x A(v)(x-v)v^{-\sigma-1}\,dv\,.$$

From (27.21) with $\sigma = 0$ we see that $B(x) = \int_0^x A(u)\,du$. We integrate the two integrals above by parts (integrating A and differentiating the rest) to obtain (27.20). We now take $a_n = \chi(n)n^{-it}$ in (27.20). From (27.19) we deduce that

$$\sum_{n\le x}(x-n)\chi(n)n^{-s} = W_1(x)(q\tau)^{1/2}\log q\tau \;+\; W_2(x)E_0(\chi)\tau^{-2}$$

where

$$W_j(x) = x^{j-\sigma} + 2\sigma\int_1^x v^{j-\sigma-1}\,dv + \sigma(\sigma+1)\int_1^x (x-v)v^{j-\sigma-2}\,dv\,.$$

We need to show that $W_j(x) \ll x^j$ for $j = 1, 2$. The bound (27.18) is trivial if $\sigma \ge 2$, so we may suppose that $0 \le \sigma \le 2$. For $j = 1$ we see that

$$W_1(x) \ll x + \int_1^x 1\,dv + \sigma\int_1^x (x-v)v^{-\sigma-1}\,dv\,.$$

It is necessary to estimate the last term above only when $\sigma > 0$, and for such σ we that it is

$$\le \sigma x \int_1^x v^{-\sigma-1}\,dv = x\bigl(1 - x^{-\sigma}\bigr) \ll x\,.$$

Finally,

$$W_2(x) \ll x^2 + \int_1^x v\,dv + x\int_1^x 1\,dv \ll x^2\,.$$

□

In the same way that we derived theorem 27.3 from (27.14), the following more general results are immediate.

T:HMqT **Theorem 27.6** *Suppose that for $r = 1, 2, \ldots R$ we have a character χ_r modulo q and a point $s_r = \sigma_r + it_r$ with $\sigma_r \ge 0$, $0 \le t_r \le T$, and with the further property that $\bigl|t_{r_1} - t_{r_2}\bigr| \ge 1$ if $\chi_{r_1} = \chi_{r_2}$. Let*

$$D(s,\chi) = \sum_{n=1}^{N} a_n \chi(n) n^{-s}, \tag{27.22}$$ E:DefD(s,chi)

and let G be defined as in (27.11)*. Then*

$$\sum_{r=1}^{R} |D(s_r,\chi_r)|^2 \ll \bigl(N + R(qT)^{1/2}\log qT\bigr)G\,.$$

If $t_r = 0$ for all r, then we find that

$$\sum_{r=1}^{R}\Bigl|\sum_{n=1}^{N} a_n\chi_r(n)\Bigr|^2 \ll \bigl(N + Rq^{1/2}\log q\bigr)G,$$

which is an exact q-analogue of (27.15).

T:HMQT **Theorem 27.7** *Suppose that for $r = 1, 2, \ldots R$ we have a primitive character χ_r modulo q_r with $1 \le q_r \le Q$ and a point $s_r = \sigma_r + it_r$ with $\sigma_r \ge 0$, $0 \le t_r \le T$, and with the further property that $\bigl|t_{r_1} - t_{r_2}\bigr| \ge 1$ if $\chi_{r_1} = \chi_{r_2}$. Let $D(s,\chi)$ be defined as in* (27.22)*, and let G be defined as in* (27.11)*. Then*

$$\sum_{r=1}^{R} |D(s_r,\chi_r)|^2 \ll \bigl(N + RQT^{1/2}\log QT\bigr)G\,.$$

The device by which we derived Theorem 27.4 from Theorem 27.3 was invented by M. N. Huxley, and is known as '*Huxley's Trick*'. It seems that there is no way to similarly partition characters into useful subsets, so we lack a q analogue of Theorem 27.4. However, to address this issue, Huxley introduced further ideas, now known as the *Huxley Reflection Method*, which we now pursue. Roughly speaking, the idea is to start a sum of length N, and then use the functional equations of L-functions to create a corresponding sum of length Q^2T/N. The product of these two sums will be of length Q^2T, and hence the mean square can be efficiently estimated by means of Theorem 26.17.

Suppose that χ is a primitive character modulo q. The functional equation of $L(s,\chi)$ (*cf* Corollary 10.8) asserts that

$$L(s,\chi)\Gamma\Big(\frac{s+\kappa}{2}\Big)\Big(\frac{q}{\pi}\Big)^{\frac{s}{2}} = \varepsilon(\chi)L(1-s,\overline{\chi})\Gamma\Big(\frac{1-s+\kappa}{2}\Big)\Big(\frac{q}{\pi}\Big)^{\frac{1-s}{2}}$$

where $\kappa = (1-\chi(-1))/2$ and $\varepsilon(\chi) = \tau(\chi)/(i^\kappa\sqrt{q})$. Hence

$$L(s,\chi) = \varepsilon(\chi)\Big(\frac{q}{\pi}\Big)^{\frac12 - s}\frac{\Gamma\big(\frac{1-s+\kappa}{2}\big)}{\Gamma\big(\frac{s+\kappa}{2}\big)}L(1-s,\overline{\chi}).$$

Now suppose that χ is a character modulo q, but not necessarily primitive. Then χ is induced by a primitive character $\chi^\star$ where $\chi^\star$ is a character modulo d for some $d|q$, and

$$L(s,\chi) = L(s,\chi^\star)\prod_{p|q}\Big(1-\frac{\chi^\star(p)}{p^s}\Big).$$

Thus

$$L(s,\chi) = \varepsilon(\chi)L(1-s,\overline{\chi})\gamma(s,\chi)P(s,\chi) \tag{27.23}$$ E:asymFE

where

$$\gamma(s,\chi) = \pi^{s-1/2}\frac{\Gamma\big(\frac{1-s+\kappa}{2}\big)}{\Gamma\big(\frac{s+\kappa}{2}\big)} = (2\pi)^s\Gamma(1-s)\sin\frac{\pi}{2}(s+\kappa), \tag{27.24}$$ E:Defgam(s)

$$P(s,\chi) = d^{\frac12 - s}\prod_{p|q}\frac{1-\frac{\chi^\star(p)}{p^s}}{1-\frac{\overline{\chi^\star}(p)}{p^{1-s}}}. \tag{27.25}$$ E:DefP(s)

If $\kappa = 0$, then $\gamma(s,\chi)$ has simple zeros at $0,-2,-4,\ldots$ and simple poles at $1,3,5,\ldots$. If $\kappa = 1$, then $\gamma(s,\chi)$ has simple zeros at $-1,-3,-5,\ldots$ and simple poles at $2,4,6,\ldots$. Thus in either case $\gamma(s,\chi)$ is analytic for $\sigma < 1$. In the product that defines $P(s,\chi)$ we may restrict p to those $p|q$ such that $p\nmid d$, for if $p|d$ then $\chi^\star(p) = 0$. Suppose that $p|q$ and $p\nmid d$.

Then $\chi^\star(p)$ is a root of unity. Choose θ so that $\chi^\star(p) = e^{i\theta}$. Then

$$\frac{1 - \dfrac{\chi^\star(p)}{p^s}}{1 - \dfrac{\overline{\chi^\star}(p)}{p^{1-s}}} \tag{27.26}$$ E:pfactor

has simple zeros on the imaginary axis and simple poles on the 1-line at the points

$$\frac{i(\theta + 2k\pi)}{\log p}, \qquad 1 + \frac{i(\theta + 2k\pi)}{\log p}, \qquad (k \in \mathbb{Z})$$

respectively. Thus $P(s, \chi)$ is analytic for $\sigma < 1$. We are especially interested in the size of $\gamma(s, \chi)$ and of $P(s, \chi)$ when $\sigma \le 1/2$. In Chapter 10 we observed that $|\gamma(s, \chi_0)| \asymp \tau^{1/2-\sigma}$ uniformly for $|t| \ge 1$, $-A \le \sigma \le A$. We now refine this.

L:gam(s)est **Lemma 27.8** *Let $\gamma(s, \chi)$ be defined as in* (27.24)*. Then $|\gamma(1/2+it, \chi)| = 1$ for all real t,*

$$\gamma(s, \chi) \ll \left(\frac{|1-s|}{2\pi}\right)^{1/2-\sigma} \tag{27.27}$$ E:gam(s)est1

uniformly for all s in the halfplane $\sigma \le 1/2$, and

$$\gamma(s, \chi) \ll \tau^{1/2-\sigma} \tag{27.28}$$ E:gam(s)est2

when $-\tau^{2/3} \le \sigma \le 1/2$.

Proof The first assertion follows from the first formula for $\gamma(s, \chi)$ in (27.24), since the denominator of the fraction is the complex conjugate of the numerator, when $\sigma = 1/2$. For the two estimates we may assume that $t \ge 0$, since $\gamma(\overline{s}, \chi) = \overline{\gamma(s, \chi)}$. We now argue from the second formula for $\gamma(s, \chi)$ in (27.24). We note that

$$\left|\sin \tfrac{\pi}{2}(s+\kappa)\right| = \left|\frac{e^{i\frac{\pi}{2}(s+\kappa)} - e^{-i\frac{\pi}{2}(s+\kappa)}}{2i}\right| = \tfrac{1}{2}e^{\pi t/2} + O\big(e^{-\pi t/2}\big).$$

Stirling's formula, as stated in Theorem C.1, applies to $\Gamma(1-s)$ uniformly for $\sigma \le 1/2$. In the logarithmic form it asserts that

$$\log \Gamma(1-s) = \Big(\frac{1}{2} - s\Big)\log(1-s) + s - 1 + \frac{1}{2}\log 2\pi + O\Big(\frac{1}{|1-s|}\Big).$$

Hence

$$\log|\gamma(s, \chi)| = \sigma \log 2\pi + \Big(\frac{1}{2} - \sigma\Big)\log|1-s| + \sigma + t\arg(1-s) + \frac{\pi}{2}t + O(1).$$

Now

$$\arg(1-s) = -\arctan\frac{t}{1-\sigma} = -\frac{\pi}{2} + \arctan\frac{1-\sigma}{t} \le -\frac{\pi}{2} + \frac{1-\sigma}{t}.$$

Thus

$$\log|\gamma(s,\chi)| \le \Big(\frac{1}{2}-\sigma\Big)\log\frac{|1-s|}{2\pi} + O(1),$$

which gives (27.27).

We note that

$$\begin{aligned}\log|1-s| &= \frac{1}{2}\log\big((1-\sigma)^2+t^2\big)\\ &= \log t + \frac{1}{2}\log\Big(1+\Big(\frac{1-\sigma}{t}\Big)^2\Big) \le \log t + \frac{1}{2}\Big(\frac{1-\sigma}{t}\Big)^2.\end{aligned}$$

Hence

$$\log|\gamma(s,\chi)| \le \Big(\frac{1}{2}-\sigma\Big)\log\frac{t}{2\pi} + \frac{(1-\sigma)^3}{2t^2} + O(1).$$

This gives (27.28) when $\sigma \ge -t^{2/3}$ and $t \ge 1$. Finally, $\gamma(s,\chi) \ll 1$ when $-4 \le \sigma \le 1/2$ and $0 \le t \le 1$.

With further analysis, outlined in Exercise 10, one finds that (27.28) fails when $(1-\sigma)/t^{2/3}$ is unbounded as $t \to \infty$. □

L:P(s)est **Lemma 27.9** *For $P(s,\chi)$ defined by* (27.25) *we have $|P(1/2+it,\chi)| = 1$ for all real t, and $|P(s,\chi)| \le q^{\frac{1}{2}-\sigma}$ for $\sigma \le 1/2$.*

Proof The first assertion is clear, since the denominator in (27.26) is the complex conjugate of the numerator, when $\sigma = 1/2$. Set

$$f(s) = f_p(s) = p^{s-1/2}\frac{1+p^{-s}}{1+p^{s-1}}.$$

Since the fraction (27.26) is periodic with period $2\pi i/\log p$, the fraction above is the same apart from a translation. We note that

$$f(\sigma) = \frac{p^{\sigma/2}+p^{-\sigma/2}}{p^{(1-\sigma)/2}+p^{(\sigma-1)/2}} = \frac{\cosh\frac{\sigma}{2}\log p}{\cosh\frac{1-\sigma}{2}\log p}.$$

Now $\cosh x$ is an even function and is strictly increasing for $x \ge 0$. Since $|\sigma/2| < (1-\sigma)/2$ for $\sigma < 1/2$, it follows that $0 < f(\sigma) < 1$ for $\sigma < 1/2$. We also note that

$$\frac{1-p^\sigma}{1+p^{\sigma-1}}p^{-1/2} \le |f(s)| \le \frac{1+p^\sigma}{1-p^{\sigma-1}}p^{-1/2}$$

when $\sigma \le 0$. The two fractions tend to 1 as $\sigma \to -\infty$ so $|f(s)|$ is uniformly close to $p^{-1/2}$ when σ is large and negative. By applying the maximum modulus principle to f on a rectangle with vertices $-C, 1/2, 1/2 + 2\pi i/\log p, -C + 2\pi i/\log p$ we deduce that $|f(s)| \le 1$ throughout this rectangle, and by periodicity for all s with $\sigma \le 1/2$. Let r denote the product of those primes that divide q but do not divide d. We have shown that $|P(s,\chi)| \le (dr)^{1/2-\sigma}$ when $\sigma \le 1/2$. Since $dr|q$, we have the stated bound. □

L:Lfcnbnds **Lemma 27.10** *Let χ be a character* mod q. *Then*

$$L(s,\chi) - E(\chi)\frac{\varphi(q)s}{q(s-1)} \ll \min\Big(\frac{1}{\sigma-1}, \log q\tau\Big) \tag{27.29}$$ E:Lbnd1

uniformly for $\sigma \ge 1$,

$$L(s,\chi) - E(\chi)\frac{\varphi(q)s}{q(s-1)} \ll (q\tau)^{\frac{1-\sigma}{2}} \log q\tau \tag{27.30}$$ E:Lbnd2

uniformly for $0 \le \sigma \le 1$, and

$$L(s,\chi) \ll \big(q|1-s|\big)^{\frac{1}{2}-\sigma} \min\Big(\frac{-1}{\sigma}, \log q\tau\Big) \tag{27.31}$$ E:Lbnd3

for $\sigma \le 0$.

Proof Let

$$S(x;\chi) = \sum_{1\le n\le x} \chi(n), \qquad R(x;\chi) = S(x;\chi) - E(\chi)\frac{\varphi(q)}{q}x.$$

Thus $R(x;\chi) \ll q$ uniformly for $x > 0$. By integrating by parts (as in the proof of Theorem 1.12) we see that if $x > 0$ and $\sigma > 1$, then

$$\begin{aligned} L(s,\chi) &= \sum_{n\le x} \frac{\chi(n)}{n^s} + \int_x^\infty u^{-s}\, dS(u;\chi) \\ &= \sum_{n\le x} \frac{\chi(n)}{n^s} + E(\chi)\frac{\varphi(q)}{q}\cdot\frac{x^{1-s}}{s-1} \\ &\quad - \frac{R(x;\chi)}{x^s} + s\int_x^\infty \frac{R(u;\chi)}{u^{s+1}}\, du. \end{aligned} \tag{27.32}$$ E:Lform0

This formula provides an analytic continuation to the halfplane $\sigma > 0$. For the present we assume that $\sigma \ge 1$. In this case,

$$L(s,\chi) - E(\chi)\frac{\varphi(q)}{q}\cdot\frac{1}{s-1} \ll \int_1^x \frac{du}{u^\sigma} + \frac{q\tau}{x^\sigma}.$$

We take $x = q\tau$. For the integral we have two bounds:

$$\int_1^x \frac{du}{u^\sigma} = \frac{1 - x^{1-\sigma}}{\sigma - 1} \le \frac{1}{\sigma - 1}, \qquad \int_1^x \frac{du}{u^\sigma} \le \int_1^x \frac{du}{u} = \log x.$$

This gives the bound (27.29).

To obtain (27.31) we appeal to (27.23), from which we see that if $\sigma \le 0$, then

$$L(s,\chi) = \varepsilon(\chi)\Big(L(1-s,\overline{\chi}) + E(\chi)\frac{\varphi(q)}{qs}\Big)\gamma(s,\chi)P(s,\chi) \\ - E(\chi)\frac{\varphi(q)}{sq}\gamma(s,\chi)P(s,\chi).$$

From (27.29), (27.27), and Lemma 27.9 we see that the first term on the right above is $\ll \big(q|1-s|\big)^{1/2-\sigma}\min(\frac{1}{-\sigma}, \log q\tau)$. The second term on the right above occurs only when $\chi = \chi_0$. Since χ_0 is an even character, it follows that $\gamma(0,\chi_0) = 0$. Thus $\gamma(s,\chi_0)/s \ll |1-s|^{1/2-\sigma}/(1+|s|)$ for $\sigma \le 0$. Thus we have (27.31).

To obtain the bound (27.30) we appeal to the bounds (27.29), (27.31), and argue 'by convexity', which is to say by using the Phragmén–Lindelöf Theorem, which states that if $f(s)$ is analytic in the strip $S = \{s : 0 < \sigma < 1\}$, if f is continuous on $\overline{S}$, if $|f(s)| \le 1$ for $s \in \partial S$, and if there are constants $A > 0$ and α, $0 < \alpha < \pi$ such that

$$|f(s)| \le \exp\big(A\exp(\alpha|t|)\big), \tag{27.33}$$ E:P-LCritBnd

then $|f(s)|$ is bounded and $\sup_{s\in S}|f(s)| = \sup_{s\in\partial S}|f(s)|$. We take

$$f(s) = \Big(L(s,\chi) - E(\chi)\frac{\varphi(q)}{q}\cdot\frac{s}{s-1}\Big)\frac{q^{\frac{s-1}{2}}\Gamma\big(\frac{s}{2}+1\big)\cos\frac{\pi s}{4}}{(2-s)\log q(3-s)}.$$

From (27.29) and (27.31) it follows that $f(1+it) \ll 1$ and $f(it) \ll 1$, in view of the orders of magnitude indicated of the following functions, each of which is analytic in the strip $-1/2 \le \sigma \le 3/2$:

$$\big|q^{(s-1)/2}\big| \asymp q^{(\sigma-1)/2}, \qquad \big|\Gamma\big(\tfrac{s}{2}+1\big)\big| \asymp \tau^{(\sigma+1)/2}\exp\big(-\tfrac{\pi\tau}{4}\big), \\ \big|\cos\tfrac{\pi s}{4}\big| \asymp \exp\big(\tfrac{\pi\tau}{4}\big), \qquad |2-s| \asymp \tau, \qquad |\log q(3-s)| \asymp \log q\tau.$$

Thus (27.30) follows from the Phragmén–Lindelöf Theorem, provided that we can show a weak upper bound of the form

$$L(s,\chi) - E(\chi)\frac{\varphi(q)}{q}\cdot\frac{s}{s-1} \ll (q\tau)^A. \tag{27.34}$$ E:Lweakbnd

To this end, we take $x = 1^-$ in (27.32) and note that $R(1^-;\chi) = -E(\chi)\varphi(q)/q$. Thus

$$L(s,\chi) - E(\chi)\frac{\varphi(q)}{q}\cdot\frac{s}{s-1} \ll q\tau\int_1^\infty u^{-\sigma-1}\,du \ll q\tau$$

uniformly for $1/2 \le \sigma \le 1$. For $0 \le \sigma \le 1/2$ we again appeal to (27.23), from which with Lemmas 27.8 and 27.9 we see that

$$\begin{aligned} L(s,\chi) - E(\chi)\frac{\varphi(q)}{q}\cdot\frac{s}{s-1} &\ll |L(1-s,\overline{chi})\gamma(s,\chi)P(s,\chi)| + 1 \\ &\ll (q\tau)^{\frac{3}{2}-\sigma} + E(\chi)\Big|\frac{1-s}{-s}\Big||\gamma(s,\chi)P(s,\chi)|. \end{aligned}$$

As we already noted, $\gamma(s,\chi_0)/s$ is bounded for s near 0. Thus we have (27.34) with $A = 3/2$, so the proof is complete. □

We also prepare a handy kernel:

L:kernel **Lemma 27.11** *For $u \ge 0$ let*

$$\textit{k}(u) = \sum_{\substack{0\le j\le 4\\ 2u-1\le j}} (-1)^j\binom{4}{j}(j+1-2u)^3, \qquad (27.35)$$ Defk(u)

and set

$$K(s) = \frac{48\sum_{j=0}^{4}(-1)^j\binom{4}{j}\Big(\frac{j+1}{2}\Big)^{s+3}}{s(s+1)(s+2)(s+3)}. \qquad (27.36)$$ E:DefK

Then $K(s)$ is an entire function, and if $\alpha(s) = \sum_{n=1}^\infty a_n n^{-s}$ has abscissa of convergence σ_c and $x > 0$, then

$$\sum_{n=1}^\infty a_n \textit{k}\Big(\frac{n}{x}\Big) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\alpha(s)K(s)x^s\,ds \qquad (27.37)$$ E:k(u)<->K(s)

for $c > \max(0,\sigma_c)$.

To prepare for the proof of this lemma, we construct notation to express forward differences. If $f(x)$ is defined on the real line, then $\Delta f(x) = f(x+1)-f(x)$. If a forward difference with a step size $h > 0$ is desired, we write $\Delta_h f(x) = f(x+h) - f(x)$. These operations can be iterated; for example, $(\Delta^2 f)(x) = \Delta(\Delta f(x)) = \big(f(x+2) - f(x+1)\big) - \big(f(x+1) - f(x)\big) = f(x) - 2f(x+1) + f(x+2)$. In general, $\Delta^k f = \Delta(\Delta^{k-1} f)$.

We refer to this as a k^{th} order forward difference. By an easy induction we see that

$$(\Delta_h^k f)(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x + jh).$$

We note that if f is a polynomial of degree d, then Δf is a polynomial of degree $d - 1$, and hence that $(\Delta^k f)(x) = 0$ identically if $k > d$.

Proof In §5.1 we remarked that the Cesàro partial sums of order k of a Dirichlet series $\alpha(s)$ are given by the formulæ

$$C_k(x) = \frac{1}{k!} \sum_{n \le x} a_n (x - n)^k = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \alpha(s) \frac{x^{s+k}}{s(s+1)\cdots(s+k)}\, ds$$

for $c > \max(0, \sigma_c)$. We take $k = 3$ and form the 4^{th}-order forward difference of both sides of the above, starting at $\frac{1}{2}x$ and taking steps of size $\frac{1}{2}x$. The left hand side then becomes

$$\begin{aligned} &\sum_{j=0}^{4} (-1)^j \binom{4}{j} C_3\big(\tfrac{1}{2}(j+1)x\big) \\ &\quad = \frac{1}{6} \sum_n a_n \sum_{j=0}^{4} (-1)^j \binom{4}{j} \big(\max(0, \tfrac{1}{2}(j+1)x - n)\big)^3 \\ &\quad = \frac{1}{6} \sum_n a_n \sum_{\substack{0 \le j \le 4 \\ (j+1)x > 2n}} (-1)^j \binom{4}{j} \big(\tfrac{1}{2}(j+1)x - n\big)^3 = \frac{x^3}{48} \sum_n a_n \kappa\big(\tfrac{n}{x}\big). \end{aligned}$$

The sum that defines $\kappa(u)$ is empty when $u > 5/2$. When $u < 1/2$, j runs over the full range $0 \le j \le 4$, and then the expression represents a fourth order forward difference of u^3. Thus $\kappa(u) = 0$ for $u < 1/2$. The weight $\kappa(u)$ is piecewise polynomial, and is equal to a single polynomial in each of the intervals $(-\infty, \frac{1}{2}]$, $[\frac{1}{2}, 1], [1, \frac{3}{2}], [\frac{3}{2}, 2], [2, \frac{5}{2}], [\frac{5}{2}, \infty)$. At each of the transition points, the summand that is introduced or removed vanishes to the third order at that point. Hence $\kappa''(u)$ is continuous and piecewise linear.

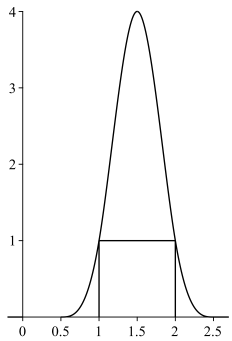

Figure 22.3 Graphs of $\kappa(u)$ and $\chi_{[1,2]}(u)$.

Thomas Simpson (1757) showed that if $X_1, X_2, \ldots, X_n$ are independent random variables, each one uniformly distributed on $[0,1]$, and $X = X_1 + X_2 + \cdots + X_n$, then X is distributed with the density

$$f_n(x) = \frac{1}{(n-1)!} \sum_{x \le j \le n} (-1)^{n-j} \binom{n}{j} (j-x)^{n-1}$$

for $n > 1$. Thus $\mathcal{K}(u) = 6f_4(2u-1)$. With this interpretation, we see that $\mathcal{K}(u) = \mathcal{K}(3-u)$, that $\int_{1/2}^{5/2} \mathcal{K}(u)\,du = 3$, and that $\mathcal{K}(u)$ is increasing for $1/2 \le u \le 3/2$. These same observations can also be derived by means of elementary calculations, without recourse to the results of Simpson.

When we apply the same fourth order forward difference to the integral representation of $C_3(x)$, we find that the integrand becomes

$$\frac{\alpha(s)}{s(s+1)(s+2)(s+3)} \sum_{j=0}^{4} (-1)^j \binom{4}{j} \Big(\frac{j+1}{2}x\Big)^{s+3} = \frac{x^3}{48}\alpha(s)K(s)x^s.$$

Thus we have (27.37).

Let $S(s)$ denote the sum over j in the definition (27.36) of $K(s)$. To show that $K(s)$ is entire, it suffices to show that $S(-r) = 0$ for $r = 0, 1, 2, 3$. But this is obvious, since $S(-r)$ is a forward difference of order 4 of x^{3-r}, which is a polynomial of degree $3 - r < 4$. □

L:G2est **Lemma 27.12** *Suppose that*

$$A(s) = \sum_{m=1}^{M} a_m m^{-s}, \; B(s) = \sum_{n=1}^{N} b_n n^{-s}, \; C(s) = A(s)B(s) = \sum_{k=1}^{MN} c_k k^{-s}.$$

Then

$$\sum_{k=1}^{MN} |c_k|^2 \le \Big(\sum_{m=1}^{M} d(m)|a_m|^2\Big)\Big(\sum_{n=1}^{N} d(n)|b_n|^2\Big).$$

Proof The sequence $\{c_k\}$ is the Dirichlet convolution $a * b$ of the sequences $\{a_m\}$ and $\{b_n\}$:

$$c_k = \sum_{\substack{m,n \\ mn=k}} a_m b_n.$$

By Cauchy's inequality,

$$|c_k|^2 \le \Big(\sum_{\substack{m,n \\ mn=k}} 1\Big)\Big(\sum_{\substack{m,n \\ mn=k}} |a_m|^2|b_n|^2\Big) = \sum_{\substack{m,n \\ mn=k}} d(mn)|a_m|^2|b_n|^2.$$

Write $m = \prod_p p^\mu$ and $n = \prod_p p^\nu$. Since $\mu + \nu + 1 \le (\mu+1)(\nu+1)$ for nonnegative μ, ν, it follows that $d(mn) \le d(m)d(n)$. We substitute this inequality in the above, and sum over k to obtain the stated result. □

T:HMqT2 **Theorem 27.13** *Let q be a positive integer, and $T \ge 2$ be real. Suppose that for $r = 1, 2, \ldots R$ we have a character χ_r modulo q and a point $s_r = \sigma_r + it_r$ with $\sigma_r \ge 0$, $0 \le t_r \le T$, and with the further property that $\left|t_{r_1} - t_{r_2}\right| \ge 1$ if $\chi_{r_1} = \chi_{r_2}$. Let $D(s,\chi)$ be defined as in (27.22), and set*

$$G = \sum_{n=1}^{N} |a_n|^2, \qquad G_2 = \sum_{n=1}^{N} d(n)|a_n|^2. \tag{27.38}$$ E:DefG,G2

If $N \le qT$, then

$$\sum_{r=1}^{R} |D(s_r,\chi_r)|^2 \ll NG + R^{2/3}N^{1/3}q^{1/3}T^{1/3}G^{2/3}G_2^{1/3}\log qT. \tag{27.39}$$ E:HMqT2

When $N \ge qT$ we already know that the sum is $\ll NG \log N$ by the estimate (26.26) found in Theorem 26.17.

Proof We first establish the desired bound under the assumptions that $\sigma_r = 0$ for all r and that the sum runs not from 1 to N but rather from $N+1$ to $2N$. Thus our first object is to show that

$$\sum_{r=1}^{R} \Big| \sum_{n=N+1}^{2N} a_n \chi_r(n) n^{-it_r} \Big|^2 \ll NG + R^{2/3}N^{1/3}q^{1/3}T^{1/3}G^{2/3}G_2^{1/3}\log qT. \tag{27.40}$$ E:Step1

In this context,

$$G = \sum_{n=N+1}^{2N} |a_n|^2, \qquad G_2 = \sum_{n=N+1}^{2N} d(n)|a_n|^2. \tag{27.41}$$ E:tempDefG,G2

Let b_r be determined by the equation

$$\overline{b_r} = \sum_{n=N+1}^{2N} a_n \chi_r(n) n^{-it_r}. \tag{27.42}$$ E:Defbr

Thus the left hand side of (27.40) is

$$= \sum_{r=1}^{R} \sum_{n=N+1}^{2N} a_n \chi_r(n) n^{-it_r} b_r = \sum_{n=N+1}^{2N} a_n \sum_{r=1}^{R} b_r \chi_r(n) n^{-it_r}.$$

Since the left hand side of (27.40) is equal to the right hand side of the above, their squares are also equal. That is,

$$\Big(\sum_{r=1}^{R}\Big|\sum_{n=N+1}^{2N} a_n\chi_r(n)n^{-it_r}\Big|^2\Big)^2 = \Big(\sum_{n=N+1}^{2N} a_n\sum_{r=1}^{R} b_r\chi_r(n)n^{-it_r}\Big)^2.$$

We apply Cauchy's inequality to the right hand side above, to see that it is

$$\le G\sum_{n=N+1}^{2N}\Big|\sum_{r=1}^{R} b_r\chi_r(n)n^{-it_r}\Big|^2. \tag{27.43}$$

`E:CritIneq`

Let $\mathscr{k}(u)$ be defined as in Lemma 27.11. Then $\mathscr{k}(u)\ge 0$ for all $u\ge 0$, and $\mathscr{k}(u)\ge 1$ for $1\le u\le 2$, so the sum over n in (27.43) is

$$\le \sum_{n=1}^{\infty}\Big(\frac{2N}{n}\Big)^{1/2}\mathscr{k}\Big(\frac{n}{N}\Big)\Big|\sum_{r=1}^{R} b_r\chi_r(n)n^{-it_r}\Big|^2.$$

We expand the modulus-squared and take the sum over n inside to see that the above is

$$\begin{aligned}&= (2N)^{1/2}\sum_{r_1=1}^{R}\sum_{r_2=1}^{R} b_{r_1}\overline{b_{r_2}}\sum_{n=1}^{\infty}\mathscr{k}\Big(\frac{n}{N}\Big)\chi_{r_1}\overline{\chi}_{r_2}(n)n^{-\frac12-i(t_{r_1}-t_{r_2})}\\ &\ll N^{1/2}\sum_{r_2=1}^{R}|b_{r_2}|\sum_{r_1=1}^{R}|b_{r_1}|\Big|\sum_{n=1}^{\infty}\mathscr{k}\Big(\frac{n}{N}\Big)\chi_{r_1}\overline{\chi}_{r_2}(n)n^{-\frac12-i(t_{r_1}-t_{r_2})}\Big|.\end{aligned} \tag{27.44}$$

`E:Exp2`

From (27.37) we deduce that

$$\begin{aligned}&\sum_{n=1}^{\infty}\mathscr{k}\big(\tfrac{n}{N}\big)\chi_{r_1}\overline{\chi}_{r_2}(n)n^{-\frac12-i(t_{r_1}-t_{r_2})}\\ &= \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} L\big(\tfrac12+i(t_{r_1}-t_{r_2})+w,\chi_{r_1}\overline{\chi}_{r_2}\big)K(w)N^w\,dw.\end{aligned} \tag{27.45}$$

`E:Exp3`

Write $w=u+iv$. The above integrand has a pole at $w=\frac12-i\big(t_{r_1}-t_{r_2}\big)$ if $\chi_{r_1}=\chi_{r_2}$. By moving the contour to the abscissa $u=-1$ we find that the above is

$$\begin{aligned}&= \frac{1}{2\pi i}\int_{-1-i\infty}^{-1+i\infty} L\big(\tfrac12+i(t_{r_1}-t_{r_2})+w,\chi_{r_1}\overline{\chi}_{r_2}\big)K(w)N^w\,dw\\ &\qquad + E\big(\chi_{r_1}\overline{\chi}_{r_2}\big)\frac{\varphi(q)}{q}K\big(\tfrac12-i\big(t_{r_1}-t_{r_2}\big)\big)N^{\frac12-i(t_{r_1}-t_{r_2})}\end{aligned} \tag{27.46}$$

`E:Residue`

where $E(\chi) = 1$ if $\chi = \chi_0$, and $E(\chi) = 0$ otherwise. Concerning the convergence of this integral as $|t| \to \infty$, we note that

$$L\big(\tfrac{1}{2} + i\big(t_{r_1} - t_{r_2}\big) + w, \chi_{r_1}\overline{\chi}_{r_2}\big) \ll q(T + |v|)$$

for $u \ge -1$, by Lemma 27.10. This suffices, since $K(w) \ll v^{-4}$. By the functional equation in the form (27.23), we see that the integral above is

$$\begin{aligned} = \frac{\varepsilon(\chi_{r_1}\overline{\chi}_{r_2})}{2\pi i} \int_{-1-i\infty}^{-1+i\infty} & L(\tfrac{1}{2} - i(t_{r_1} - t_{r_2}) - w, \overline{\chi}_{r_1}\chi_{r_2}) \\ & \times \gamma(\tfrac{1}{2} + i(t_{r_1} - t_{r_2}) + w, \chi_{r_1}\overline{\chi}_{r_2}) \qquad (27.47) \\ & \times P(\tfrac{1}{2} + i(t_{r_1} - t_{r_2}) + w, \chi_{r_1}\overline{\chi}_{r_2})K(w)N^w\,dw. \end{aligned}$$

E:Exp4

For $\sigma > 1$ we write

$$\begin{aligned} L(s, \overline{\chi}_{r_1}\chi_{r_2}) &= \sum_{m=1}^{\infty} \overline{\chi}_{r_1}\chi_{r_2}(m)m^{-s} = \sum_{m \le M} + \sum_{m > M} \\ &= S_1(s, \overline{\chi}_{r_1}\chi_{r_2}) + S_2(s, \overline{\chi}_{r_1}\chi_{r_2}), \qquad (27.48) \end{aligned}$$

E:DefS1S2

say, where M is an integer such that $MN \asymp qT$. When we replace L by S_i in (27.47) we obtain an integral $I_i(r_1, r_2)$ for $i = 1, 2$. To complete the estimation of (27.44) we estimate the contributions made by the I_i and the residue in (27.46).

To estimate I_1 we move the contour from the abscissa $u = -1$ to $u = 0$. This is justified by our bounds for $\gamma(s)$, $P(s)$, and $K(s)$. Since $|\gamma(s)| = |P(s)| = 1$ on the new contour, we deduce that

$$I_1(r_1, r_2) \ll \int_{-\infty}^{\infty} \big|S_1(\tfrac{1}{2} - i(t_{r_1} - t_{r_2}) - iv, \overline{\chi}_{r_1}\chi_{r_2})\big||K(iv)|\,dv.$$

After replacing b_{r_1} and S_1 by their complex conjugates, we see that

$$\begin{aligned} &|b_{r_1}|\big|S_1(\tfrac{1}{2} - i(t_{r_1} - t_{r_2}) - iv, \overline{\chi}_{r_1}\chi_{r_2})\big| \\ &= \Big|\Big(\sum_{n=N+1}^{2N} a_n\chi_{r_1}(n)n^{-it_{r_1}}\Big)\Big(\sum_{m=1}^{M} m^{-\frac{1}{2}-i(t_{r_1}-t_{r_2})-iv}\chi_{r_1}(m)\overline{\chi}_{r_2}(m)\Big)\Big| \\ &= \Big|\sum_{k=N}^{2MN} c_k\chi_{r_1}(k)k^{-it_{r_1}}\Big| = \big|C\big(it_{r_1}, \chi_{r_1}\big)\big| \end{aligned}$$

say, where

$$c_k = c_{k,r_2,v} = \sum_{\substack{1 \le m \le M \\ N < n \le 2N \\ mn = k}} a_n m^{-\frac{1}{2}+it_{r_2}-iv}\,\overline{\chi}_{r_2}(m).$$

Hence

$$|c_k| \le \sum_{\substack{1\le m\le M\\ N<n\le 2N\\ mn=k}} |a_n| m^{-1/2}$$

for all r_2 and v. By Cauchy's inequality,

$$\sum_{r_1=1}^{R} |b_{r_1}| \big|S_1\big(\tfrac12 - i(t_{r_1}-t_{r_2}) - iv, \overline{\chi}_{r_1}\chi_{r_1}\big)\big|$$
$$= \sum_{r_1=1}^{R} \big|C\big(it_{r_1},\chi_{r_1}\big)\big| \le R^{1/2}\Big(\sum_{r_1=1}^{R} \big|C\big(it_{r_1},\chi_{r_1}\big)\big|^2\Big)^{1/2}.$$

From the estimate (26.26) of Theorem 26.17 we see that the above is

$$\ll R^{1/2}\Big(qT(\log qT)\sum_{k=1}^{2MN}|c_k|^2\Big)^{1/2}.$$

By Lemma 27.12 we deduce that

$$\sum_{k=N}^{2MN}|c_k|^2 \le \Big(\sum_{n=N+1}^{2N} d(n)|a_n|^2\Big)\Big(\sum_{m=1}^{M}\frac{d(m)}{m}\Big) \ll G_2(\log qT)^2.$$

Thus

$$\sum_{r_1=1}^{R} |b_{r_1}| \big|S_1\big(\tfrac12 - i(t_{r_1}-t_{r_2}) - iv, \overline{\chi}_{r_1}\chi_{r_2}\big)\big| \ll (RqTG_2)^{1/2}(\log qT)^{3/2}$$

uniformly for all r_2 and v, so the contribution of I_1 to (27.44) is

$$\ll N^{1/2}\Big(\sum_{r_2=1}^{R}|b_{r_2}|\Big)(RqTG_2)^{1/2}(\log qT)^{3/2}\int_{-\infty}^{\infty}|K(iv)|\,dv.$$

From the definition (27.36) of $K(s)$ we see that $K(iv) \ll (|v|+1)^{-4}$, and hence that the integral above is bounded. By a further application of Cauchy's inequality we see that the above is

$$\ll (NqT)^{1/2}RG_2^{1/2}(\log qT)^{3/2}\Big(\sum_{r=1}^{R}|b_r|^2\Big)^{1/2}. \qquad (27.49)$$ E:I1contrib

Apart from the need to partition $S_2(s,\chi)$ into subsums, we treat I_2

in the same way that we treated I_1. By our bounds from Lemmas 27.8, 27.9, 27.11 we see that

$$I_2(r_1,r_2) \ll \frac{1}{N}\int_{-\infty}^{\infty} q(T+|v|)(|v|+1)^{-4}\big|S_2\big(\tfrac{3}{2}-i(t_{r_1}-t_{r_2})-iv,\overline{\chi}_{r_1}\chi_{r_2}\big)\big|dv.$$

Our main task is to show that

$$\sum_{r_1=1}^{R}|b_{r_1}|\big|S_2\big(\tfrac{3}{2}-i(t_{r_1}-t_{r_2})-iv,\overline{\chi}_{r_1}\chi_{r_2}\big)\big| \\ \ll R^{\frac{1}{2}}M^{-\frac{1}{2}}N^{\frac{1}{2}}G_2^{\frac{1}{2}}\log 2MN \tag{27.50}$$

E:sum|br||S2|est

uniformly for all r_2 and v. Since $MN \asymp qT$, it then follows that

$$\sum_{r_1=1}^{R}|b_{r_1}|\big|I_2(r_1,r_2)\big| \ll (RqTG_2)^{\frac{1}{2}}\log qT,$$

which implies that

$$N^{\frac{1}{2}}\sum_{r_2=1}^{R}|b_{r_2}|\sum_{r_1=1}^{R}|b_{r_1}|\big|I_2(r_1,r_2)\big| \ll \Big(\sum_{r_2=1}^{R}|b_{r_2}|\Big)(RNqTG_2)^{\frac{1}{2}}\log qT,$$

and by a further application of Cauchy's inequality we see that this is

$$\ll R(NqTG_2)^{\frac{1}{2}}\Big(\sum_{r_2=1}^{R}|b_{r_2}|\Big)^{1/2}\log qT.$$

This quantity is slightly smaller than our estimate (27.49) for the contribution of I_1 in (27.44).

To prove (27.50) we first put $M_j = 2^j M$ for $j = 0,1,2,\ldots$, and set

$$S_{2,j} = S_{2,j}(r_1,r_2,v) = \sum_{M_{j-1}<m\le M_j}\overline{\chi}_{r_1}\chi_{r_2}(m)m^{-\frac{3}{2}+i(t_{r_1}-t_{r_2})+iv}$$

for $j = 1,2,\ldots$. By the triangle inequality,

$$|b_{r_1}|\big|S_2\big(\tfrac{3}{2}-i(t_{r_1}-t_{r_2})-iv,\overline{\chi}_{r_1}\chi_{r_2}\big)\big| \le \sum_{j=1}^{\infty}|b_{r_1}||S_{2,j}|.$$

On replacing b_{r_1} and $S_{2,j}$ by their complex conjugates, we see that

$$\begin{aligned}|b_{r_1}||S_{2,j}| &= \Big|\sum_{n=N+1}^{2N} a_n\chi_{r_1}(n)n^{-it_{r_1}}\Big| \\ &\quad\times\Big|\sum_{M_{j-1}<m\le M_j} m^{-\frac{3}{2}-it_{r_1}}\chi_{r_1}(m)\Big(m^{i(v+it_{r_2})}\overline{\chi}_{r_2}\Big)\Big| \\ &= |C_j(r_1)| = |C_j(r_1,r_2,v)|\end{aligned}$$

where

$$C_j(r_1) = C_j(r_1,r_2,v) = \sum_{M_{j-1}N<m\le 2M_jN} c_{j,k}\chi_{r_1}(k)k^{-it_{r_1}}$$

and

$$c_{j,k} = c_{j,k,r_2,v} = \sum_{\substack{M_{j-1}<m\le M_j\\ N<n\le 2N\\ mn=k}} a_n m^{-\frac{3}{2}}\big(m^{i(v+t_{r_2})}\overline{\chi}_{r_2}(m)\big).$$

Hence

$$|c_{j,k}| \le \sum_{\substack{M_{j-1}<m\le M_j\\ N<n\le 2N\\ mn=k}} |a_n| m^{-\frac{3}{2}}$$

for all r_2 and v. Thus by Cauchy's inequality,

$$\sum_{r_1=1}^{R}\sum_{j=1}^{\infty}|b_{r_1}||S_{2,j}| \le \Big(\sum_{r_1,j} j^{-2}\Big)^{1/2}\Big(\sum_{j=1}^{\infty} j^2\sum_{r_1=1}^{R}|C_j(r_1)|^2\Big)^{1/2}.$$

Since $M_jN \gg qT$, we see by the estimate (26.26) of Theorem 26.17 that the above is

$$\ll R^{1/2}\Big(\sum_{j=1}^{\infty} j^2\big(\log M_jN\big)M_jN\sum_{M_{j-1}N<k\le 2M_jN}|c_{j,k}|^2\Big)^{1/2}. \qquad (27.51)$$ E:Exp5

By Lemma 27.12 we see that

$$\begin{aligned}\sum_{M_{j-1}N<k\le 2M_jN}|c_{j,k}|^2 &\le \Big(\sum_{M_{j-1}<m\le M_j} d(m)m^{-3}\Big)\Big(\sum_{n=N+1}^{2N} d(n)|a_n|^2\Big) \\ &\ll M_j^{-2}\big(\log M_j\big)G_2.\end{aligned}$$

Thus the expression (27.51) is

$$\begin{aligned}&\ll R^{1/2}\Big(\sum_{j=1}^{\infty} j^2(j+\log MN)2^{-j}M^{-1}N(j+\log M)G_2\Big)^{1/2}\\&\ll R^{\frac12}M^{-\frac12}N^{\frac12}G_2^{\frac12}\log qT,\end{aligned}$$

and hence we have (27.50).

From the definition (27.36) of $K(s)$ it is clear that $K\big(\frac12+it\big)\ll\tau^{-4}$. Hence the contribution to (27.44) made by the residue in (**??**) is

$$\ll N\sum_{\substack{1\le r_1,r_2\le R\\ \chi_{r_1}=\chi_{r_2}}}\frac{\big|b_{r_1}b_{r_2}\big|}{\big(1+\big|t_{r_1}-t_{r_2}\big|\big)^4}.$$

By the arithmetic–geometric mean inequality we know that $\big|b_{r_1}b_{r_2}\big|\le \frac12\big|b_{r_1}\big|^2+\frac12\big|b_{r_2}\big|^2$. Hence the above is

$$\le N\sum_{r_1=1}^{R}\big|b_{r_1}\big|^2\sum_{\substack{1\le r_2\le R\\ \chi_{r_1}=\chi_{r_2}}}\big(1+\big|t_{r_1}-t_{r_2}\big|\big)^{-4}.$$

For a given r_1, the numbers t_{r_2} for which $\chi_{r_2}=\chi_{r_1}$ are spaced apart from each other by a distance of at least 1. Hence the sum over r_2 above is $\ll 1$, and so the quantity above is

$$\ll N\sum_{r=1}^{R}|b_r|^2.$$

Thus the quantity (27.44) is bounded by the above plus the estimate (27.49) for the contribution of I_1, since our estimate for the contribution of I_2 is smaller. From the definition (27.42) of b_r we know that

$$\sum_{r=1}^{R}|b_r|^2=\sum_{r=1}^{R}\Big|\sum_{n=N+1}^{2N}a_n\chi_r(n)n^{-it_r}\Big|^2.$$

Hence when we insert our bound for the expression (27.44) into the inequality (27.43) we find that

$$\begin{aligned}\Big(\sum_{r=1}^{R}|b_r|^2\Big)^2&\ll R(NqT)^{1/2}GG_2^{1/2}(\log qT)^{3/2}\Big(\sum_{r=1}^{R}|b_r|^2\Big)^{1/2}\\&\quad+NG\sum_{r=1}^{R}|b_r|^2.\end{aligned}\tag{27.52}$$

E:Exp6

If the second term on the right hand side is larger than the first, then it majorizes the left hand side, so we have

$$\sum_{r=1}^{R} |b_r|^2 \ll NG.$$

If the first term on the right hand side of (27.52) is at least as large as the second one, then it majorizes the left hand side, with the result that

$$\Big(\sum_{r=1}^{R} |b_r|^2\Big)^{3/2} \ll R(NqT)^{1/2} G G_2^{1/2} (\log qT)^{3/2}$$

and so

$$\sum_{r=1}^{R} |b_r|^2 \ll R^{2/3} N^{1/3} q^{1/3} T^{1/3} G^{2/3} G_2^{1/3} \log qT$$

in this case. Hence in either case we have (27.40)

We now drop the condition that $\sigma_r = 0$ for all r, and instead allow $\sigma_r \ge 0$, while still restricting n to lie between $N+1$ and $2N$. Thus our new goal is to show that

$$\begin{aligned}\sum_{r=1}^{R} \Big| \sum_{n=N+1}^{2N} a_n \chi_r(n) n^{-\sigma_r - it_r}\Big|^2 \\ \ll NG + R^{2/3} N^{1/3} q^{1/3} T^{1/3} G^{2/3} G_2^{1/3} (\log 2qT)^A.\end{aligned} \qquad (27.53)$$

E:Step2

Here G and G_2 are still defined as in (27.41). We employ the same integration by parts method that we used to prove Theorems 26.9 and 26.18. This approach is more successful in the present setting because the sum over n is shorter. For a given character χ and real number t, let

$$S(u) = S(u; t, \chi) = \sum_{N+1 \le n \le u} a_n \chi(n) n^{-it}.$$

Then by integrating by parts we see that if $\sigma \ge 0$, then

$$\sum_{n=N+1}^{2N} a_n \chi(n) n^{-\sigma - it} = \frac{S(2N)}{(2N)^\sigma} + \sigma \int_N^{2N} \frac{S(u)}{u^{\sigma+1}}\, du.$$

Hence

$$\Big| \sum_{n=N+1}^{2N} a_n \chi(n) n^{-\sigma - it}\Big| \le |S(2N)| + \frac{\sigma}{N^{\sigma+1}} \int_N^{2N} |S(u)|\, du.$$

For $\sigma \geq 0$, the quantity σ/N^{sigma} achieves its maximum when $\sigma = 1/\log N$. Thus the above is

$$\ll |S(2N)| + \frac{1}{N \log N} \int_N^{2N} |S(u)|\, du.$$

By squaring both sides, and the applying the Cauchy–Schwarz inequality we deduce that

$$\Big| \sum_{n=N+1}^{2N} a_n \chi(n) n^{-\sigma-it} \Big|^2 \ll |S(2N)|^2 + \frac{1}{N(\log N)^2} \int_N^{2N} |S(u)|^2\, du.$$

We now take $\chi = \chi_r$, $\sigma = \sigma_r$, $t = t_r$, and sum over r. The contribution of the $|S(2N; s_r, \chi_r)|^2$ is the same as in (27.40). From (27.40) we see that if $N \leq u \leq 2N$, then

$$\sum_{r=1}^{R} |S(u; t_r, \chi_r)|^2 \ll NG(u) + R^{2/3} q^{1/3} T^{1/3} N^{1/3} G(u)^{2/3} G_2(u)^{1/3} \log 2qT$$

where

$$G(u) = \sum_{N+1 \leq n \leq u} |a_n|^2 \leq G, \qquad G_2(u) = \sum_{N+1 \leq n \leq u} d(n)|a_n|^2 \leq G_2,$$

Thus we have the same bound as in (27.40) for all u, and hence we have (27.53).

Finally, let $D(s,\chi)$ be defined as in (27.22), and write

$$D(s,\chi) = \sum_{j=1}^{J} D_j(s,\chi)$$

where $J = \lfloor (\log N)/\log 2 \rfloor$

$$D_j(s,\chi) = \sum_{N/2^{j+1} < n \leq N/2^j} a_n \chi(n) n^{-s},$$

$$G(j) = \sum_{N/2^{j+1} < n \leq N/2^j} |a_n|^2, \qquad G_2(j) = \sum_{N/2^{j+1} < n \leq N/2^j} d(n)|a_n|^2.$$

Let δ be fixed, $0 < \delta < 1$. Then by Cauchy's inequality,

$$\begin{aligned} |D(s_r,\chi_r)|^2 = \Big| \sum_{j=0}^{J} D_j(s_r,\chi_r) \Big|^2 &\leq \Big(\sum_{j=0}^{J} \delta^j \Big) \Big(\sum_{j=0}^{J} \delta^{-j} |D_j(s_r,\chi_r)|^2 \Big) \\ &\ll \sum_{j=0}^{J} \delta^{-j} |D_j(s_r,\chi_r)|^2. \end{aligned}$$

We sum over r and apply (27.53) to see that

$$\sum_{r=1}^{R} |D(s_r,\chi_r)|^2 \ll C_1 NG(j) + C_2 R^{2/3} N^{1/3} q^{1/3} T^{1/3} G(j)^{2/3} G_2(j)^{1/3} \log qT$$

where

$$C_1 = \sum_{j=0}^{J} (2\delta)^{-j}, \qquad C_2 = \sum_{j=0}^{J} \left(2^{1/3}\delta\right)^{-j}.$$

We observe that $G(j) \le G$ and $G_2(j) \le G_2$ for all j where G and G_2 are defined as in (27.38). We set $\delta = 9/10$, and note that $2\delta > 2^{1/3}\delta = 1.1339\ldots > 1$, so that $C_1 \ll 1$ and $C_2 \ll 1$. Thus we have (27.39), and the proof is complete. □

Cor:HMLVqT2 **Corollary 27.14** *Suppose that q is a positive integer, and that $T \ge 2$ is real. Suppose that $D(s,\chi)$ is defined as in* (27.22)*, and that for $r = 1, 2, \ldots, R$ we have pairs (s_r, χ_r) with the property that $s_r = \sigma_r + it_r$, $\sigma_r \ge 0$, $0 \le t_r \le T$, that χ_r is a character* mod q, and that $|t_{r_1} - t_{r_2}| \ge 1$ if $\chi_{r_1} = \chi_{r_2}$. Let G and G_2 be defined as in (27.38). Then

$$R \ll \frac{NG}{V^2} + \frac{NqTG^2G_2(\log qT)^3}{V^6}. \qquad (27.54)$$ E:HMLVqT2

The bound here is strikingly similar to that obtained by a totally different method for $q = 1$ in Theorem 27.4.

Proof In (27.39), the left hand side is $\ge RV^2$. If NG is the larger of the two terms on the right hand side, then it follows that $R \ll NG/V^2$. If NG is the smaller of the two terms on the right hand side, then $RV^2 \ll R^{2/3} N^{1/3} q^{1/3} T^{1/3} G^{2/3} G_2^{1/3} \log qT$, which gives

$$R \ll \frac{NqTG^2G_2(\log qT)^3}{V^6}.$$

Thus (27.54) holds in either case. □

T:HMQT2 **Theorem 27.15** *Suppose that for $r = 1, 2, \ldots R$ we have a primitive character χ_r modulo q_r with $1 \le q_r \le Q$ and a point $s_r = \sigma_r + it_r$ with $\sigma_r \ge 0$, $0 \le t_r \le T$, and with the further property that $\left|t_{r_1} - t_{r_2}\right| \ge 1$ if $\chi_{r_1} = \chi_{r_2}$. Let $D(s,\chi)$ be defined as in* (27.22)*, and let G and G_2 be defined as in* (27.38)*. Then*

$$\sum_{r=1}^{R} |D(s_r,\chi_r)|^2 \ll NG + R^{2/3} N^{1/3} Q^{2/3} T^{1/3} G^{2/3} G_2^{1/3} \log QT. \qquad (27.55)$$ E:HMQT2

Proof This theorem is proved in the same way as the preceding one, with just a few obvious changes. For example, since χ_{r_s} and χ_{r_2} are primitive, if $\chi_{r_1}\overline{\chi}_{r_2} = \chi_0$, then $\chi_{r_1} = \chi_{r_2}$. The character $\chi_{r_1}\overline{\chi}_{r_2}$ is a character modulo $[q_{r_1}, q_{r_2}] \le q_{r_1}q_{r_2} \le Q^2$. In places where we appealed to (26.26) in Theorem 26.17, we instead appeal to (26.28), which is found in the same theorem. □

??.1 Exercises

1. Suppose that f is a measurable function on $[0,1]$, that $V > 0$, and that $m(V)$ denotes the measure of the set of those $x \in [0,1]$ for which $|f(x)| \ge V$. Suppose further that we know that

$$\int_0^1 |f(x)|\,dx \le 1, \qquad \text{and that} \qquad \int_0^1 |f(x)|^3\,dx \le 4\,.$$

   (a) Show that

$$\int_0^1 |f(x)|^2\,dx \le 2\,.$$

   (b) Show that $m(V) \le 1$, that $m(V) \le 1/V$, that $m(V) \le 4/V^3$, and that $m(V) \le 2/V^2$ for all $V > 0$.

   (c) Show that $m(V) \le \min(1, 1/V, 4/V^3) \le 2/V^2$ for all $V > 0$.

Exer:M(alpha,4T) 2. HLM71 (Montgomery, 1971) Let α be fixed, $0 < \alpha < 1$, and set

$$M(\alpha,T) = \max_{\substack{\sigma\ge\alpha\\ 0\le t\le T\\ |s-1|\ge 1}} |\zeta(s)|\,. \qquad (27.56)$$ E:DefM(alpha,T)

   (a) Let $w_n = \max(0, 2 - n/N)$. With $W(s)$ defined as in the proof of Theorem 27.1, show that if $|t| \le T$, then $W(it) \ll M(\alpha, 4T)N^\alpha + N/\tau^2$.

   (b) Let $D(s)$ be a Dirichlet polynomial as in (27.1), and let $t_1, t_2, \ldots, t_R$ be real numbers in the interval $[0,T]$ such that $|t_{r_1} - t_{r_2}| \ge 1$ whenever $r_1 \ne r_2$. Show that if G is defined as in (27.11), then

$$\sum_{r=1}^R |D(it_r)|^2 \ll \big(N + M(\alpha, 4T)N^\alpha R\big)G\,.$$

   (c) Show that there is a constant $C = C(\alpha) > 0$ such that if $V^2 \ge CM(\alpha, 4T)N^\alpha$ and $|D(it_r)| \ge V$ for all r, then (27.10) holds.

3. (a) Show that when Bourgain's Example is inserted into the bilinear form inequality (27.2), the conclusion is that $\Delta^2 \gg RH$. By duality, this must hold also in (27.3). Our object now is to give a direct proof of this latter inequality.
   (b) Let $\mathcal{T}_\delta$ be defined as in Bourgain's Example. Show that $\mathcal{T}_\delta - \mathcal{T}_\delta = \mathcal{T}_\delta + \mathcal{T}_\delta \subseteq \mathcal{T}_{2\delta}$. (By definition, $\mathcal{A} + \mathcal{B} = \{a + b : a \in \mathcal{A}, b \in \mathcal{B}\}$.)
   (c) Note that the left hand side of (27.3) is

$$\begin{aligned} &\geq \sum_{N-H<n\leq N} \Big|\sum_{r=1}^{R} y_r n^{-it_r}\Big|^2 \\ &= \sum_{1\leq r_1,r_2\leq R} y_{r_1}\overline{y_{r_2}} \sum_{N-H<n\leq N} n^{-i(t_{r_1}-t_{r_2})}\,. \end{aligned}$$

   (d) Explain why this last sum over n is $= HN^{-i(t_{r_1}-t_{r_2})}(1 + O(\delta))$.
   (e) Take $y_r = N^{it_r}$ for all r. Explain why the lower expression displayed above is $\gg HR^2$, while the right hand side of (27.3) is $= \Delta^2 R$. Deduce that $\Delta^2 \gg HR$.
4. We now develop a variant of Halász's Method. Let $D(s)$, $\mathcal{T}$, and G be defined as in Theorem 27.1. Let $t_1, t_2, \ldots, t_R$ denote the members of $\mathcal{T}$, and let y_r be a unimodular number with the property that $|D(it_r)| = y_r D(it_r)$.
   (a) Show that

$$\sum_{r=1}^{R} |D(it_r)| = \sum_{n=1}^{N} a_n \sum_{r=1}^{R} y_r n^{-it_r}\,.$$

   (b) Deduce that

$$\Big(\sum_{r=1}^{R} |D(it_r)|\Big)^2 \leq G\Big(\sum_{n=1}^{N}\Big|\sum_{r=1}^{R} y_r n^{-it_r}\Big|^2\Big)\,.$$

   (c) Suppose that $w_1, w_2, \ldots$ are nonnegative numbers such that $w_n \geq 1$ for $1 \leq n \leq N$, and $\sum_{n=1}^{\infty} w_n < \infty$. Put $W(s) = \sum_{n=1}^{\infty} w_n n^{-s}$ for $\sigma \geq 0$. Explain why the second factor on the right hand side above is

$$\leq \sum_{1\leq r_1,r_2\leq R} \big|W\big(i(t_{r_1} - t_{r_2})\big)\big|\,.$$

   (d) Conclude that

$$\Big(\sum_{r=1}^{R} |D(it_r)|\Big)^2 \leq G \sum_{1\leq r_1,r_2\leq R} \big|W\big(i(t_{r_1} - t_{r_2})\big)\big|\,.$$

(e) Suppose that $|D(it_r)| \ge V$ for all r. Deduce that

$$R \le \frac{W(0)G}{V^2} + \frac{G}{V^2 R} \sum_{\substack{1 \le r_i \le R \\ r_1 \ne r_2}} \left|W\bigl(i(t_{r_1} - t_{r_2})\bigr)\right| .$$

(f) Note that

$$\frac{1}{R} \sum_{\substack{1 \le r_i \le R \\ r_1 \ne r_2}} \left|W\bigl(i(t_{r_1} - t_{r_2})\bigr)\right| \le \max_{1 \le r_1 \le R} \sum_{\substack{1 \le r_2 \le R \\ r_2 \ne r_1}} \left|W\bigl(i(t_{r_1} - t_{r_2})\bigr)\right| .$$

When combined with the bound from the preceding part, we obtain the same bound that we achieved from our original method. In the case that the numbers $t_{r_1} - t_{r_2}$ are distinct and well-spaced, we might be able to derive an upper bound for the left hand side above than the one on the right hand side, and thus obtain a better overall bound for, such t_r.

5. Suppose that $D(s)$ is defined as in (27.1). Suppose that $N^{1/2} \le N \le T$. From Corollary 26.2 and Lemma 27.12 we know that

$$\int_0^T |D(it)|^2 \, dt \ll T \sum_{n=1}^N |a_n|^2,$$

$$\int_0^T |D(it)|^4 \, dt \ll N^2 \Bigl(\sum_{n=1}^N d(n) |a_n|^2 \Bigr)^2 .$$

It would be helpful if we could derive similar estimates for fractional exponents between 2 and 4. In particular, let ν be determined by the equation $N^\nu = T$. Thus $1 \le \nu \le 2$. It would be useful if it were the case that

$$\int_0^T |D(it)|^{2\nu} \, dt \ll T \Bigl(\sum_{n=1}^N d(n) |a_n|^2 \Bigr)^\nu .$$

Suppose that $D(s)$ is taken as in Bourgain's Example, with $H \le \delta N$. Take $T = N^2/H^2$. Show that in the interval $[0, T]$ there are $\asymp N/H^2$ disjoint intervals, each of length $\asymp N/H$ on which $|D(it)| \gg H$. Deduce that the left hand side above is $\gg N^2 H^{2\nu - 3}$ and that the right hand side is $\ll N^2 H^{\nu - 2} (\log N)^\nu$. Note the resulting contradiction.

One of course might conjecture that if $|a_n| \le 1$ for all n, then

$$\int_0^T |D(it)|^{2\nu} \, dt \ll_\varepsilon T^{2+\varepsilon} . \qquad (27.57)$$

E:Conj|D|2nu

6. Let $\mathcal{T}$ be a finite set of real numbers. Show that the following two statements about a number $\Delta(N, \mathcal{T})$ are equivalent:
   (i) If $|a_n| \le 1$ for all $n \le N$, then
$$\sum_{t\in\mathcal{T}} \Big|\sum_{n=1}^{N} a_n n^{-it}\Big| \le \Delta(N, \mathcal{T})\,.$$
   (ii) If $|y_t| \le 1$ for all $t \in \mathcal{T}$, then
$$\sum_{n=1}^{N} \Big|\sum_{t\in\mathcal{T}} y_t n^{-it}\Big| \le \Delta(N, \mathcal{T})\,.$$

7. To prove the identity (27.20), it suffices to note that both sides are linear forms in the a_n, and then to note that for each n the coefficient of a_n is the same on both sides.
   (a) Show that if $n > x$, then a_n makes no contribution to either side of (27.20).
   (b) Show that if $n \le x$, then the contribution of a_n to the right hand side of (27.20) is a_n multiplied by
$$\frac{x-n}{x^\sigma} + 2\sigma\int_n^x (v-n)v^{-\sigma-1} + \sigma(\sigma+1)\int_n^x (v-n)(x-v)v^{-\sigma-2}\,dv\,.$$
   (c) Show that the above expression is equal to $(x-n)n^{-\sigma}$.

Exer:HMGLH

8. The *Generalized Lindelöf Hypothesis* (GLI) asserts that if χ is a character modulo q, then $L(1/2+it,\chi) \ll_\varepsilon (q\tau)^\varepsilon$. If $L(s,\chi)$ is an L-function whose nontrivial zeros all lie on the 1/2-line, then it also satisfies this estimate.
   (a) Adopt the notation and hypotheses of Theorem 27.6. Suppose, in addition, that GLI is valid for all L-functions mod q. Show that
$$\sum_{r=1}^{R} |D(s_r,\chi_r)|^2 \ll \big(N + C(\varepsilon)RN^{1/2}(qT)^\varepsilon\big)G\,.$$
   (b) Adopt the notation and hypotheses of Theorem 27.7. Suppose, in addtion, that GLI is valid for all L-functions. Show that
$$\sum_{r=1}^{R} |D(s_r,\chi_r)|^2 \ll \big(N + C(\varepsilon)RN^{1/2}(QT)^\varepsilon\big)G\,.$$

DRHB79a
9. (?) The q-analgue of Halász's bound is a bilinear form inequality of the form

$$\sum_{r=1}^{R}\Big|\sum_{n=1}^{N} a_n\chi_r(n)\Big|^2 \le \Delta^2 \sum_{n=1}^{N} |a_n|^2 .$$

Our object is to establish an analogue of Bourgain's Example for the bilinear form above. Let $p_1, p_2, \ldots, p_R$ denote the first R odd primes. Let χ_r denote the quadratic character modulo p_r. Take $a_n = 1$ if n is a square, and $a_n = 0$ otherwise.

(a) Show that the left hand side above is $\asymp NR$.

(b) Show that the right hand side above is $\asymp \Delta^2 N^{1/2}$.

(c) Deduce that $\Delta^2 \gg RN^{1/2}$. Compare this with the result of Exercise 8(a).

Exer:gam(s)est3
10. (a) Show that $\arctan\delta = \delta - \frac{1}{3}\delta^3 + O\big(\delta^5\big)$ for $0 \le \delta \le 1$.

(b) Show that $\log(1+\delta) = \delta + O\big(\delta^2\big)$ for $0 \le \delta \le 1$.

(c) Deduce that there is a small positive constant c such that

$$|\gamma(s,\chi)| \asymp \Big(\frac{t}{2\pi}\Big)^{\frac12-\sigma} \exp\Big(\frac{(1-\sigma)^3}{8t^2}\Big)$$

uniformly for $t^{2/3} \le -\sigma \le ct$.

11. (a) Show that if $s = \sigma + it$, then

$$\sin\pi s = (\sin\pi\sigma)\cosh\pi t + i(cos\pi\sigma)\sinh\pi t.$$

(b) Let $f(s) = \exp(\sin\pi s)$. Show that $|f(it)| = |f(1+it)| = 1$ for all t.

(c) Show that $f\big(\frac12 + it\big) = \exp(\cosh\pi t) > \exp\big(\frac12\exp(\pi|t|)\big)$.

(d) One of the hypotheses in the Phragmén–Lindelöf Theorem is that $0 < \alpha < \pi$. Show that this constraint on α cannot be relaxed.

12. (a) For a positive integer k, let $f(x) = \frac{1}{k!}\max\big(0,(1-x)^k\big)$. Let $F(s) = \int_0^\infty f(x)x^{s-1}\,dx$ be the Mellin transform of f. By induction on k, show that the integral that defines F converges for $\sigma > 0$, and that in that halfplane

$$F(s) = \frac{1}{s(s+1)\cdots(s+k)}.$$

(b) For a positive integer k let $F(s)$ be defined as above, and let $f(x)$ be the inverse Mellin transform of F,

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)x^{-s}\,ds$$

where $c > 0$. Show that $f(x) = 0$ if $x \ge 1$, and that $f(x) = \frac{1}{k!}(1-x)^k$ if $0 \le x \le 1$.

13. Let $\kappa(u)$ be defined as in (**??**).

(a) Show that

$$\kappa(u) = \sum_{\substack{0\le j\le 4\\ 2u-1\ge j}} (-1)^j \binom{4}{j}(2u-j-1)^3.$$

(b) By swapping j with $4-j$, show that

$$\kappa(u) = \sum_{\substack{0\le j\le 4\\ 5-2u\le j}} (-1^j \binom{4}{j}(2u+j-5)^3.$$

(c) Deduce that $\kappa(3-k) = \kappa(u)$.
(d) Show that $\kappa(u) = (2u-1)^3$ for $1/2 \le u \le 1$.
(e) Show that $\kappa(u) = 31 - 90u + 84u^2 - 24u^3$ for $1 \le u \le 3/2$.
(f) Show that $\kappa(u) = -131 + 234u - 132u^2 + 24u^3$ for $3/2 \le u \le 2$.
(g) Show that $\kappa(u) = (5-2u)^3$ for $2 \le u \le 5/2$.
(h) Show that $\kappa(1) = \kappa(2) = 1$.
(i) Show that $\kappa'(u)6(3-2u)(6u-5) > 0$ for $1 \le u < 3/2$.

14. Suppose that $D_i(s) = \sum_{k_i=1}^{K_i} a_i(k_i)k_i^{-s}$ for $i = 1, 2, 3$, and that $D(s) = D_1(s)D_2(s)D_3(s) = \sum_{n=1}^{K_1K_2K_3} c(n)n^{-s}$.

(a) Show that

$$c(n) = \sum_{\substack{k_1,k_2,k_3\\ k_1k_2k_3=n}} a_1(k_1)a_2(k_2)a_3(k_3).$$

(b) Show that

$$|c(n)|^2 \le d_3(n) \sum_{\substack{k_1,k_2,k_3\\ k_1k_2k_3=n}} |a_1(k_1)|^2|a_2k_2|^2|a_3(k_3)|^2.$$

(c) Show that if $k_1k_2k_3 = n$, then $d_3(n) \le d)3(k_1)d_3(k_2)d_3(k_3)$.

(d) Deduce that

$$\sum_{n=1}^{K_1K_2K_3} |c(n)|^2 \le \prod_{i=1}^{3}\Bigg(\sum_{k_i=1}^{K_i} d_3(k_i)|a_i(k_i)|^2\Bigg).$$

S:LargeVals Notes

27.1 Notes

For an account of the Phragmń–Lindelöf Theoem see, for example, Theorem 5.1.9 in BS2A Simon (2015).

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28
Zero Density Theorems

C:ZDT

28.1 Zero counting functions

S:ZDT ZCF

There are many situations where, in lieu of the Riemann Hypothesis, an unconditional conclusion can be achieved *via* a bound for one of the functions

$$N(\sigma, T) = \text{card}\{\rho = \beta + i\gamma : \zeta(\rho) = 0, \beta \geq \sigma, |\gamma| \leq T\}, \qquad (28.1)$$ E:NT

$$N_1(\sigma, q, T) = \sum_{\chi \bmod q} N(\sigma, \chi, T), \qquad (28.2)$$ E:NqT

or

$$N_2(\sigma, Q, T) = \sum_{q \leq Q} \sum_{\chi \bmod q}{}^{*} N(\sigma, \chi, T), \qquad (28.3)$$ E:NQT

where

$$N(\sigma, \chi, T) = \text{card}\{\rho = \beta + i\gamma : L(\rho; \chi^*) = 0, \beta \geq \sigma, |\gamma| \leq T\}, \qquad (28.4)$$ E:NchiT

χ^* denotes the primitive character inducing χ and $\sum^*$ denotes a sum restricted to primitive characters. The underlying methods for dealing with each are closely related, and we will mostly work out the details for (28.3). Later we will see that in some aspects the theory for (28.1) can be pushed a bit further. It is also possible to cover all possible bases by considering

$$N(\sigma, q, Q, T) = \sum_{\substack{k \leq Q \\ (k,q)=1}} \sum_{\chi \bmod qk}{}^{\dagger} N(\sigma, \chi, T), \qquad (28.5)$$ E:NqQT

where $\sum^\dagger$ indicates a sum over primitive characters with conductor dk and $d|q$.

In earlier chapters estimates for the number of zeros of an analytic function of interest in a given region have depended upon standard theorems from classical complex analysis. To make progress here more sophisticated methods are required which depend on the special properties of L-functions.

When $\sigma > 1$ the lack of zeros is an immediate consequence of the Euler product. It can also be seen through the observation that

$$L(s;\chi)^{-1} = \sum_{n=1}^{\infty} \frac{\mu(n)\chi(n)}{n^s} \tag{28.6}$$

E:L-1

converges when $\sigma > 1$ and so is analytic in that half-plane. In particular

$$L(s;\chi)L(s;\chi)^{-1} = 1.$$

When $\sigma \le 1$ we do not have the luxury of immediately knowing that (28.6) converges. However if we write

$$M(s;\chi) = \sum_{n \le K} \mu(n)\chi(n)n^{-s}, \tag{28.7}$$

E:MSchi

then we can have some expectation that for a suitable parameter K the expression

$$L(s;\chi)M(s;\chi) - 1 \tag{28.8}$$

E:LM-1

is small most of the time. Such a function M is often termed a *mollifier*. On the other hand if $L(s;\chi) = 0$, then the expression is -1 and so the expectation is that this happens at most infrequently.

In order to quantify this observation we will need to approximate $L(s;\chi)$ by a series of the kind

$$\sum_n \chi(n)n^{-s}w(n)$$

with a weight function $w(n)$ which is close to 1 for smaller n. One such example which we have used in the past is given by

$$e^{-1/x} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^w \Gamma(w)dw$$

where $c > 0$, and its inverse transform

$$\Gamma(w) = \int_0^\infty x^{w-1}e^{-x}dx = \int_0^\infty x^{-w-1}e^{-1/x}dx.$$

Although this is simple and useful, it would be handier if we could move part of our contour well to the left. However the gamma function has the disadvantage of possessing singularities at the negative integers. A small modification removes this deficiency.

E:Melling

28.2 A Mellin transfom

THIS SECTION SHOULD BE IN AN APPENDIX.

Let

$$g(w) = \frac{1}{\varpi w}\int_0^\infty x^{-w-1}\exp(-x-x^{-1})dx \tag{28.9}$$ E:gfn

where

$$\varpi = \int_0^\infty x^{-1}\exp(-x-x^{-1})dx. \tag{28.10}$$ E:resg

Then integral in (28.9) is an entire function, so that the function g is analytic for $w \ne 0$ and has a simple pole at $w = 0$ with residue 1. For $x \ge 0$, let

$$f(x) = \frac{1}{\varpi}\int_x^\infty y^{-1}\exp(-y-y^{-1})dy. \tag{28.11}$$ E:ffn

Then, by integration by parts and a change of variable,

$$g(w) = \int_0^\infty x^{-w-1}f(x)dx = \int_0^\infty y^{w-1}f(1/y)dy. \tag{28.12}$$ E:gintf

A change of variable also shows that

$$f(1/x) = \frac{1}{\varpi}\int_0^x y^{-1}\exp(-y-y^{-1})dy. \tag{28.13}$$ E:f1overx

Thus

$$f(x) + f(1/x) = 1 \tag{28.14}$$ E:f+f

and so

$$f(x) \ll x^{-1}e^{-x}, \quad f(1/x) = 1 + O\big(x^{-1}e^{-x}\big) \tag{28.15}$$ E:f=1+o

The function $g(w)$ behaves in a similar way to the gamma function, but with the added advantage that its only singularity is at 1. Thus we have the following estimate, which is similar to bounds following from Stirling's formula. It is perhaps not the most precise bound that can be established, but it will suffice for our purposes.

T:omegag

Lemma 28.1 *Suppose that $w \in \mathbb{C}$, $\operatorname{Re} w = u$ and $\operatorname{Im} w = v$. Then there is a constant $C \ge 1$ such that*

$$wg(w) \ll (C + C|w|)^{\max(1,|u|)}\exp(-\pi|v|/2).$$

Proof If necessary by considering the complex conjugate we may suppose that $v \geq 0$. Let $\delta = (1+|u|)/(2+|w|)$, $R > 0$ and $\mathcal{C}_R$ denote the contour consisting of the line segments joining

$$\{-R, R, R+i(\pi/2-\delta), -R+i(\pi/2-\delta), -R\}.$$

Then, by Cauchy's theorem, we have

$$\int_{\mathcal{C}_R} \exp\big(wz - e^z - e^{-z}\big)dz = 0.$$

Moroever

$$\int_{\pm R}^{\pm R+i(\pi/2-\delta)} \exp\big(wz - e^z - e^{-z}\big)dz \ll \exp\big(\pm Ru - e^R \sin\delta\big) \to 0 \text{ as } R \to \infty.$$

Thus

$$\begin{aligned} \varpi w g(w) &= \int_0^\infty y^{w-1}\exp(-y-1/y)dy \\ &= \int_{-\infty}^\infty \exp\big(wx - e^x - e^{-x}\big)dx \\ &= \int_{-\infty}^\infty \exp\big(wx + w(i\tfrac{\pi}{2} - i\delta) - 2\cosh(x + i\tfrac{\pi}{2} - i\delta)\big)dx. \end{aligned}$$

Hence

$$\varpi|wg(w)| \leq \int_{-\infty}^\infty \exp\big(ux - v\pi/2 + v\delta - e^x\sin\delta - e^{-x}\sin\delta\big)dx$$

Therefore

$$\varpi|wg(w)| \leq 2\exp(-v\pi/2 + v\delta)\int_0^\infty \exp(|u|x - e^x\sin\delta)dx.$$

The integral here is

$$\int_1^\infty t^{|u|-1}e^{-t\sin\delta}dt. \qquad (28.16)$$

E:sindelta

When $|u| \geq 1$ it is

$$\leq (\sin\delta)^{-|u|}\Gamma(|u|) \ll \left(\frac{|u|}{e\sin\delta}\right)^{|u|}|u|^{-1/2}$$

and

$$\sin\delta = \delta - \int_0^\delta\int_0^\theta \sin\beta d\beta d\theta \geq \delta - \frac{1}{2}\delta^2\sin\delta$$

so that

$$\sin\delta \ge \frac{2\delta}{2+\delta^2}.$$

Also $v\delta \le 1+|u|$. Hence

$$e^{v\delta}\left(\frac{|u|}{e\sin\delta}\right)^{|u|} \ll \left(\frac{|u|(2+\delta^2)}{\delta}\right)^{|u|} \ll (C+C|w|)^{|u|}$$

as desired.

When $|u| \le 1$ the integral (28.16) is

$$\le \frac{1}{\sin\delta}$$

and we have

$$e^{v\delta}(\sin\delta)^{-1} \ll 1+|w|$$

which once more suffices. □

We also need to know more about the inverse Mellin transform of g.

T:f1X **Theorem 28.2** *Suppose that $X>0$ and $\nu>0$. Then*

$$\frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty} X^w g(w)dw = f(1/X)$$

Proof By (28.9) and the lemma, the integral above is

$$\frac{1}{\varpi}\int_0^\infty x^{-1}\exp(-x-x^{-1})\frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty}\frac{(X/x)^w}{w}dwdx.$$

The inner integral is $2\pi i$ when $x<X$ and 0 when $x>X$. Thus the above is

$$\frac{1}{\varpi}\int_0^X x^{-1}\exp(-x-x^{-1})dx = f(1/X)$$

by (28.13) □

28.3 A bound for $L(s;\chi)$

THIS SECTION SHOULD BE IN CHAPTER 22.

Let χ be a primitive character modulo q and suppose that $\frac{1}{2}\le\sigma\le 1$ and $Y>0$. We need an effective upper bound for $L(s;\chi)$ in terms of Dirichlet polynomials on the σ-line. We start by imitating the process used to establish approximate functional equations.

When $c = \operatorname{Re} w > \sigma$, by Theorem 28.2 we have

$$\sum_{m=1}^{\infty} \frac{\chi(m)}{m^s} f(m/Y) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} L\big(s+w;\chi\big) Y^w g(w)dw$$

We now move the path to the line $\operatorname{Re} w = -1$, picking up the residue at 0, and another one at $w = 1-s$ when $q = 1$ and apply the functional equation for $L(s;\chi)$ (Corollary 10.9). Thus the above is

$$L(s;\chi) + E(\chi)x^{1-s}g(1-s) + \frac{1}{2\pi i}\int_{-1-i\infty}^{-1+i\infty} \psi(s+w;\chi)L\big(1-s-w;\overline{\chi}\big)Y^w g(w)dw \quad (28.17)$$ eq:E:Lapprox1

where

$$\psi(z;\chi) = \epsilon(\chi)2^z\pi^{z-1}q^{1/2-z}\Gamma(1-z)\sin\big(\pi(z+\kappa)/2\big). \quad (28.18)$$ E:fepsi

and $E(\chi) = 0$ unless $q = 1$ in which case $E(\chi) = 1$.

We apply Stirling's formula, Theorem C.1, (C.18), to bound ψ. When $\operatorname{Re} w \le -1$ and $\frac{1}{2} \le \sigma \le 1$ we have $|1-s-w| \ge 1$ and $|\arg(1-s-w)| \le \pi/2$ so that uniformly for such $s+w$ we have

$$\psi(s+w;\chi) \ll (3q + q|s+w|)^{\frac{1}{2}-\sigma-u}. \quad (28.19)$$ eq:E:psibound

where $w = u + iv$.

The third term in (28.17) is

$$\sum_{m=1}^{\infty} \overline{\chi}(m)m^{s-1}h(mY;s,\chi) \quad (28.20)$$ E:sumhnY

where, for $V > 0$,

$$h(V;s,\chi) = \frac{1}{2\pi i}\int_{-1-i\infty}^{-1+i\infty} \psi(s+w;\chi)V^w g(w)dw. \quad (28.21)$$ E:hVschi

Let $\lambda \ge 3$ be a parameter at our disposal. We move the contour to the vertical line $u = -\lambda$. Then, by Lemma 28.1, for a suitable constant $c_1 \ge 1$ the integrand is

$$\ll (4q + q|t| + q\lambda + q|v|)^{\frac{1}{2}-\sigma+\lambda}V^{-\lambda}(c_1 + c_1\lambda + c_1|v|)^{\lambda}e^{-\frac{\pi|v|}{2}}.$$

We suppose henceforward that

$$|t| \le T$$

where $T \ge 3$. Then the integrand is

$$\ll P^{\lambda}V^{-\lambda}(\lambda + |v|)^{2\lambda}e^{-\pi|v|/2}$$

where

$$P \geq CqT \tag{28.22}$$ eq:E:psibound1

and C is a suitably large absolute constant. Hence

$$h(V; s, \chi) \ll P^\lambda V^{-\lambda} \lambda^{5\lambda}.$$

Suppose that $V \geq (3e)^5 P$. Then the function of a positive real variable ν,

$$F(\nu) = P^\nu V^{-\nu} \nu^{5\nu} = \exp(5\nu \log \nu - \nu \log(V/P))$$

has a minimum when $\nu = \nu_0$ where

$$\nu_0 = e^{-1}(V/P)^{1/5} \geq 3$$

and

$$F(\nu_0) = \exp(5\nu_0) = \exp\big(-5e^{-1}(V/P)^{1/5}\big).$$

Let $\lambda = \nu_0$. Then

$$h(V; s, \chi) \ll \exp\big(-5e^{-1}(V/P)^{1/5}\big). \tag{28.23}$$ E:hVschi1

Suppose that $Y \leq P/\log P$ and $Z \geq 2PY^{-1}(3e \log P)^5$. The function

$$\nu(\theta) = 5e^{-1}(\theta Y/P)^{1/5} - 2\log\theta$$

is increasing for $\theta \geq PY^{-1}(2e)^5$ and so if $m > Z$, then we have

$$\begin{aligned} \nu(m) &> \nu(Z) \\ &\geq 15.2^{1/5} \log P - 2\log\left(\frac{2P}{Y}(3e\log P)^5\right) \\ &> 3\log P. \end{aligned}$$

Recall that $P > C$ is suitably large. Thus when $m > Z$ we have

$$5e^{-1}(mY/P)^{1/5} > 3\log P + 2\log m.$$

Hence, by (28.23),

$$\begin{aligned} \sum_{m>Z} \overline{\chi}(m) m^{s-1} h(mY; s, \chi) \ll \sum_{m>Z} &\exp\big(-5e^{-1}(mY/P)^{1/5}\big) \\ \sum_{m>Z} &m^{-2} \exp(-\nu(m)) \end{aligned}$$

so that

$$\sum_{m>Z} \overline{\chi}(m) m^{s-1} h(mY; s, \chi) \ll P^{-3}. \tag{28.24}$$ E:sumhV

We now rewrite the contribution from $m \le Z$ as

$$\sum_{m\le Z} \overline{\chi}(m)n^{s-1}h(mY;s,\chi) = \frac{1}{2\pi i}\int_{-1-i\infty}^{-1+i\infty} \sum_{m\le Z} \overline{\chi}(m)m^{s+w-1}\psi(s+w;\chi)Y^w g(w)dw.$$

When $\sigma \ge \theta > \frac{1}{2}$ we then move the path to the line $\operatorname{Re} w = \theta + \frac{1}{2} - 2\sigma$. Thus

$$-1 < \theta - \frac{3}{2} \le \operatorname{Re} w \le \frac{1}{2} - \theta < 0,$$

and to summarize so far, we have

$$\sum_{m=1}^{\infty} \frac{\chi(m)}{m^s} f(m/Y) = L(s;\chi) + E(\chi)Y^{1-s}g(1-s) + \frac{1}{2\pi i}\int_{\theta-\frac{1}{2}-i\infty}^{\theta-\frac{1}{2}+i\infty} \sum_{m\le Z} \frac{\overline{\chi}(m)}{m^{\overline{s}-w}}\psi(1-\overline{s}+w;\chi)Y^{1-2\sigma+w}g(1-2\sigma+w)dw + O(P^{-3}). \quad (28.25)$$ E:sumfm/Y

Suppose that $X \ge 3Y\log(YP)$. Then, by (28.15)

$$\sum_{m>X} \chi(m)m^{-1/2-it}f(m/Y) \ll Y(YP)^{-3} \ll P^{-3}. \quad (28.26)$$ E:sumfm>X

Thus we obtain

$$L(s;\chi) = \sum_{m\le X} \frac{\chi(m)}{m^s} f(m/Y) - E(\chi)Y^{1-s}g(1-s) - \frac{1}{2\pi i}\int_{\theta-\frac{1}{2}-i\infty}^{\theta-\frac{1}{2}+i\infty} \sum_{m\le Z} \frac{\overline{\chi}(m)}{m^{\overline{s}-w}}\psi(1-\overline{s}+w;\chi)Y^{1-2\sigma+w}g(1-2\sigma+w)dw + O(P^{-3}).$$

If $E(\chi) \ne 0$, then $q = 1$. Thus, by Lemma 28.1

$$E(\chi)Y^{1-s}g(1-s) \ll Y^{1-\sigma}|1-s|^{-1}(1+|s|)\exp(-\pi|t|/2).$$

Thus we have

$$L(s;\chi) = \sum_{m\le X} \frac{\chi(m)}{m^s} f(m/Y) -$$

$$\frac{1}{2\pi i}\int_{\theta-\frac12-i\infty}^{\theta-\frac12+i\infty} \sum_{m\le Z} \frac{\overline{\chi}(m)}{m^{\overline{s}-w}} \psi(1-\overline{s}+w;\chi) Y^{1-2\sigma+w} g(1-2\sigma+w)\,dw$$

$$+ O\left(P^{-3} + E(\chi)Y^{1-\sigma}|1-s|^{-1}e^{-|t|}\right). \quad (28.27)$$ E:sumfm<X

When $\operatorname{Re} w = \theta - \frac12$ we also have

$$\psi(1-\overline{s}+w;\chi)Y^{1-2\sigma+w}g(1-2\sigma+w)$$

$$\ll \left|\theta - \frac12\right|^{-1} Y^{\theta+\frac12-2\sigma}P^{\theta-\sigma}e^{-|v|}.$$

Putting it all together we have the following.

T:Lapprox **Theorem 28.3** *There is a constant $C \ge 3$ such that whenever $T \ge 2$, χ is a primitive character modulo q, $P \ge CqT$,*

$$Y \le P/\log P, \quad X \ge 3Y\log(YP) \text{ and } Z \ge 2PY^{-1}(3e\log P)^5$$

we have

$$L(s;\chi) - \sum_{m\le X}\frac{\chi(m)}{m^s}f(m/Y) \ll$$

$$Y^{\frac12-\theta}(Y^2P)^{\theta-\sigma}\left|\theta-\frac12\right|^{-1}\int_{-\infty}^{\infty}\left|\sum_{m\le Z}\chi(m)m^{-s+\theta-\frac12+iv}\right|e^{-|v|}dv$$

$$+ E(\chi)Y^{\theta+\frac12-2\sigma}|1-s|^{-1}\exp(-|t|) + P^{-3}$$

uniformly for $|t| \le T$, $\frac12 < \theta \le \sigma \le 1$, $E(\chi)s \ne 1$.

28.4 Zero density estimates

S:ZDE1

We can now apply the results in the previous sections to give non-trivial bounds for the functions N, N_1, N_2. The basic bounds we give here have many applications.

T:ZDE1a **Theorem 28.4** *Suppose $q \ge 1$ and $T \ge 1$. Then*

$$N_1(\theta,q,T) \ll (qT)^{\frac{3(1-\theta)}{2-\theta}}(\log(2qT))^6 \quad (28.28)$$ E:ZDE1a

uniformly when $\frac{1}{2} \le \theta \le 1$.

Suppose further that $\eta > 0$. *Then there is a positive number* $C(\eta)$ *such that*

$$N_1(\theta, q, T) \ll_\eta \big(\log(2qT)\big)^{C(\eta)} \big((qT)^{(2+\eta)(1-\theta)} + (qT)^{\frac{3(1-\theta)}{3\theta-1}}\big) \quad (28.29)$$ E:ZDE2a

uniformly when $\frac{2}{3} \le \theta \le 1$,

T:ZDE1b **Corollary 28.5** *There is a positive constant* c *such that, whenever* $\frac{1}{2} \le \theta \le 1$,

$$N_1(\theta, q, T) \ll (qT)^{\frac{12}{5}(1-\theta)}(\log 2qT)^c.$$

T:ZDE1 **Theorem 28.6** *Suppose* $q \ge 1$ *and* $T \ge 1$. *Then*

$$N_2(\theta, Q, T) \ll (Q^2T)^{\frac{3(1-\theta)}{2-\theta}}(\log(2QT))^6 \quad (28.30)$$ E:ZDE1

uniformly when $\frac{1}{2} \le \theta \le 1$.

Suppose further that $\eta > 0$. *Then there is a positive number* $C(\eta)$ *such that*

$$N_2(\theta, Q, T) \ll_\eta (\log(2QT))^{C(\eta)}\big((QT)^{(2+\eta)(1-\theta)} + (Q^2T)^{\frac{3(1-\theta)}{3\theta-1}}\big) \quad (28.31)$$ E:ZDE2

uniformly when $\frac{2}{3} \le \theta \le 1$.

T:ZDE1c **Corollary 28.7** *There is a positive constant* c *such that whenever* $\frac{1}{2} \le \theta \le 1$

$$N_2(\theta, Q, T) \ll (Q^2T)^{\frac{12}{5}(1-\theta)}(\log 2QT)^c.$$

The conjecture that the above corollaries hold with 12/5 replaced by ξ for any $\xi > 2$ is known at the density hypothesis. On inspection of the Theorems one can see that this does indeed hold in the restricted range $\theta \ge 5/6$.

We concentrate on the proof of Theorem 28.6. Theorem 28.4 follows in the same way using the concomitant mean value theorems.

By Corollary 14.7,

$$N(1/2, Q, T) \ll Q^2T \log(2QT).$$

Also, by Corollary 11.10 we know that there is a positive constant c such that there is at most one zero $\rho = \beta + i\gamma$ to be counted with $\beta \ge 1 - c/\log 2QT$. Thus in the proof of Theorem 28.6 we may suppose that

$$\frac{1}{2} - \frac{\log\log(2QT)}{\log(2QT)} \le \theta \le 1 - \frac{1}{c\log 2QT}. \quad (28.32)$$ E:ZDEalpha

Similar considerations to the above pertain with regard to Theorem 28.4.

We consider first (28.30) as this has the simplest proof and yet illustrates the main ideas.

28.4.1 The Ingham Bound (28.30)

For brevity write

$$H = CQ^2T \tag{28.33}$$ E:IngD

where C is a large constant, and for

$$C \leq K \leq H \tag{28.34}$$ E:IngK

let $M(s;\chi)$ be as in (28.7), and define

$$a(n) = \sum_{\substack{l|n \\ l\leq K}} \mu(l) \tag{28.35}$$ E:Ingan

so that

$$|a(n)| \leq d(n). \tag{28.36}$$ E:Ingand

Let

$$Y \geq 1 \tag{28.37}$$ E:IngY

and suppose that $\frac{1}{2} \leq \sigma \leq 1$. Then, by (5.25)

$$\sum_{n=1}^{\infty} a(n)\chi(n)n^{-s}e^{-n/Y} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} L(s+w;\chi)M(s+w;\chi)Y^w\Gamma(w)dw.$$

The integrand has singularities at 0, and, when $L(s;\chi) = \zeta(s)$, at $w = 1-s$. In view of the bounds given by Corollaries 10.5 and 10.10, Lemma 10.15 and (C.19) of Theorem C.1 we are able to move the path of integration to the line $\operatorname{Re} w = \frac{1}{2} - \sigma$ and pick up the residues at 0 and, when $L(s;\chi) = \zeta(s)$, at $w = 1-s$. Thus

$$\sum_{n=1}^{\infty} a(n)\chi(n)n^{-s}e^{-n/Y} =$$

$$E(\chi)M(1;\chi)Y^{1-s}\Gamma(1-s) + L(s;\chi)M(s;\chi)+$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} L(\tfrac{1}{2}+it+iv;\chi)M(\tfrac{1}{2}+it+iv;\chi)Y^{\frac{1}{2}-\sigma+iv}\Gamma(\tfrac{1}{2}-\sigma+iv)dv.$$

Let

$$Z = Y(\log H)^2. \tag{28.38}$$ E:IngZ

Then

$$\sum_{n>Z} a(n)\chi(n)n^{-s}e^{-n/Y} \ll H^{-1}.$$

When $E(\chi) = 1$ and $|t| \ge (\log H)^2$, by Stirling's formula (C.19),

$$E(\chi)M(1;\chi)Y^{1-s}\Gamma(1-s) \ll H^{-1}.$$

We also have $a(n) = 0$ when $2 \le n \le K$, and $a(1) = 1$ and $e^{-1/Y} = 1 + O(1/Y)$. Hence

$$1 + \sum_{K<n\le Z} a(n)\chi(n)n^{-s}e^{-n/Y} = L(s;\chi)M(s;\chi) + O(1/H) + \frac{1}{2\pi}\int_{-\infty}^{\infty} L(\tfrac12 + it + iv;\chi)M(\tfrac12 + it + iv;\chi)Y^{\frac12-\sigma+iv}\Gamma(\tfrac12 - \sigma + iv)dv. \tag{28.39}$$ E:ZDELM1

By Corollary 14.3 the number of zeros $\rho = \beta + i\gamma$ of the Riemann zeta function with $|\gamma| \le (\log H)^2$ is $\ll (\log H)^3$ which is an acceptable bound in our theorem. Thus we can exclude such zeros from our subsequent analysis and (28.39) holds for all remaining zeros s. When χ is a primitive character modulo q, let $\mathcal{R}_0(\theta,\chi,T)$ denote the set of zeros ρ of $L(s;\chi)$ with $\beta \ge \theta$, $|\gamma| \le T$, except that when $q = 1$ we exclude the zeros with $|\gamma| \le (\log H)^2$. Then

$$N_2(\theta,Q,T) \le \sum_{q\le Q}\ \sideset{}{^*}\sum_{\chi \bmod q} \operatorname{card} R_0(\theta,\chi,T) + O\big((\log H)^3\big)$$

and for $\rho \in \mathcal{R}_0(\theta,\chi,T)$ we have

$$1 \ll \Big|\sum_{K<n\le Z} a(n)\chi(n)n^{-\rho}e^{-n/Y}\Big| + Y^{\frac12-\beta}\int_{-\infty}^{\infty} |L(\tfrac12 + i\gamma + iv;\chi)M(\tfrac12 + i\gamma + iv;\chi)|e^{-|v|}dv.$$

By Corollary 14.7, given $q \le Q$, a primitive character χ modulo q and $|t| \le T$ there are at most $\ll \log 2QT$ zeros ρ of $L(s;\chi)$ with $|t - \operatorname{Im}\rho| \le 1$. Thus we can partition $\mathcal{R}_0(\theta,\chi,T)$ into $\ll \log H$ subsets in each of which any distinct pair ρ and ρ' of zeros of a given $L(s;\chi)$ satisfy $|\operatorname{Im}(\rho -$

$\rho')| \ge 1$. Let $\mathcal{R}(\theta,\chi,T)$ denote such a subset with the largest number of elements. Thus

$$N_2(\theta,Q,T) \ll (\log H)\sum_{q\le Q}\sideset{}{^*}\sum_{\chi \bmod q} \operatorname{card}\mathcal{R}(\theta,\chi,T) + O\big((\log H)^3\big). \quad (28.40)$$ E:NQTlog

For each $\rho \in \mathcal{R}(\theta,Q,T)$ at least one of the following holds

$$1 \ll \Big|\sum_{K<n\le Z} a(n)\chi(n)n^{-\rho}e^{-n/Y}\Big|^2,$$

$$1 \ll Y^{\frac{2}{3}-\frac{4}{3}\theta}\int_{-\infty}^{\infty} |L(\tfrac12+i\gamma+iv;\chi)M(\tfrac12+i\gamma+iv;\chi)|^{4/3}e^{-|v|}dv.$$

The number of zeros in the first case is

$$\ll \sum_{q\le Q}\sideset{}{^*}\sum_{\chi \bmod q}\sum_{\rho\in\mathcal{R}(\theta,\chi,T)}\Big|\sum_{K<n\le Z} a(n)\chi(n)n^{-\rho}e^{-n/Y}\Big|^2$$

and in the second is $\ll$

$$Y^{\frac{2}{3}-\frac{4}{3}\theta}\int_{-\infty}^{\infty}\sum_{q\le Q}\sideset{}{^*}\sum_{\chi \bmod q}\sum_{\rho\in\mathcal{R}(\theta,\chi,T)} |L(\tfrac12+i\gamma+iv;\chi)M(\tfrac12+i\gamma+iv;\chi)|^{\frac43}\tfrac{dv}{e^{|v|}}.$$

By (26.28) the first expression is

$$\begin{aligned}
&\ll (\log H)\sum_{K<n\le Z} a(n)^2 n^{-2\theta}e^{-2n/Y}(n+H)\\
&\ll (\log H)^4\big((1-\theta)^{-1}Y^{2-2\theta}+(2\theta-1)^{-1}HK^{1-2\theta}\big)\\
&\ll (\log H)^5\big(Y^{2-2\theta}+HK^{1-2\theta}\big)
\end{aligned}$$

and, by Hölder's inequality and Corollary 26.31 and (26.28), the second is

$$\begin{aligned}
&\ll Y^{\frac{2}{3}-\frac{4}{3}\theta}\int_{-\infty}^{\infty}\Bigg(\sum_{q\le Q}\sideset{}{^*}\sum_{\chi \bmod q}\sum_{\rho\in\mathcal{R}(\theta,\chi,T)} |L(\tfrac12+i\gamma+iv;\chi)|^4\Bigg)^{1/3}\times\\
&\qquad\Bigg(\sum_{q\le Q}\sideset{}{^*}\sum_{\chi \bmod q}\sum_{\rho\in\mathcal{R}(\theta,\chi,T)} |M(\frac12+i\gamma+iv;\chi)|^2\Bigg)^{2/3} e^{-|v|}dv\\
&\ll Y^{\frac{2}{3}-\frac{4}{3}\theta}(\log H)^{5/3}H^{1/3}\Bigg((\log H)\sum_{n\le K} n^{-1}(n+H)\Bigg)^{2/3}\\
&\ll Y^{\frac{2}{3}-\frac{4}{3}\theta}(\log H)^3H^{1/3}(K+H)^{2/3}.
\end{aligned}$$

The choices

$$K = H, \quad Y = H^{\frac{3}{4-2\theta}}$$

with (28.40) secure the required bound (28.28).

28.4.2 The Huxley Bound (28.31)

We now concentrate on (28.31). Since the bound in (28.30) is smaller when $\theta \le 3/4$ we may suppose that

$$\theta \ge 3/4. \qquad (28.41)$$ E:ZDEalphalb

In (28.31) the first term on the right only comes into play when, essentially, $\theta > \frac{5}{6}$.

Define H as before and

$$P = CQT \qquad (28.42)$$ E:ZDE31

where C is as in Theorem 28.3, and in that theorem take

$$X = Z = C_1 P^{1/2}(\log P)^3, Y = P^{1/2}(\log P)^2 \qquad (28.43)$$ E:ZDEXYU

where C_1 is a sufficiently large constant.

We may certainly suppose that

$$0 < \eta < 1/4, \quad K = P^{\eta/5} \qquad (28.44)$$ E:HuxX

and we use the same mollifier (28.7) as before.

When $\sigma \ge \theta$ and $|t| \le T$, by Theorem 28.3 and (28.41),

$$L(s;\chi)M(s;\chi) - \sum_{n \le KX} a(n)\chi(n)n^{-s} \ll$$
$$E(\chi)Y^{\frac{1}{2}-\theta}K^{1-\theta}(\log K)|1-s|^{-1}\exp(-|t|)+$$
$$Y^{\frac{1}{2}-\theta}\int_{-\infty}^{\infty}\left|\sum_{n \le KX} b(n;v)\chi(n)n^{-s}\right| e^{-|v|}dv + P^{-1}$$

where

$$a(n) = \sum_{\substack{l \le K, m \le X \\ lm=n}} \mu(l) f(m/Y) \qquad (28.45)$$ E:ZDEan

and

$$b(n;v) = \sum_{\substack{l \le K, m \le X \\ lm=n}} \mu(l) m^{\theta-\frac{1}{2}+iv}. \qquad (28.46)$$ E:ZDEbn

By (28.15) we have $f(m/Y) = 1 + O\big(mY^{-1}\exp(-Y/m)\big)$. By (28.43) and (28.44) we have $K \le Y(\log P)^{-2} < X$ and, when $n \le K$,

$$\begin{aligned} a(n) &= \sum_{lm=n} \mu(l) f(m/Y) \\ &= \sum_{lm=n} \mu(l) + O\big(d(n)\exp(-(\log P)^2)\big). \end{aligned}$$

The main term here is 0 unless $n = 1$ in which case it is 1. Thus

$$\begin{aligned} &L(s;\chi)M(s;\chi) - 1 \ll \\ &\qquad \left| \sum_{K<n\le KX} a(n)\chi(n)n^{-s} \right| + H^{-1} + \\ &\qquad E(\chi)Y^{\frac12-\theta}K^{1-\theta}(\log K)|1-s|^{-1}\exp(-|t|) + \\ &\qquad\qquad Y^{\frac12-\theta}\int_{-\infty}^{\infty} \left| \sum_{n\le KX} \chi(n)n^{-s}b(n;v) \right| e^{-|v|}dv \end{aligned}$$

When $n \le K$, since $\sigma \ge \theta$, we have, by (28.46),

$$\begin{aligned} \sum_{n\le K} \chi(n)n^{-s}b(n;v) &\ll \sum_{l\le K} l^{-\sigma} \sum_{m\le K/l} m^{\theta-\sigma-\frac12} \\ &\ll \sum_{l\le K} l^{-\theta} \sum_{m\le K/l} m^{-\frac12}. \end{aligned}$$

Thus

$$\sum_{n\le K} \chi(n)n^{-s}b(n;v) \ll \sum_{l\le K} l^{-\frac12-\theta}K^{\frac12} \ll K^{\frac12}\log 2K.$$

By (28.41) and (28.43) we have

$$(X/Y)^{\theta-1/2}(\log H)^{-1/2} \ll 1.$$

Hence

$$\begin{aligned} L(s;\chi)M(s;\chi) - 1 &\ll \left| \sum_{K<n\le KX} a(n)\chi(n)n^{-s} \right| + P^{-1} \\ &\quad + Y^{\frac12-\theta}K^{\frac12}(\log K)\left(1 + E(\chi)|1-s|^{-1}\right) \\ &\qquad + \int_{-\infty}^{\infty} \left| \sum_{K<n\le KX} b^*(n;v)\chi(n)n^{-s} \right| e^{-|v|}dv \end{aligned}$$

where

$$b^*(n;v) = (\log H)^{1/2} \sum_{\substack{l \le K, m \le X \\ lm = n}} \mu(l)(m/Z)^{\theta - \frac{1}{2} + iv} \tag{28.47}$$

E:ZDEb*n

and we have used (28.43).

We record for future use that

$$a(n) \ll d(n), \quad b^*(n;v) \ll (\log P)^{1/2} d(n). \tag{28.48}$$

E:ZDEab

Now we suppose that if $q = 1$, then $|1 - s| \gg 1$, which we are certainly entitled to do when s is a zero since in this case $L(s;\chi) = \zeta(s)$. Hence

$$\begin{aligned} L(s;\chi)M(s;\chi) - 1 \ll & \left| \sum_{K < n \le KX} a(n)\chi(n)n^{-s} \right| + H^{-1} \\ & + Y^{\frac{1}{2} - \theta} K^{\frac{1}{2}} \log P \\ & + \int_{-\infty}^{\infty} \left| \sum_{K < n \le KX} b^*(n;v)\chi(n)n^{-s} \right| e^{-|v|} dv \end{aligned}$$

By (28.41) and (28.43) we have

$$Y^{\frac{1}{2} - \theta} K^{\frac{1}{2}} \log P \ll (\log P)^{-1}.$$

Therefore, if $\rho = \beta + i\gamma$ is a zero of $L(s;\chi)$ with $\beta = \operatorname{Re} \rho \ge \theta$, then

$$\begin{aligned} 1 \ll & \left| \sum_{K < n \le KX} a(n)\chi(n)n^{-\rho} \right| \\ & + \int_{-\infty}^{\infty} \left| \sum_{K < n \le KX} b^*(n;v)\chi(n)n^{-\rho} \right| e^{-|v|} dv \end{aligned}$$

We now partition the interval $(K, KX]$ into

$$\ll \log H$$

dyadic intervals $I_j = (K_j, K_j']$ where $K_j = K2^{j-1}$ and

$$K_j' = \min(K2^j, KX).$$

Thus

$$1 \ll \sum_{j \ll \log H} \left(\left| \sum_{n \in I_j} a(n)\chi(n)n^{-\rho} \right| + \int_{-\infty}^{\infty} \left| \sum_{n \in I_j} b^*(n;v)\chi(n)n^{-\rho} \right| e^{-|v|} dv \right)$$

Let $\mathcal{P}(\chi)$ be the set of zeros ρ of $L(s;\chi)$ being counted. Then for each such ρ there is a $j \ll \log H$ such that

$$1 \ll (\log H) \left(\left| \sum_{n \in I_j} a(n)\chi(n)n^{-\rho} \right| + \int_{-\infty}^{\infty} \left| \sum_{n \in I_j} b^*(n;v)\chi(n)n^{-\rho} \right| e^{-|v|} dv \right).$$

Let $\mathcal{P}_j(\chi)$ be the set of $\rho \in \mathcal{P}(\chi)$ for which this holds. Then, for any fixed k and $\rho \in \mathcal{P}_j(\chi)$,

$$1 \ll (\log H)^{2k} \left(\left| \sum_{n \in I_j} a(n)\chi(n)n^{-\rho} \right|^{2k} + \int_{-\infty}^{\infty} \left| \sum_{n \in I_j} b^*(n;v)\chi(n)n^{-\rho} \right|^{2k} e^{-|v|} dv \right).$$

Hence

$$1 \ll (\log H)^{3k} \left(\left| \sum_{n \in J_j} c_1(n)\chi(n)n^{-\rho} \right|^{2} + \int_{-\infty}^{\infty} \left| \sum_{n \in J_j} c_2(n;v)\chi(n)n^{-\rho} \right|^{2} e^{-|v|} dv \right).$$

where

$$J_j \subset (K_j^k, {K'_j}^k], \quad |c_1(n)| \ll d_{2k}(n), \quad |c_2(n;v)| \ll d_{2k}(n).$$

Let

$$N_j = \sum_{q \le Q} \sideset{}{^*}\sum_{\chi \bmod q} \operatorname{card} \mathcal{P}_j(\chi).$$

Then, by Theorem 27.15 we have

$$N_j \ll (\log H)^{C_1(k)}\big(K_j^{k(2-2\theta)} + K_j^{k(\frac{4}{3}-2\theta)} H^{\frac{1}{3}} N_j^{\frac{2}{3}}\big).$$

Thus

$$N_j \ll (\log 2QT)^{C(k)}\big(K_j^{k(2-2\theta)} + K_j^{k(4-6\theta)} H\big).$$

If $K_j \le H^{\frac{1/2}{3\theta-1}}$, then we choose k so that

$$H^{\frac{1}{3\theta-1}} \le K_j^k \le H^{\frac{3/2}{3\theta-1}}.$$

Note that by (28.44), $K_j \ge K = P^{\eta/5}$. Thus $k \ll 1$ as required and

$$N_j \ll (\log H)^{C_2(k)} H^{\frac{3(1-\theta)}{3\theta-1}}.$$

If $K_j > H^{\frac{1/2}{3\theta-1}}$, then we take $k=2$. By 28.43 and 28.44, $KX \le P^{\frac{1}{2}+\eta/4}$. Thus

$$\begin{aligned} N_j &\ll (\log H)^{C(k)}\big(P^{(2+\eta)(1-\theta)} + H^{\frac{4-6\theta}{3\theta-1}} H\big) \\ &\ll (\log H)^{C(k)}\big(P^{(2+\eta)(1-\theta)} + H^{\frac{3(1-\theta)}{3\theta-1}}\big). \end{aligned}$$

Since there are $\ll \log H$ possibilities for j, (28.31) follows.

Theorem 28.3 follows in the same way. We then define $P = H = CqT$ and use Theorems 26.18 and 27.13.

28.4.3 Exercises

1. Suppose that there is no exceptional zero of any L-function formed from a character with conductor dividing q. Prove that there are positive constants c_1 and c_2 such that if

$$q \le \exp\left(\frac{\log x}{c_2 \log\log x}\right),$$

then

$$\psi(x;q,a) = \frac{x}{\phi(q)}\left(1 + \exp\left(-\frac{(\log x)}{c_1\big(\log q + (\log x)^{\frac{2}{5}}(\log\log x)^{\frac{1}{5}}\big)}\right)\right).$$

28.5 Primes in Short Intervals

S:PinShorts

One of the most important applications of zero density estimates, and a driving force for their development, concerns the distribution of primes in short intervals. It is a routine consequence of the Riemann Hypothesis that when $2 \le h \le x$ we have

$$\pi(x+h) - \pi(x) = \int_x^{x+h} \frac{du}{\log u} + O\big(x^{1/2} \log x\big).$$

It is, perhaps, surprising that without any unproven hypothesis the exponent $\frac{7}{12}$ below is so close to $\frac{1}{2}$.

T:HuxleyinShorts **Theorem 28.8** *There is a positive number c such that if $G(x) \ge 1$ and $x^{7/12}(\log x)^c G(x)^{7/2} \le h \le x$, then*

$$\psi(x+h) - \psi(x) = h + O\left(\frac{h}{G(x)} + h \exp\left(- \frac{(\log x)^{\frac{1}{3}}}{(c \log\log x)^{\frac{1}{3}}}\right)\right) \tag{28.49}$$ eq:Huxpsi

and

$$\pi(x+h) - \pi(x) - \int_x^{x+h} \frac{du}{\log u} \ll \frac{h}{G(x)} + h \exp\left(- \frac{(\log x)^{\frac{1}{3}}}{(c \log\log x)^{\frac{1}{3}}}\right). \tag{28.50}$$ eq:Huxpi

If Corollary 28.5 were to hold with $12/5$ replaced by ξ for some $\xi > 2$, then the above holds with $7/12$ replaced by $1 - 1/\xi$. Thus one sees that the density hypothesis is practically as good as the Riemann Hypothesis in this context.

Proof By Theorem 12.5, when $2 \le T \le x^{1/2}$ and $x \ge c$ we have

$$\psi(x+h) - \psi(x) = h - \sum_{\substack{\rho \\ |\gamma| \le T}} \int_x^{x+h} u^{\rho - 1} du + O\big(xT^{-1}(\log x)^2\big).$$

Here the sum is over zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with $0 < \beta < 1$. By Corollary 14.3, Theorem 6.6 and Corollary 10.3 we may restrict the sum to zeros with $\beta \ge 1/2$. Hence

$$\psi(x+h) - \psi(x) - h \ll hx^{-1} \sum_{\substack{|\gamma| \le T \\ \beta \ge 1/2}} x^\beta + xT^{-1}(\log x)^2. \tag{28.51}$$ E:HuxEst

We recall that, by (28.1) and (28.2), $N(\theta, T) = N_1(\theta, 1, T)$. The sum above is

$$x^{1/2} N(1/2, T) + \int_{1/2}^1 x^u N(u, T)(\log x) du.$$

By Theorem 24.18, $N(u, T) = 0$ when $u \ge 1 - \delta$ where

$$\delta = \frac{1}{c_1 (\log T)^{2/3} (\log\log T)^{1/3}} \tag{28.52}$$ E:Huxdelta

and c_1 is a positive constant. Thus the above is

$$x^{1/2}N(1/2,T)+\int_{1/2}^{1-\delta}x^u N(u,T)(\log x)du.$$

Then, by Theorem 28.4 and Corollary 28.5 with $q=1$, this is

$$\begin{aligned}&\ll \int_{1/2}^{5/6}x^uT^{12(1-u)/5}(\log x)^{c_2}+\int_{5/6}^{1-\delta}x^uT^{11(1-u)/5}(\log x)^{c_2}du\\&\ll (\log x)^{c_2}\big(x^{1/2}T^{6/5}+(\log x)^{c_2}x^{5/6}T^{2/5}+x(T^{11/5}/x)^{\delta}\big).\end{aligned}$$

We now make the choice

$$T=xh^{-1}(\log x)^{-2}G(x)$$

so that

$$T\le x^{5/12}(\log x)^{-c}G(x)^{-7/2}.$$

Then, by (28.51)

$$\psi(x+h)-\psi(x)-h\ll hG(x)^{-1}+h(\log x)^{c_2}(T^{11/5}/x)^{\delta}.$$

By (28.52),

$$(T^{11/5}/x)^{\delta}\le \exp\left(-\frac{\log x}{c_3(\log x)^{2/3}(\log\log x)^{1/3}}\right),$$

which gives (28.49).

It is immediate from Chebyshev's inequality that

$$\psi(x+h)-\psi(x)-\vartheta(x+h)+\vartheta(x)\ll x^{1/2},$$

and so (28.49) holds with ψ replaced by ϑ. Moreover

$$\pi(x+h)-\pi(x)=\frac{\vartheta(x+h)-\vartheta(x)}{\log(x+h)}+\int_x^{x+h}\frac{\vartheta(t)-\vartheta(x)}{t\log^2 t}dt.$$

Then substituting (28.49) with ψ replaced by ϑ when

$$t\ge x+x^{7/12}(\log x)^cG(x)^{7/2}$$

gives (28.50) as required. □

Whilst it is speculated that results of the above kind persist for h significantly smaller than $\frac{7}{12}$, or $\frac{1}{2}$, we can only establish that such results hold for most, but not necessarily all, pairs x and h with h smaller.

T:2momshorts **Theorem 28.9** *There is a positive number c such that if $x \ge 2$ and $h \ge 1$, then*

$$\int_0^x \big(\psi(y+h)-\psi(y)-h\big)^2 dy$$
$$\ll xh^2(x^{1/6}h^{-1})^{2/5}(\log x)^c + xh^2 \exp\left(-\frac{(\log x)^{\frac{1}{3}}}{(c\log\log x)^{\frac{1}{3}}}\right). \quad (28.53)$$ eq:2momshorts

Proof We will show that

$$\int_{x/2}^x \big(\psi(y+h)-\psi(y)-h\big)^2 dy$$
$$\ll xh^2(x^{1/6}h^{-1})^{2/5}(\log x)^c + xh^2 \exp\left(-\frac{(\log x)^{\frac{1}{3}}}{(c'\log\log x)^{\frac{1}{3}}}\right). \quad (28.54)$$ E:shortdyadic

Then it follows that

$$\int_{x^{\frac{2}{3}}}^x \big(\psi(y+h)-\psi(y)-h\big)^2 dy$$
$$\le \sum_{k\le\frac{\log x}{3\log 2}} \int_{x2^{-k}}^{x2^{1-k}} \big(\psi(u+h)-\psi(u)-h\big)^2 du$$
$$\ll \sum_{k\le\frac{\log x}{3\log 2}} \frac{x}{2^k}h^2(x^{1/6}h^{-1})^{2/5}(\log x)^c$$
$$+ \frac{x}{2^k}h^2 \exp\left(-\frac{(\log x)^{\frac{1}{3}}}{(c\log\log x)^{\frac{1}{3}}}\right)$$

which is acceptable, and we also have trivially

$$\int_0^{x^{2/3}} \big(\psi(y+h)-\psi(y)-h\big)^2 dy \ll x^{2/3}h^2(\log x)^2.$$

Thus the theorem would follow from (28.54).

To prove (28.54) we can suppose that $h \le x/4$, since otherwise the conclusion follows from Theorem 6.9. Suppose that $h/2 \le g \le h$. Then the left hand side of (28.54) is

$$\int_{\frac{x}{2}-g}^{x-g} \big(\psi(y+g+h)-\psi(y+g)-h\big)^2 dy$$

and on integrating over g we see that it is

$$2h^{-1}\int_{h/2}^h \int_{\frac{x}{2}-g}^{x-g} \big(\psi(y+g+h)-\psi(y+g)-h\big)^2 dy dg.$$

We have

$$\big(\psi(y+g+h)-\psi(y+g)-h\big)^2 \ll \big(\psi(y+g+h)-\psi(y)-g-h\big)^2+\big(\psi(y+g)-\psi(y)-g\big)^2.$$

Hence the left hand side of (28.54) is

$$\ll h^{-1}\int_{h/2}^{2h}\int_{x/4}^{x}\big(\psi(y+g)-\psi(y)-g\big)^2 dydg.$$

On inverting the order of integration and making the substitution $g = hu$ we obtain

$$\begin{aligned}
&h^{-1}\int_{x/4}^{x}\int_{h/2}^{2h}\big(\psi(y+g)-\psi(y)-g\big)^2 dgdy\\
&= h^{-1}\int_{x/4}^{x}\int_{h/(2y)}^{2h/y}\big(\psi(y+yu)-\psi(y)-yu\big)^2 ydudy\\
&\ll xh^{-1}\int_{x/4}^{x}\int_{h/(2x)}^{8h/x}\big(\psi(y+yu)-\psi(y)-yu\big)^2 dudy\\
&= h^{-1}\int_{h/(2x)}^{8h/x}\int_{x/4}^{x}(2x-y)\big(\psi_0(y+yu)-\psi_0(y)-yu\big)^2 dydu
\end{aligned}$$

where $\psi_0(v) = \psi(v)$ unless v is a positive integer, in which case it is $\psi(v) - \frac{1}{2}\Lambda(v)$.

By Theorem 12.5 and the discussion following that theorem we see that when $v \geq 2$ we have

$$\psi_0(v) = v - \lim_{T\to\infty}\sum_{|\gamma|\leq T}\frac{v^\rho}{\rho} - \log 2\pi - \frac{1}{2}\log(1-v^{-2})$$

and the series is boundedly convergent. We also have

$$\frac{1}{2}\log\left(\frac{1-(y+yu)^{-2}}{1-y^{-2}}\right) = \frac{1}{2}\log\left(1-\frac{(1+u)^2-1}{y^2(1+u)^2-1}\right) \ll uy^{-2}.$$

Hence the above is

$$\ll x + h^2x^{-5} + h^{-1}\int_{h/(2x)}^{8h/x}\int_{x/4}^{x}(2x-y)\left|\sum_\rho y^\rho\frac{(1+u)^\rho-1}{\rho}\right|^2 dydu$$

Therefore the left hand side of (28.54) is

$$\ll x + h^{-1}\int_{h/(2x)}^{8h/x}\int_{0}^{2x}(2x-y)\left|\sum_\rho y^\rho\frac{(1+u)^\rho-1}{\rho}\right|^2 dydu$$

and expanding the integrand and integrating by parts we obtain

$$x + \frac{1}{h}\sum_{\rho_1}\sum_{\rho_2}\int_{\frac{h}{2x}}^{\frac{8h}{x}} \frac{(2x)^{2+\rho_1+\overline{\rho}_2}\big((1+u)^{\rho_1}-1\big)\big((1+u)^{\overline{\rho}_2}-1\big)}{(1+\rho_1+\overline{\rho}_2)(2+\rho_1+\overline{\rho}_2)\rho_1\overline{\rho}_2}du.$$

We have $\rho = \beta + i\gamma$ with $0 < \beta < 1$ and $|\gamma| > 1$. Hence

$$\left|\frac{(1+u)^\rho - 1}{\rho} = \int_1^{1+u} v^{\rho-1}dv\right| \ll \min(u, |\gamma|^{-1}).$$

Then the inequality $|z_1z_2| \le |z_1|^2 + |z_2|^2$ and the symmetry of ρ_1 and ρ_2 establishes that the above is

$$\ll x + \frac{1}{h}\sum_{\rho_1}\int_{h/(2x)}^{8h/x}\sum_{\rho_2}\frac{x^{2+2\beta_1}\min(u^2, |\gamma_1|^{-2})}{(1+|\gamma_1-\gamma_2|)^2}du$$
$$\ll x + x\sum_{\rho_1} x^{2\beta_1}\min(h^2x^{-2}, \gamma_1^{-2})\sum_{\rho_2}\frac{1}{(1+|\gamma_1-\gamma_2|)^2}.$$

By Theorem 10.13 the innermost sum is $\ll \log|\gamma_1|$. Thus the left hand side of (28.54) is

$$\ll x + x\sum_{\rho} x^{2\beta}(\log|\gamma|)\min(h^2x^{-2}, \gamma^{-2}).$$

The sum here is

$$\ll h^2x^{-1}(\log x)\sum_{\substack{\rho \\ |\gamma|\le x/h}} x^{2\beta} + x\sum_{\substack{\rho \\ |\gamma|>x/h}}\frac{x^{2\beta}\log|\gamma|}{\gamma^2}.$$

By the symmetry of the zeros we can suppose that $\beta \ge 1/2$. The first term here is

$$\ll h^2(\log x)N(1/2, x/h) + h^2x^{-1}(\log x)\int_{1/2}^1 x^{2\theta}(2\log x)N(\theta, x/h)d\theta$$
$$\ll xh(\log x)^2 + \int_{\frac{1}{2}}^{1-\delta} x^{2\theta}\frac{h^2}{x}(\log x)^c\Big(\frac{x}{h}\Big)^{\lambda(\theta)}d\theta$$

where

$$\lambda(\theta) = \begin{cases} \frac{12}{5}(1-\theta) & (1/2 \le \theta \le 5/6), \\ (2+\eta)(1-\theta) & (5/6 \le \theta \le 1-\delta), \end{cases}$$

and the second is

$$\begin{aligned}
&\ll x \sum_{\substack{\rho \\ \beta \geq 1/2 \\ |\gamma| > x/h}} \left(x + \int_{1/2}^{\beta} x^{2\theta} (2\log x) d\theta \right) \left(\int_{|\gamma|}^{\infty} \frac{2\log t - 1}{t^3} dt \right) \\
&\ll x^2 \int_{x/h}^{\infty} N(1/2; t) \frac{2\log t - 1}{t^3} dt \\
&\quad + x(\log x) \int_{1/2}^{1} x^{2\theta} \int_{x/h}^{\infty} N(\theta; t) \frac{2\log t - 1}{t^3} dt d\theta \\
&\ll xh(\log x)^2 + \int_{\frac{1}{2}}^{1-\delta} x^{2\theta+1} \int_{\frac{x}{h}}^{\infty} (\log t)^c t^{\lambda(\theta)-3} dt d\theta
\end{aligned}$$

By (28.31) with $\eta = 1/5$ this is

$$\begin{aligned}
&\ll xh^2 (\log x)^c \left(\frac{1}{h} \int_{\frac{1}{2}}^{\frac{5}{6}} \left(\frac{x^{\frac{1}{6}}}{h} \right)^{\frac{12(1-\theta)}{5}} d\theta + \int_{\frac{5}{6}}^{1-\delta} \left(\frac{x^{\eta}}{h^{2+\eta}} \right)^{1-\theta} d\theta \right) \\
&\ll xh^2 (\log x)^c \left(\left(\frac{x^{\frac{1}{6}}}{h} \right)^{\frac{6}{5}} + \left(\frac{x^{\frac{1}{6}}}{h} \right)^{\frac{2}{5}} + xh^2 \left(\frac{x^{\frac{1}{11}}}{h} \right)^{\frac{11}{30}} + xh^2 \left(\frac{x^{\eta}}{h^{2+\eta}} \right)^{\delta} \right).
\end{aligned}$$

If $h \leq x^{1/6}$, then (28.54) is trivial. Thus we may suppose $h > x^{1/6}$. Then the above is

$$\ll xh^2 (x^{1/6}/h)^{2/5} (\log x)^c + xh^2 \exp\left(- \frac{(\log x)^{\frac{1}{3}}}{(c' \log\log x)^{\frac{1}{3}}} \right)$$

as required, and this completes the proof of the theorem. □

28.5.1 Exercises

1. (a) Prove that if $x \geq 2$ and

$$x^{-5/6} \exp\left(\frac{(\log x)^{1/3}}{(c \log\log x)^{1/3}} \right) < u \leq 1$$

and $x \geq 2$, then

$$\int_{x/4}^{x} (\psi_0(y + yu) - \psi_0(y) - yu)^2 dy \ll x^3 u^2 \exp\left(- \frac{(\log x)^{1/3}}{(c' \log\log x)^{1/3}} \right).$$

(b) Assume the Riemann Hypothesis. Prove that if $x \ge 2$ and $0 < u \le 1$, then

$$\int_{x/4}^{x} (\psi_0(y + yu) - \psi_0(y) - yu)^2 dy \ll x^2 u \left(\log \frac{2}{u}\right)^2 .$$

2. (a) Prove that if the Riemann Hypothesis is true and $2 \le h \le x$, then

$$\int_0^x (\vartheta(y + h) - \vartheta(y) - h)^2 \ll xh\big(\log(2x/h)\big)^2.$$

(b) Prove that if p_n is the n-th prime in order of magnitude, then

$$\sum_{\substack{p_n \le x \\ p_{n+1} - p_n > h}} (p_{n+1} - p_n) \ll xh^{-1}(\log x)^2.$$

and

$$\sum_{p_n \le x} (p_{n+1} - p_n)^2 \ll x(\log x)^3.$$

3. Prove that for all large x the interval $(x, x + x^{\frac{7}{72}+\varepsilon})$ contains a sum of two primes. If the Riemann Hypothesis is true show that the interval $(x, x + c(\log x)^2)$ contains a sum of two primes.

4. (a) Prove that there is a positive constant c such that if $q \ge 1$ and $x \ge 2$, then

$$\sum_{\chi \bmod q} \sup_{y \le x} |\psi(y; \chi)| \ll (\log x)^c (x^{1/2} q + x^{7/11} q^{9/11} + x^{3/4} q^{3/5} + x).$$

(b) Prove that if $(a, q) = 1$, then

$$\sum_{p \le x} (\log p) e(ap/q) \ll (\log x)^c (xq^{-1/2} + x^{7/8} + x^{1/2} q^{1/2}.$$

(c) Prove that if $f(\theta)$ has the property that there is an $x \ge 1$ and $\alpha > 0$ such that whenever $(a, q) = 1$ and $|\theta - a/q| \le q^{-2}$ we have

$$|f(\theta)| \le xq^{-\alpha} + x^{1-\alpha} q^{\alpha},$$

then for any pair a, q we have

$$f(\theta) \ll x(q + x|\theta q - a|)^{-\alpha} + x^{1-\alpha}(q + x|\theta q - a|)^{\alpha}$$

(d) Prove that, if $(q,a)=1$, then

$$\sum_{p\le x}(\log p)e(\theta p) \ll (\log x)^c(x\Delta^{-1/2}+x^{7/8}+x^{1/2}\Delta^{1/2})$$

where $\Delta = q + x|q\theta - a|$. Compare with Lemma **??** and with Exercise 19.1.7.

5. Prove that there is a positive constant c such that if $Q\ge 1$ and $x\ge 2$, then

$$\sum_{q\le Q}\ \sideset{}{^*}\sum_{\chi \bmod q}\sup_{y\le x}|\psi(y;\chi)| \ll (\log x)^c(x^{1/2}Q^2+x^{7/11}Q^{18/11}+x^{3/4}Q^{6/5}+x).$$

Compare this with Theorem **??** and give an alternative proof of the Bombieri-Vinogradov theorem, Theorem **??**.

S:Near1

28.6 Zeros near the 1-line

There are several ways in which one can obtain significantly smaller bounds near the 1-line. One way in the special case of the zeta function is to make use of the Korobov-Vinogradov-Richert bound Theorem 24.15 for $\zeta(s)$ near 1.

T:ZDEVin

Theorem 28.10 *Let*

$$\Theta(\sigma,U) = 1 + \sup_{0<t\le U}\left|\zeta(\sigma+it) - \frac{1}{\sigma+it}\right|.$$

Then, for $\frac34 \le \theta \le 1$

$$N(\theta,T) \ll \Theta(3\theta-2,2T)^3(\log T)^{15}$$

T:ZDEVincor

Corollary 28.11 *There is a positive constant* c *such that if* $\frac34\le\theta\le 1$, *then*

$$N(\theta,T) \ll T^{c(1-\theta)^{3/2}}(\log T)^c$$

Corollary 28.11 follows by combining the theorem with Theorem 24.15.

Proof By Theorem 24.18 we may certainly suppose that

$$\frac34 \le \theta \le 1 - \frac{1}{C(\log T)^{2/3}(\log\log T)^{1/3}}$$

for some positive constant C. We then follow the proof of (28.30), with

χ identically 1, as far as (28.39) except that we instead move the path of integration to the line $\operatorname{Re} w = \alpha - \beta$ where $\alpha \le 2\theta - 1$, and we suppose that $s = \rho = \beta + i\gamma$ is a zero of ζ with $\beta \ge \theta$ and $|\gamma| \ge (\log T)^2$. Note that then

$$|\alpha - \beta| = \beta - \alpha \ge \theta - (2\theta - 1) = 1 - \theta \gg 1/\log T.$$

Thus we obtain

$$1 + \sum_{K<n\le Z} a(n)n^{-\rho}e^{-n/Y} = O(1/T) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \zeta(\alpha + i\gamma + iv;\chi)M(\alpha + i\gamma + iv)Y^{\alpha-\beta+iv}\Gamma(\alpha - \beta + iv)\,dv \tag{28.55}$$

E:ZDELM1a

and we can observe that

$$\Gamma(\alpha - \beta + iv) \ll \left(1 + \frac{1}{|\alpha - \beta + iv|}\right)e^{-|v|} \ll (\log T)e^{-|v|}.$$

At this stage the only constraints on K, Y, Z are $C \le K \le T$, $Y \ge 1$ and $Z = Y(\log T)^2$, and in addition we have $Q = 1$, $H = CT$. Hence

$$1 \ll \left|\sum_{K<n\le Z} a(n)n^{-\rho}e^{-n/Y}\right| + (\log T)\Theta(\alpha, 2T)Y^{\alpha-\theta}\int_{-(\log T)^2}^{(\log T)^2} |M(\alpha + i\gamma + iv)|e^{-|v|}\,dv.$$

Let $\mathcal{N} = \{Z2^{-j} : 0 \le j \le (\log(Z/K))/\log 2\}$. Then

$$\left|\sum_{K<n\le Z} a(n)n^{-\rho}e^{-n/Y}\right| \le \sum_{N\in\mathcal{N}}\left|\sum_{N'<n\le N} a(n)\chi(n)n^{-\rho}e^{-n/Y}\right|$$

where $N' = \max(N/2, K)$. Hence

$$1 \ll \sum_{N\in\mathcal{N}}\left|\sum_{N'<n\le N} a(n)n^{-\rho}e^{-n/Y}\right| + (\log T)\Theta(\alpha, 2T)Y^{\alpha-\theta}\int_{-(\log T)^2}^{(\log T)^2} |M(\alpha + i\gamma + iv)|e^{-|v|}\,dv. \tag{28.56}$$

E:Ubound

LEMMAS 28.12 AND 28.13 SHOULD GO IN CHAPTER 22.

T:HalLem **Lemma 28.12** *Suppose that* $\boldsymbol{\xi} = (\xi_1, \dots, \xi_N)$ *and* ϕ_{rn} $(1 \le r \le R)$, $(r \in \mathbb{N})$ *are complex numbers and* $\boldsymbol{\phi}_r = (\phi_{r1}, \dots, \phi_{rN})$. *Suppose further that* $b_n > 0$ $(1 \le n \le N)$ *and* $b_n \ge 0$ $(n > N)$. *Let* $(\boldsymbol{\xi}, \boldsymbol{\phi}_r)$ *denote the inner product*

$$\sum_{n=1}^{N} \xi_n \overline{\phi}_{rn}.$$

Then

$$\sum_{r=1}^{R} |(\boldsymbol{\xi}, \boldsymbol{\phi}_r)| \le \left(\sum_{1 \le n \le N} |\xi_n|^2 b_n^{-1} \right)^{1/2} \left(\sum_{q=1}^{R} \sum_{r=1}^{R} \left| \sum_{n=1}^{\infty} b_n \phi_{qn} \overline{\phi}_{rn} \right| \right)^{1/2}.$$

Proof Choose θ_r so that $e^{i\theta_r}(\boldsymbol{\xi}, \boldsymbol{\phi}_r) = |(\boldsymbol{\xi}, \boldsymbol{\phi}_r)|$. Then the sum in question is

$$\sum_{n=1}^{N} \xi_n b_n^{-1/2} b_n^{1/2} \sum_{r=1}^{R} e^{i\theta_r} \overline{\phi}_{rn}.$$

Hence, by the Cauchy-Schwarz inequality, it is

$$\le \left(\sum_{n=1}^{N} |\xi_n|^2 b_n^{-1} \right)^{1/2} \left(\sum_{n=1}^{\infty} b_n \left| \sum_{r=1}^{R} e^{i\theta_r} \overline{\phi}_{rn} \right|^2 \right)^{1/2}.$$

We square out the expression on the right to obtain

$$\sum_{n=1}^{\infty} b_n \left| \sum_{r=1}^{R} e^{i\theta_r} \overline{\phi}_{rn} \right|^2 = \sum_{q=1}^{R} \sum_{r=1}^{R} e^{i(\theta_r - \theta_q)} \sum_{n=1}^{\infty} b_n \phi_{qn} \overline{\phi}_{rn}$$

and the lemma follows. □

T:HalLemD **Lemma 28.13** *Let*

$$D(s; u, v) = \sum_{u < n \le v} c(n) n^{-s}$$

where the $c(n)$ *are complex numbers, and let* s_r $(r = 1, \dots, R)$ *denote a set of complex numbers* $s_r = \sigma_r + it_r$ *with the property that* $0 \le \nu \le \sigma_r \le 1$, $|t| \le T$ *and* $|t_q - t_r| \ge 1$ *when* $q \ne r$. *Suppose also that*

$N/2 \le N' \le N \le T^2$ and $\frac{1}{4} \le \alpha < 1$. Then

$$\sum_{r=1}^{R} |D(s_r; N', N)| \ll$$

$$N^{-\nu}\big((RN)^{1/2} + R\Theta(\alpha; 2T)^{1/2} N^{\alpha/2}\big) \left(\sum_{N'<n\le N} |c(n)|^2\right)^{1/2}$$

Proof To establish the lemma we note that

$$D(s; N', N) = N^{-\sigma} D(it; N', N) + \int_{N'}^{N} \sigma u^{-\sigma-1} D(it; N', u)\,du$$

and so when $0 \le \sigma \ge 1$ we have

$$D(s; N', N) \ll N^{-\nu} |D(it; N', N)| + N^{-\nu-1} \int_{N'}^{N} |D(it; N', u)|\,du.$$

Therefore

$$\sum_{r=1}^{R} |D(s_r; N', N)| \ll$$

$$N^{-\nu} \sum_{r=1}^{R} |D(it_r, N', N)| + N^{-\nu-1} \int_{N'}^{N} \sum_{r=1}^{R} |D(it_r; N', u)|\,du. \quad (28.57)$$ E:Dbound

We now apply the previous lemma with $\xi_n = c(n)$, $b_n = e^{-n/N}$, $\phi_{rn} = n^{it_r}$. Then for $N' \le u \le N$ we have

$$\sum_{r=1}^{R} |D(it_r; N', u)| \le$$

$$\left(\sum_{N'<n\le N} |c(n)|^2 e^{n/N}\right)^{1/2} \left(\sum_{q=1}^{R} \sum_{r=1}^{R} \left|\sum_{n=1}^{\infty} e^{-n/N} n^{it_q - it_r}\right|\right)^{1/2}. \quad (28.58)$$ E:2Dbound

For brevity, put $t = t_r - t_q$. Then the sum over n here is

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta(w+it) N^w \Gamma(w)\,dw$$

We now move the line to the path $\operatorname{Re} w = \alpha$. In doing so we pick up a residue

$$N^{1-it}\Gamma(1-it) \ll N e^{-|t|}$$

at $w = 1 - it$.

On the α-line we write

$$\zeta(w+it) = \zeta(w+it) - \frac{1}{w+it-1} + \frac{1}{w+it-1}.$$

For the third term here we move the path to $\operatorname{Re} w = -\frac{1}{2}$ picking up the residue

$$\frac{1}{it-1} \ll 1$$

at 0 and the bound

$$\ll \int_{-\infty}^{\infty} \frac{N^{-1/2}e^{-|v|}}{1+|v+t|} dv \ll \frac{N^{-1/2}}{1+|t|}$$

from the $-\frac{1}{2}$ -line.

We also have, by Corollary 1.17,

$$\zeta(\alpha+iv+it) - \frac{1}{\alpha+iv+it-1} \ll 2+|v+t|.$$

Thus

$$\int_{|v|\geq(\log T)^2} \left|\zeta(\alpha+iv+it) - \frac{1}{\alpha+iv+it-1}\right| N^\alpha |\Gamma(\alpha+iv)| dv$$
$$\ll N^\alpha \int_{|v|\geq(\log T)^2} (2+|v+t|)e^{-|v|} dv \ll N^\alpha T^{-1}.$$

Since $|t| \leq T$ we have

$$\int_{|v|\leq(\log T)^2} \left|\zeta(\alpha+iv+it) - \frac{1}{\alpha+iv+it-1}\right| N^\alpha |\Gamma(\alpha+iv)| dv$$
$$\ll \Theta(\alpha, 2T) N^\alpha.$$

Putting these estimates together we find that

$$\sum_{n=1}^{\infty} e^{-n/N} n^{it_q - it_r} \ll \Theta(\alpha, 2T) N^\alpha + N e^{-|t_q - t_r|}.$$

Hence, by (28.58),

$$\sum_{r=1}^{R} |D(it_r; N', u)| \ll \left(\sum_{N'<n\leq N} |c(n)|^2 e^{n/N} \right)^{1/2} \times$$
$$\left(RN + R^2 \Theta(\alpha, 2T) N^\alpha + \sum_{q=1}^{R} \sum_{\substack{r=1 \\ r\neq q}}^{R} N e^{-|t_q - t_r|} \right)^{1/2}.$$

Inserting this in (28.57) establishes the lemma. □

Let $\mathcal{R}$ be a maximal subset of the zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with $\beta \geq \theta$, $(\log T)^2 \leq |\gamma| \leq T$ and the γ spaced at least 1 apart, and let $R = \operatorname{card} \mathcal{R}$. Then

$$N(\theta, T) \ll (\log T)^3 + (\log T)R$$

and, by (28.56),

$$R \ll \sum_{N \in \mathcal{N}} \sum_{\rho \in \mathcal{R}} \left| \sum_{N' < n \leq N} a(n) n^{-\rho} e^{-n/Y} \right| \\ + (\log T)\Theta(\alpha, 2T) Y^{\alpha - \theta} \int_{-(\log T)^2}^{(\log T)^2} e^{-|v|} \sum_{\rho \in \mathcal{R}} |M(\alpha + i\gamma + iv)| dv.$$

By Lemma 28.13 with $\nu = \theta$ and the s_r the elements of $\mathcal{R}$, the first sum over $\mathcal{R}$ is bounded by

$$N^{-\theta}\big((RN)^{1/2} + R\Theta(\alpha, 2T)^{1/2} N^{\alpha/2}\big) e^{-N/(2Y)} \left(\sum_{N' < n \leq N} d(n)^2 \right)^{1/2} \\ \ll (\log N)^{3/2} R^{1/2} N^{1-\theta} e^{-N/(2Y)} + \\ (\log N)^{3/2} R \Theta(\alpha, 2T)^{1/2} N^{\frac{1+\alpha}{2} - \theta} e^{-N/(2Y)}$$

and summing this over $N \in \mathcal{N}$ gives the bound

$$\ll (\log T)^{3/2} R^{1/2} Y^{1-\theta} + (\log T)^{5/2} R \Theta(\alpha, 2T)^{1/2} K^{\frac{1+\alpha}{2} - \theta}$$

provided that $1 + \alpha \leq 2\theta$.

The second sum over $\mathcal{R}$ above is

$$\sum_{\rho \in \mathcal{R}} |M(\alpha + i\gamma + iv)| \leq \\ \sum_{0 \leq j \leq \frac{\log K}{\log 2}} \sum_{\rho \in \mathcal{R}} \left| \sum_{K2^{-j-1} \leq k \leq K2^{-j}} \mu(k) k^{-\alpha - i\gamma - iv} \right|.$$

By Lemma 28.13 with $\nu = \alpha$ and the $s_r = \alpha + i \operatorname{Im} \rho$ where $\rho \in \mathcal{R}$, this is

$$\ll (\log T) R^{1/2} K^{1-\alpha} + (\log T) R \Theta(\alpha; 2T)^{1/2} K^{(1-\alpha)/2}.$$

Hence

$$R \ll (\log T)^{3/2} R^{1/2} Y^{1-\theta} + (\log T)^{5/2} R \Theta(\alpha, 2T)^{1/2} K^{\frac{1+\alpha}{2}-\theta}$$
$$+ (\log T)^2 R^{1/2} Y^{\alpha-\theta} \Theta(\alpha; 2T) K^{1-\alpha}$$
$$+ (\log T)^2 R Y^{\alpha-\theta} \Theta(\alpha; 2T)^{3/2} K^{\frac{1-\alpha}{2}}.$$

Let

$$\alpha = \theta - \lambda(1-\theta)$$

where $\lambda > 1$ is a parameter at our disposal. Then $(1+\alpha)/2 - \theta = -(\lambda-1)(1-\theta)/2$, $\alpha - \theta = -\lambda(1-\theta)$ and $1-\alpha = (1+\lambda)(1-\theta)$. Let δ be a sufficiently small constant and choose

$$K = \left(\delta^{-1} (\log T)^{5/2} \Theta(\alpha, 2T)^{1/2}\right)^{\frac{2}{(\lambda-1)(1-\theta)}}. \tag{28.59}$$ E:defK

Then the second term above is $\ll \delta R$. Likewise if we choose

$$Y = \left(\delta^{-1} (\log T)^2 \Theta(\alpha, 2T)^{3/2} K^{\frac{1-\alpha}{2}}\right)^{\frac{1}{\lambda(1-\theta)}}, \tag{28.60}$$ E:defY

then the fourth term above is

$$\ll \delta R.$$

Hence

$$R \ll (\log T)^{3/2} R^{1/2} Y^{1-\theta} + (\log T)^2 R^{1/2} Y^{\alpha-\theta} \Theta(\alpha; 2T) K^{1-\alpha}$$

and so

$$R \ll (\log T)^3 Y^{2-2\theta} + (\log T)^4 Y^{2\alpha-2\theta} \Theta(\alpha; 2T)^2 K^{2-2\alpha}.$$

To tidy things up, by the choices (28.59) and (28.60)

$$Y^{2-2\theta} \ll (\log T)^{\frac{9\lambda+1}{\lambda(\lambda-1)}} \Theta(\alpha; 2T)^{\frac{4\lambda-2}{\lambda(\lambda-1)}}$$

and

$$Y^{2\alpha-2\theta} K^{2-2\alpha} \ll (\log T)^{\frac{\lambda+9}{(\lambda-1)}} \Theta(\alpha; 2T)^{\frac{4-2\lambda}{(\lambda-1)}}.$$

Hence

$$R \ll (\log T)^{3+\frac{9\lambda+1}{\lambda(\lambda-1)}} \Theta(\alpha; 2T)$$

The optimal choice of λ depends on whatever bound for $\Theta(\alpha; 2T)$ that we may insert. If θ is restricted to being close to 1, then large values of λ are possible and exponents above would be small, but in the current state of knowledge we have no better bound for Θ than

$$\Theta(\alpha; 2T) = \Theta(\theta - \lambda(1-\theta)) \ll (\log T)^{c'} T^{c(\lambda+1)^{3/2}(1-\theta)^{3/2}}.$$

Thus for simplicity we suppose that $\lambda = 2$. Note that then the requirement $\frac{1}{4} \le \alpha = \theta - \lambda(1-\theta) = 3\theta - 2$ does permit $\frac{3}{4} \le \theta$ as required. Thus

$$R \ll (\log T)^{13}\Theta(3\theta - 2, 2T)^3 + (\log T)^{15}\Theta(3\theta - 2; 2T)^2.$$

□

28.6.1 Exercises

1 Assume the Lindelöf Hypothesis in the form

$$\Theta(\alpha; T) \ll_\varepsilon T^\varepsilon$$

uniformly for $\alpha \ge \frac{1}{2}$. Prove that whenever $\theta > \frac{3}{4}$

$$N(\theta, T) \ll_\varepsilon T^\varepsilon.$$

28.7 A logarithm free bound and the Deuring-Heibronn phenomenon

S:logfree

Another way in which significant improvements can be made is to use Turán's power sum method to remove the logarithmic power. This also comes into play if there were to be an "exceptional zero" of some L -function close to 1, as it can be used to show that the non-exceptional zeros are repelled to the left.

We need to remind ourselves of some basic results concerning the concept of an "exceptional" zero.

Exceptional Zero Statement. *By Corollary 11.10 of Volume 1 there is a positive constant* c_1 *such that*

$$F(s,T) = \prod_{q \le T} \prod_{\chi \bmod q}^{*} L(s,\chi) \tag{28.61}$$

E:LprodT

has at most one zero s *with* $\operatorname{Re} s > 1 - \frac{1}{c_1 \log T}$*, of necessity real and if this "exceptional zero"* β_1 *exists, then the corresponding character* χ_1 *is quadratic and, by Corollary 11.12, there is a positive constant* c_2 *such that* $\delta_1 = 1 - \beta_1$ *satisfies*

$$\frac{1}{c_2 q_1^{1/2} (\log q_1)^2} \le \delta_1 < \frac{1}{c_1 \log T} \tag{28.62}$$

E:XZlb

where q_1 *is the conductor of* χ_1*.*

It is convenient to write $E_1 = 0$ if there is no exceptional zero and

$E_1 = 1$ if there is an exceptional zero and to reserve χ_1, β_1, q_1 to denote the corresponding exceptional character, zero and conductor.

Let

$$N^*(\theta, T)$$

denote the number of zeros $\rho = \beta + i\gamma$ of (28.61), with $\beta \geq \theta$ and $|\gamma| \leq T$, *other* than any exceptional zero.

One can observe that if as T varies there are only a finite number of exceptional moduli, then in principle one could simply adjust the constant c_1 and eliminate the concept of "exceptional". On the other hand if the exceptional moduli form an infinite sequence $\{q_j\}$ and $\{\beta_j\}$ are the corresponding exceptional zeros, then by the same token one would have to have

$$\limsup_{j \to \infty} (1 - \beta_j) \log q_j = 0.$$

Thus in principle one could take c_1 to be as small as one pleases. However, this leads inexorably to the non-computability of c_1, which here we would prefer to avoid, especially in connection with bounding the least prime in an arithmetic progression.

With this in the background we can establish

T:ZDETur **Theorem 28.14** *There are positive constants c and c_0 such that when $\frac{1}{2} \leq \theta \leq 1$ and $T \geq 2$ we have*

$$N^*(\theta, T) \leq c_0 T^{c(1-\theta)}, \tag{28.63}$$ E:logfree1

and if there is an exceptional real zero β_1 associated with some exceptional primitive character χ_1 with conductor $q_1 \leq T$, then

$$N^*(\theta, T) \leq c_0 \delta_1 (\log T) T^{c(1-\theta)}. \tag{28.64}$$ E:logfree2

We can immediately conclude from this an effective version of the Deuring-Heilbronn phenomenon, which essentially says that if there is an exceptional zero, then the other zeros are repelled away from 1-line.

C:DerHeil **Corollary 28.15** *There are positive constants c_0, c such that if β_1 is an exceptional zero as defined above, then any other zero $\rho = \beta + i\gamma$ with $|\gamma| \leq T$ of an L-function formed from a primitive character modulo $q \leq T$ satisfies*

$$\beta \leq 1 - \frac{\log \frac{1}{c_0 (1-\beta_1) \log T}}{c \log T}.$$

We begin by eliminating some ranges for θ. By Corollary 28.7 there is a positive constant c_2 such that

$$N^*(\theta, T) \ll T^{8(1-\theta)}(\log T)^{C_2}.$$

Suppose that $C > 8$ and $\theta \le 1 - \frac{1}{c-8}$. If β_1 exists, then, by (28.62), the above is

$$\ll T^{-1/2}(\log T)^{-1}T^{c(1-\theta)} \ll \delta_1(\log T)T^{c(1-\theta)}.$$

and when β_1 does not exist it is

$$\ll T^{c(1-\theta)}.$$

Also, from the definition above we have

$$N^*(\theta, T) = 0$$

when $\theta \ge 1 - \frac{1}{c_1 \log T}$. Hence we may assume that

$$1 - \frac{1}{c-8} \le \theta \le 1 - \frac{1}{c_1 \log T}.$$

Let

$$r = c_3(1-\theta) \tag{28.65}$$ E:rdef

where

$$c_3 = \max(c_1, 2).$$

Then

$$\frac{1}{\log T} \le r \le \frac{1}{10^5} \tag{28.66}$$ E:rbound

provided we take

$$c \ge (8 + 10^5)\max(c_1, 2).$$

Suppose $N^*(\theta, T) > 0$ and let $\rho_0 = \beta_0 + i\gamma_0$ be a non-exceptional zero counted by $N^*(\theta, T)$ so that

$$1 - \beta_0 \le 1 - \theta = \frac{r}{c_3} \le \frac{1}{2}.$$

Then consider v satisfying

$$|\gamma_0 - v| \le \frac{r}{2}, \tag{28.67}$$ E:defv

so that $|v| \le T + 1$ and

$$|\rho_0 - w| \le 1 - \beta_0 + |\gamma_0 - v| \le r, \text{ where } w = 1 + iv. \tag{28.68}$$ E:defw

In particular, when χ is principal we have $|v| \ge 2$.

The core of the proof is a lower bound for expressions of the kind

$$I(X,Y,r,\gamma_0,\chi) = \int_X^Y \int_{\gamma_0-\frac{r}{2}}^{\gamma_0+\frac{r}{2}} \left| \sum_{X<p\le y} \frac{1+E_1 p^{-\delta_1}\chi_1(p)}{p^{1+iv}}\chi(p)\right|^2 dv \frac{dy}{y}. \tag{28.69}$$ E:doubleint

Suppose that $s = \sigma + it$ satisfies

$$|s-w| \le \frac{3}{4}, \quad \frac{5}{6} \le \sigma \le 2. \tag{28.70}$$ E:esssigma

Then, by Lemma 11.1

$$\frac{L'}{L}(s,\chi) = \sum_{\substack{\rho \\ |\rho-\frac{3}{2}-it|\le\frac{5}{6}}} \frac{1}{s-\rho} + O(\log T). \tag{28.71}$$ E:L'Lapprox

Note that if χ is principal, since then $|v| \ge 2$, we have $|s-1| = |s-w+iv| \ge |v| - \frac{3}{4} \ge 1$.

It is useful to restate here an immediate consequence of Theorem 11.5 and Corollary 14.7 of Volume 1.

T:denslem **Lemma 28.16** *Suppose that λ, q and t satisfy $q \ll T$, $|t| \ll T$ and $(\log T)^{-1} \le \lambda \le 2$. Then for, each character χ modulo q, $L(s,\chi)$ has $\ll \lambda \log T$ zeros ρ with $|\rho - 1 - it| \le \lambda$.*

If $\left|\rho - \frac{3}{2} - it\right| \le 5/6$ and $|\rho - w| \ge 1$, then $|\rho - s| = |\rho - w + w - s| \ge 1 - \frac{3}{4} = \frac{1}{4}$. Thus the contribution to the sum in (28.71) from such zeros is $\ll \log T$. Moreover, if $\left|\rho - \frac{3}{2} - it\right| > \frac{5}{6}$ and $\frac{5}{6} \le \sigma \le 2$ we have $|\rho - s| = \left|\rho - \frac{3}{2} - it + \frac{3}{2} - \sigma\right| > \frac{5}{6} - \frac{2}{3} = \frac{1}{6}$ and again the contribution from such zeros is $\ll \log T$. Therefore

$$\frac{L'}{L}(s,\chi) = \sum_{\substack{\rho \\ |\rho-w|<1}} \frac{1}{s-\rho} + O(\log T).$$

Suppose that

$$|s-w| \le \frac{5}{8}, \quad \frac{5}{6} \le \sigma \le \frac{15}{8}, \tag{28.72}$$ E:newess

If $|\rho - w| > \frac{3}{4}$, then $|s-\rho| = |\rho - w - s + w| \ge \frac{1}{8}$. Hence the contribution from such zeros to the above sum is $\ll \log T$, and we have

$$\frac{L'}{L}(s,\chi) = \sum_{\substack{\rho \\ |\rho-w|\le\frac{3}{4}}} \frac{1}{s-\rho} + O(\log T).$$

Now suppose that there is an exceptional zero, and the associated character is χ_1. Then (28.71) with (28.70) holds with s replaced by $s+\delta_1$ and w replaced by $w+\delta_1$. Hence a parallel argument to the above shows that

$$\frac{L'}{L}(s+\delta_1,\chi\chi_1) = \sum_{\substack{\rho' \\ |\rho'-\delta_1-w|\le\frac{3}{4}}} \frac{1}{s+\delta_1-\rho'} + O(\log T).$$

where now the sum is over zeros ρ' of $L(s;\chi\chi_1)$.

Let

$$g(z,\chi) = \frac{L'}{L}(z,\chi) + E_1\frac{L'}{L}(z+\delta_1,\chi\chi_1) - \sum_{\substack{\rho \\ |\rho-w|\le\frac{3}{4}}} \frac{1}{z-\rho} - E_1 \sum_{\substack{\rho' \\ |\rho'-\delta_1-w|\le\frac{3}{4}}} \frac{1}{z+\delta_1-\rho'}.$$

Suppose that $|s-w| \le \frac{1}{2}$. The function $g(z,\chi)$ has only removable singularities when $|z-w| \le \frac{3}{4}$ and by Cauchy's integral formula

$$\frac{g^{(h)}(s,\chi)}{h!} = \frac{1}{2\pi i}\int_C g(z,\chi)\frac{dz}{(z-s)^{h+1}} \ll 4^h \log T$$

where $\mathcal{C}$ is the circle, centre w, of radius $\frac{3}{4}$. Let

$$f(z,\chi) = \frac{L'}{L}(z,\chi) + E_1\frac{L'}{L}(z+\delta_1,\chi\chi_1). \tag{28.73}$$ E:Bombf

Then

$$(-1)^h\frac{f^{(h)}(s,\chi)}{h!} - \sum_{\substack{\rho \\ |\rho-w|\le\frac{3}{4}}} \frac{1}{(s-\rho)^{h+1}} - E_1 \sum_{\substack{\rho' \\ |\rho'-\delta_1-w|\le\frac{3}{4}}} \frac{1}{(s+\delta_1-\rho')^{h+1}} \ll 4^h \log T.$$

Now suppose

$$s_0 = w + r \tag{28.74}$$ E:defs0

and let

$$\lambda = 200r. \tag{28.75}$$ E:deflam

Then, by (28.66),

$$\frac{200}{\log T} < \lambda \le \frac{1}{4}$$

and so by Lemma 28.16 we have

$$(-1)^h \frac{f^{(h)}(s_0,\chi)}{h!} - \sum_{\substack{\rho \\ |\rho-w|\le\lambda}} \frac{1}{(s_0-\rho)^{h+1}} - E_1 \sum_{\substack{\rho' \\ |\rho'-\delta_1-w|\le\lambda}} \frac{1}{(s_0+\delta_1-\rho')^{h+1}} \ll (\lambda\log T)\lambda^{-h-1} = \lambda^{-h}\log T.$$

Also, if N is the total number of terms in the above sums, then

$$N \ll \lambda \log T \ll \log T. \tag{28.76}$$ E:NlambdaT

Suppose that

$$L \ge N. \tag{28.77}$$ E:LgeN

Then, by Turán's Second Main Theorem, in the form of Corollary K.5, there is a k with $L+1 \le k \le 2L$ such that

$$\left| \sum_{\substack{\rho \\ |\rho-w|\le\lambda}} \frac{1}{(s_0-\rho)^k} + E_1 \sum_{\substack{\rho' \\ |\rho'-\delta_1-w|\le\lambda}} \frac{1}{(s_0+\delta_1-\rho')^k} \right| \ge 2(16e)^{-k} \max_{|\rho-w|\le\lambda} |s_0-\rho|^{-k} \ge (50)^{-k}|s_0-\rho_0|^{-k}.$$

By (28.68) we have $|s_0-\rho_0| \le |s_0-w| + |\rho_0-w| \le 2r$. Thus on taking $h=k-1$ we have $L \le h \le 2L-1$ and

$$\left|\frac{f^{(h)}(s_0,\chi)}{h!}\right| \ge (50)^{-h-1}|s_0-\rho_0|^{-h-1} - c_4\lambda^{-h}\log T \ge (100r)^{-h-1} - c_4(200r)^{-h}\log T$$

for some positive absolute constant c_4.

We choose $c_5 \ge 1$ so that when

$$L \ge c_5 r \log T \tag{28.78}$$ E:BombL

we have

$$\frac{100c_4 r\log T}{2^h} \le \frac{1}{2},$$

by (28.76) we have (28.77), and at several points below, including (28.82) and (28.83), we suppose that it is a sufficiently large absolute constant. Then

$$\left|\frac{f^{(h)}(s_0,\chi)}{h!}\right| \geq (100r)^{-h-1}\left(1-\frac{100c_4 r\log T}{2^h}\right)$$
$$\geq \frac{1}{2}(100r)^{-h-1}.$$

We now return to the definition of f, (28.73). Thus, by (28.74),

$$r^h\frac{f^{(h)}(s_0)}{h!} = (-1)^{h+1}\sum_{n=1}^{\infty}\frac{(r\log n)^h\Lambda(n)\chi(n)}{h!n^{s_0}}(1+E_1\chi_1(n)n^{-\delta_1})$$
$$= (-1)^{h+1}\sum_{n=1}^{\infty}\frac{\Lambda(n)\chi(n)}{n^w}\varpi_h(r\log n)(1+E_1\chi_1(n)n^{-\delta_1})$$

where, for $y>0$,

$$\varpi_h(y)=\frac{y^h}{h!e^y}.$$

We have

$$\log h! = h\log h - \int_1^h \frac{\lfloor u\rfloor}{u}du \geq h\log h - h.$$

Hence

$$\varpi_h(y)\leq (ey/h)^h e^{-y}.$$

Thus

$$\varpi_h(y)\leq (300)^{-h}\quad (h\geq 900y).$$

The function of y, $h\log(ey/h)-y/2$ is decreasing for $y\geq 2h$. Hence

$$h\log(ey/h)-y/2\leq h(1+\log 20-10)\quad (y\geq 20h)$$

and so

$$\varpi_h(y)\leq 300^{-h}e^{-y/2}\quad (y\geq 20h). \tag{28.79}$$ E:varpiy

We now consider any x with

$$x\geq T^{c_5} \tag{28.80}$$ E:defxC5

where c_5 is as in (28.78). Then (28.75), (28.76) and the choice $L=\lceil r\log x\rceil$ ensures that (28.78), and so (28.77) holds, and that h satisfies

$$r\log x\leq h\leq 2r\log x. \tag{28.81}$$ E:hrlogx

Thus, when $n \le x^{1/900}$ we have $r \log n \le h/900$ and when $n > x^{40}$ we have $r \log n > 20h$. Hence the contribution from these n to

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)\chi(n)}{n^w} \varpi_h(r \log n)(1 + E_1\chi_1(n)n^{-\delta_1})$$

is

$$\begin{aligned} &\ll \sum_{n \le x^{1/900}} \frac{\Lambda(n)}{n}(300)^{-h} + \sum_{n > x^{40}} \frac{\Lambda(n)}{n^{1+r/2}}(300)^{-h} \\ &\ll (\log x + 1/r)(300)^{-h} \\ &\ll 100^{-h} r^{-1} 3^{-r \log x}(r \log x + 1). \end{aligned}$$

We also require of c_5 that when $y \ge c_5$ we have

$$3^{-y}(y+1) \le \frac{1}{400} \quad (y \ge c_5). \tag{28.82}$$ E:c5ineq

Thus the total contribution is

$$< \frac{1}{4r100^{h+1}}.$$

It follows that

$$\left| \sum_{x^{1/900} < n \le x^{40}} \frac{\Lambda(n)\chi(n)}{n^w} \varpi_h(r \log n)(1 + E_1\chi_1(n)n^{-\delta_1}) \right| \ge \frac{1}{4r100^{h+1}}.$$

We also have $w_h(y) \le 1$. Hence, by Chebyshev's inequalities and (28.80), the contribution from the $n = p^k$ with $k \ge 2$ to this sum is

$$\le \sum_{\substack{x^{1/900} < p^m \le x^{40} \\ m \ge 2}} \frac{2(\log p)}{p^m} < x^{-1/2000} \quad (x \ge T^{c_5}, \tag{28.83}$$ E:Cheb

whereas, by (28.66),

$$4r100^{h+1} \le 800r100^{2r \log x} < x^{2r \log 100} < x^{2(\log 100)/10^5} < x^{1/2000}.$$

Hence

$$\left| \sum_{x^{1/900} < p \le x^{40}} \frac{(\log p)\chi(p)}{p^w} \varpi_h(r \log p)(1 + E_1\chi_1(p)p^{-\delta_1}) \right| \ge \frac{1}{800r100^h}. \tag{28.84}$$ E:Bsumlb

For convenience write

$$X = x^{1/900}, \quad Y = x^{40}. \tag{28.85}$$ E:defXY

Then

$$\sum_{x^{1/900}<p\le x^{40}} \frac{(\log p)\chi(p)}{p^w}\varpi_h(r\log p)(1+E_1\chi_1(p)p^{-\delta_1})$$
$$= F(Y,v)\varpi_h(r\log Y) - \int_X^Y F(y,v)\varpi_h'(r\log y)\frac{rdy}{y}$$

where

$$F(y,v) = \sum_{X<p\le y} \frac{(\log p)\chi(p)}{p^w}(1+E_1\chi_1(p)p^{-\delta_1}). \tag{28.86}$$ E:defF

We have, by (28.81), $r\log Y = 40r\log x \ge 20h$ and by Mertens theorem, Theorem 2.7 (b),

$$F(Y,v) \ll \log x.$$

Hence, by (28.79),

$$|F(Y,v)\varpi_h(r\log Y)| \ll (\log x)300^{-h}e^{-\frac{\log Y}{2}}$$

and since c_5 is assumed sufficiently large we have

$$|F(Y,v)\varpi_h(r\log Y)| < \frac{1}{1600r100^h}$$

We also have

$$|\varpi_h'(u)| = |\varpi_{h-1}(u) - \varpi_h(u)| \le 1.$$

Hence

$$\int_X^Y |F(y,v)|\frac{dy}{y} \gg 100^{-h}r^{-2}$$

and so by Schwarz' inequality

$$\int_X^Y |F(y,v)|^2\frac{dy}{y} \gg (\log x)^{-1}100^{-2h}r^{-4} \tag{28.87}$$ E:F2lb

To summarise what we have established so far. We are given $T \ge 2$ and r and θ satisfying (28.65) and (28.66). Then for each non-exceptional zero $\rho_0 = \beta_0 + i\gamma_0$ with $\beta_0 \ge \theta$ and $|\gamma_0| \le T$, and for each v satisfying (28.67), namely $|\gamma_0 - v| \le r/2$, we have (28.87) with (28.86) and (28.68) for any x satisfying (28.80) and some h for which (28.81) holds.

By (28.81)

$$(\log x)^{-1}100^{-2h}r^{-4} \geq (\log x)^3 100^{-4r\log x}(r\log x)^{-4}$$

By (28.80) and (28.66) and that fact that c_5 is sufficiently large, the above is

$$\gg (\log x)^3 x^{-c_6 r}$$

for some positive constant c_6. Integrating over the v satisfying (28.67) we obtain

$$\int_X^Y \int_{\gamma_0 - r/2}^{\gamma_0 + r/2} |F(y,v)|^2 dv \frac{dy}{y} \gg r(\log x)^3 x^{-c_6 r}.$$

For concision we now drop the suffix 0 and sum over the non-exceptional zeros ρ of $L(s;\chi)$ with $\beta \geq \theta$ and $|\gamma| \leq T$. Thus

$$r(\log x)^3 x^{-c_6 r} N^*(\theta;\chi,T) \ll \int_X^Y \int_{T-r}^{T+r} \sum_{\substack{\rho \\ |\rho - v| \leq r}} \left| \sum_{X<p\leq y} \frac{(\log p)\chi(p)}{p^w} b(p) \right|^2 dv \frac{dy}{y}$$

where $N^*(\theta;\chi,T)$ is the number of such zeros and

$$b(p) = 1 + E_1\chi_1(p)p^{-\delta_1}$$

. By (28.65), for a each ρ in the sum we we have $|\rho - 1 - iv| = |\beta - 1 + i(\gamma - v)| \leq 1 - \theta + r \ll r$. Hence, by Lemma 28.16 the number of ρ in the sum is $\ll r\log T$. Thus

$$(\log x)^3 x^{-c_6 r} N^*(\theta;\chi,T) \ll (\log T)\int_X^Y \int_{T-r}^{T+r} \left| \sum_{X<p\leq y} \frac{(\log p)\chi(p)}{p^w} b(p) \right|^2 dv \frac{dy}{y}.$$

Therefore, summing over all primitive characters modulo $q \leq T$ we have

$$(\log x)^3 x^{-c_6 r} N^*(\theta;T) \ll (\log T)\int_X^Y \sum_{q\leq T} \sideset{}{^*}\sum_{\chi \bmod q} \int_{T-r}^{T+r} \left| \sum_{X<p\leq y} \frac{(\log p)\chi(p)}{p^w} b(p) \right|^2 dv \frac{dy}{y}. \quad (28.88)$$ E:Nupper

We now require a lemma arising from the large sieve.

THIS SHOULD GO IN CHAPTER 18.

T:varls **Lemma 28.17** *Suppose $Q \ge 1$, $N \ge 1$ and $T \ge 2$ and that a_n $(1 \le n \le N)$ are complex numbers with the property that $a_n = 0$ when n has a prime factor $p \le Q$. Let*

$$S(s;\chi) = \sum_{n=1}^{N} a_n \chi(n) n^{-s}.$$

Then

$$\sum_{q \le Q} \left(\log \frac{Q}{q}\right) \sideset{}{^*}\sum_{\chi \bmod q} \int_0^T |S(it;\chi)|^2 \ll \sum_{n=1}^{N} |a_n|^2 (n + Q^2 T)$$

where the sum over χ is over primitive characters modulo q.

To prove the lemma, we start from the observation that, by (9.6), when $(n,q)=1$ and χ is an arbitrary character modulo q we have

$$\chi(n)\tau(\overline{\chi}) = \sum_{a=1}^{q} \overline{\chi}(a) e(an/q)$$

and so

$$\frac{1}{\phi(q)} \sum_{\chi \bmod q} |\tau(\overline{\chi}) S(0,\chi)|^2 = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} |T(a/q)|^2$$

where

$$T(\alpha) = \sum_{n=1}^{N} a_n e(\alpha n).$$

By Theorem 9.10, when χ is induced by the primitive character χ^* with conductor $d|q$ we have

$$\tau(\overline{\chi}) = \mu(q/d)\chi^*(q/d)\tau(\overline{\chi}^*)$$

and by Theorem 9.7

$$|\tau(\overline{\chi}^*)| = d^{1/2}.$$

Moreover, since $a_n = 0$ when n has a prime factor $p \le Q$, we have

$$\frac{1}{\phi(q)} \sum_{\chi \bmod q} |\tau(\overline{\chi}) S(0,\chi)|^2 = \sum_{\substack{d|q \\ (d,q/d)=1}} \frac{d}{\phi(q)} \sideset{}{^*}\sum_{\chi^* \bmod d} |S(0,\chi^*)|^2.$$

We sum over $q \le Q$ and replace q by dm. Thus, by Corollary **??**, the above is

$$\sum_{d\le Q}\frac{d}{\phi(d)}\sum_{\substack{m\le Q/d\\(m,d)=1}}\frac{\mu(m)^2}{\phi(m)}\sum_{\chi^*\bmod d}^{*}|S(0,\chi^*)|^2\le(N+Q^2)\sum_{n=1}^{N}|a_n|^2.$$

By Exercise 3.2.1.9 and (3.18)

$$\sum_{\substack{m\le R\\(n,d)=1}}\frac{\mu(m)^2}{\phi(m)}\ge\log R.$$

Therefore

$$\sum_{d\le Q}\left(\log\frac{Q}{d}\right)\sum_{\chi^*\bmod d}^{*}|S(0,\chi^*)|^2\le(N+Q^2)\sum_{n=1}^{N}|a_n|^2.$$

Now following the proof of (26.22) gives the lemma.

We now return to (28.88). The right hand side is

$$\ll\int_X^Y\sum_{q\le T^2}\left(\log\frac{T^2}{q}\right)\sum_{\chi\bmod q}^{*}\int_{T-r}^{T+r}\left|\sum_{X<p\le y}\frac{(\log p)\chi(p)}{p^w}b(p)\right|^2 dv\frac{dy}{y}.$$

Applying the lemma gives the bound

$$\begin{aligned}N^*(\theta,T)&\ll\frac{x^{c_6r}}{(\log x)^3}\int_X^Y\sum_{X<p\le y}\frac{(\log p)^2}{p^2}|b(p)|^2(p+T^5)\frac{dy}{y}\\&\ll\frac{x^{c_6r}}{(\log x)^2}\sum_{X<p\le Y}\frac{(\log p)^2}{p^2}|b(p)|^2(p+T^5).\end{aligned}$$

We now have to deal with the sum over p. If $E_1=0$, then by Mertens, Theorem 2.7 (b), the sum over p is

$$\ll(\log X)^2(1+T^5/X)$$

and we are done. Thus we can henceforward suppose that $E_1=1$.

T:hyperb **Lemma 28.18** *Let*

$$c(n)=\sum_{m|n}\chi_1(n)$$

where $\chi_1(n)$ is an exceptional character with conductor q_1. Suppose further that $U/\log U\ge q_1$. Then

$$\sum_{n\le U}\frac{c(n)}{n}=(\log U+\gamma)L(1,\chi_1)+L'(1,\chi_1)+O\big(q_1^{1/2}(\log U)^{1/2}U^{-1/2}\big).$$

The proof of the lemma is a routine application of Dirichlet's method of the hyperbola. Thus, with $V = q_1^{1/2}U^{1/2}(\log U)^{1/2}$ we have

$$\sum_{n\le U}\frac{c(n)}{n} = \sum_{m\le V}\frac{\chi_1(m)}{m}\sum_{l\le U/m}\frac{1}{l} + \sum_{l\le U/V}\frac{1}{l}\sum_{V<m\le U/l}\frac{\chi_1(m)}{m}.$$

Then applying Euler's estimate to the first sum over l and then partial summation to both sums over m we obtain the desired conclusion.

By the lemma, when $X > q_1^2$ we have

$$\sum_{n\le X}\frac{c(n)}{n} = L(1,\chi_1)\left(\log X + \gamma + \frac{L'}{L}(1,\chi_1)\right) + O\big(q_1^{\frac12}(\log X)^{\frac12}X^{-\frac12}\big).$$

By (11.8) of Theorem 11.4 of volume 1 we have

$$\frac{L'}{L}(1,\chi_1) = \frac{1}{1-\beta_1} + O(\log q_1).$$

We recall that, by (28.85), $X = x^{1/900}$ and, by (28.80), $x \ge T^{c_5} \ge q_1^{c_5}$. Also, by (11.10) of volume 1, $L(1,\chi_1) \gg 1-\beta_1$. Hence, as c_5 is sufficiently large, it follows that

$$\sum_{n\le X}\frac{c(n)}{n} \ge \frac{L(1,\chi_1)}{2(1-\beta_1)}.$$

On the other hand, also by the lemma, and Theorem 11.11 of volume 1, we have

$$\sum_{X<n\le XY}\frac{c(n)}{n} = (\log Y)L(1,\chi_1) + O(X^{-1/3}) \ll (\log Y)L(1,\chi_1)$$

Since $c(n)$ is multiplicative and non-negative we have

$$\sum_{n\le X}\frac{c(n)}{n}\sum_{X<p\le Y}\frac{c(p)}{p} \le \sum_{X<n\le XY}\frac{c(n)}{n}.$$

Combining this with the above inequalities shows that

$$\sum_{\substack{X<p\le Y\\ \chi_1(p)\ne -1}}\frac{1}{p} \le \sum_{X<p\le Y}\frac{c(p)}{p} \ll (1-\beta_1)\log Y.$$

We have

$$b(n)^2 = 1 + 2p^{-\delta_1}\chi_1(n) + p^{-2\delta_1}\chi_1(p)^2 \le 2\big(1 + p^{-\delta_1}\chi_1(n)\big)$$

Hence

$$\sum_{\substack{X<p\le Y\\ \chi_1(p)\ne -1}} \frac{b(p)^2}{p} \ll (1-\beta_1)\log Y.$$

We also have

$$1-p^{-\delta_1} \ll (1-\beta_1)\log p.$$

Hence, by Mertens once more,

$$\sum_{\substack{X<p\le Y\\ \chi_1(p)=-1}} \frac{b(p)^2}{p} \ll (1-\beta_1)\log Y.$$

The theorem now follows.

28.8 Linnik's Theorem on Primes in A.P.

S:Linnikap

An important application of the previous section is to the distribution of prime numbers in arithmetic progressions with relatively large common difference, and in particular to the least prime in an arithmetic progression.

T:PXGpinap

Theorem 28.19 (Gallagher) *There are constants $c \ge 1$ and $\kappa_0 \ge 3$ such that if κ is a constant with $\kappa \ge \kappa_0$ and q and x are such that $1 < Q^{6c} \le x$, then we have*

$$\sum_{q\le Q} \sum_{\chi \bmod q}{}^{*} |\vartheta(x;\chi) - E_0(\chi)x| \ll x\exp\left(-\frac{\log x}{\kappa\log Q}\right) + \left(\frac{\log x}{\log Q}\right)^2 xQ^{-1}$$

unless

$$F(s,q) = \prod_{q\le Q} \prod_{\chi \bmod q}{}^{*} L(s,\chi) \tag{28.89}$$

E:newLprodT

has an exceptional real zero β_1 with

$$1-\beta_1 < \frac{1}{\kappa\log Q}$$

in which case the general term on the left is to be replaced by

$$\left|\vartheta(x;\chi_1) + \frac{x^{\beta_1}}{\beta_1}\right|$$

when $\chi = \chi_1$ is the exceptional character and the right hand side is replaced by

$$(1-\beta_1)(\log x)\left(x\exp\left(-\frac{\log x}{\log Q}\right)+\frac{x\log x}{Q\log Q}\right).$$

Proof By Theorem 12.12 and Corollary 14.7, when $x \ge 2$ and $T \le x^{1/2}$ we have

$$\vartheta(x;\chi) = E_0(\chi)x - \sum_{\rho\in\mathcal{R}(\chi)} \frac{x^\rho}{\rho} + O\left(\frac{x}{T}(\log x)^2\right) \tag{28.90}$$

E:varthchi

where $\mathcal{R}(\chi)$ is the set of zeros $\rho = \beta + i\gamma$ of $L(s,\chi)$ with $\beta \ge \frac{1}{2}$ and $|\gamma| \le T$.

Let the constants c, c_0, c_1, c_2 be as in the Exceptional Zero Statement, Theorem 28.14 and Corollary 28.15 and let

$$\kappa_0 = 3\max(c, c_1, c_0e, 1, c_0e^{3c}). \tag{28.91}$$

E:kappalin

On hypothesis, $\kappa \ge \kappa_0$ and it is convenient to write

$$\kappa' = \kappa/3.$$

Let

$$T = Q^3. \tag{28.92}$$

E:Tandx

The proof divides into two cases. First we suppose that F, given by (28.89), has no zeros $\rho = \beta + i\gamma$ with $|\gamma| \le T$ and

$$\beta > 1 - \frac{1}{\kappa_0 \log T},$$

that is, either there are no exceptional zeros, or the exceptional zero exists but satisfies $1 - \beta_1 \ge \frac{1}{\kappa_0 \log T}$. By 28.90 our sum is

$$\ll QxT^{-1}(\log x)^2 + \sum_{q\le Q}\ \sideset{}{^*}\sum_{\chi \bmod q}\ \sum_{\rho\in\mathcal{R}(\chi)} x^\beta.$$

We have

$$x^\beta = x^{1/2} + \int_{1/2}^{\beta} x^u(\log x)du$$

and so the multiple sum above is

$$\le x^{1/2}N^*(1/2,T) + \int_{1/2}^{1-1/(\kappa'\log T)} x^u N^*(u,T)(\log x)du.$$

By Theorem 28.14 this is

$$\ll x^{1/2}T^{c/2} + \int_{1/2}^{1-1/(\kappa' \log T)} x^u T^{c(1-u)}(\log x)du.$$

By (28.92) and the hypothesis on x.

$$xT^{-c} = xQ^{-3c} \geq x^{1/2} \text{ and } x^{1/2}Q^{c/2} \leq x^{3/4}.$$

Hence the sum of interest is

$$\begin{aligned}
&\ll xQ^{-2}(\log x)^2 + x^{1-1/(\kappa' \log T)}T^{c/(\kappa' \log T)} \\
&= xQ^{-2}(\log x)^2 + x\exp\big(-(\log x)/(\kappa' \log Q) + c/\kappa'\big) \\
&\ll xQ^{-2}(\log x)^2 + x\exp\big(-(\log x)/(\kappa \log Q) + c/\kappa'\big).
\end{aligned}$$

The remaining case is that in which there is an exceptional zero satisfying

$$\beta_1 > 1 - \frac{1}{\kappa' \log T}.$$

Now we have

$$\vartheta(x,\chi) + E_1(\chi)\frac{x^{\beta_1}}{\beta_1} \ll xT^{-1}(\log x)^2 + \sum_{\rho \in \mathcal{R}^*(\chi)} x^\beta$$

where $\mathcal{R}^*(\chi)$ denotes the set of zeros $\rho = \beta + i\gamma$ of $L(s;\chi)$, other than β_1, with $|\gamma| \leq T$ and $\beta \geq \frac{1}{2}$.

We can proceed as above, but now the multiple sum is

$$\begin{aligned}
&\ll (1-\beta_1)(\log T)x^{1/2}T^{c/2} + \int_{1/2}^{1-\delta} (1-\beta_1)(\log T)x^u T^{c(1-u)}du \\
&\ll (1-\beta_1)(\log T)x^{1-\delta}T^{c\delta}
\end{aligned}$$

where

$$\delta = \frac{1}{c\log T}\log\frac{1}{c_0(1-\beta_1)\log T}.$$

Hence, by (28.62), the sum in question is

$$\begin{aligned}
&\ll (1-\beta_1)QxT^{-1}(\log x)^2 + (1-\beta_1)(\log x)x^{1-\delta}T^{c\delta} \\
&\ll (1-\beta_1)(\log x)\frac{x\log x}{Q\log Q} + (1-\beta_1)(\log x)x^{1-\delta}T^{c\delta}
\end{aligned}$$

where the implicit constant is absolute.

We have

$$\begin{aligned} x^{1-\delta}T^{c\delta} &= x\exp\left(\frac{\log(xT^{-c})}{c\log T}\log\left(c_0(1-\beta_1)\log T\right)\right) \\ &\le x\exp\left(-\frac{\log x-3c\log Q}{3c\log Q}\log(\kappa'/c_0)\right) \\ &\ll x\exp\left(-\frac{\log x-3c\log Q}{\log Q}\right) \\ &\ll x\exp\left(-\frac{\log x}{\log Q}\right) \end{aligned}$$

and that completes the proof. □

The following two theorems are almost immediate

T:PXGspinap **Theorem 28.20** (Gallagher) *Suppose that $1 < q^{6c} \le x$, $(a,q)=1$, κ is as in Theorem 28.19 and that there is no exceptional zero β_1 with*

$$1-\beta_1 < \frac{1}{\kappa\log q}.$$

Then

$$\vartheta(x;q,a) = \frac{x}{\phi(q)}\left(1+O\left(\exp\left(-\frac{\log x}{\kappa\log q}\right)+\frac{\log^2 x}{q\log^2 q}\right)\right).$$

Given $q\in\mathbb{N}$ and $a\in\mathbb{Z}$ with $(q,a)=1$ we define $p(q,a)$ to be the least prime number p such that $p\equiv a \pmod q$.

T:Leastpinap **Theorem 28.21** (Linnik) *There is a positive constant A such that whenever $q\in\mathbb{N}$, $a\in\mathbb{Z}$ and $(q,a)=1$ we have $p(q,a)\le q^A$.*

Proof We have

$$\begin{aligned} \vartheta(x;q,a) &= \frac{1}{\phi(q)}\sum_{\chi \bmod q}\overline{\chi}(a)\vartheta(x,\chi) \\ &= \frac{1}{\phi(q)}\sum_{m|q}\sideset{}{^*}\sum_{\chi \bmod q}\overline{\chi}(a)\vartheta(x,\chi)+O\big((\log q)/\phi(q)\big) \qquad (28.93) \end{aligned}$$ E:varther

and Theorem 28.20 follows immediately from Theorem 28.19

The proof of Theorem 28.21 divides into two cases. First, when there is no exceptional zero β_1 with

$$1-\beta_1 < \frac{1}{\kappa\log q},$$

by Theorem 28.19 we have, for some constant c'

$$\left|\vartheta(x;q,a) - \frac{x}{\phi(q)}\right| < c' \frac{x}{\phi(q)} \left(\exp\left(-\frac{\log x}{\kappa \log q}\right) + \frac{\log^2 x}{q \log^2 q}\right)$$

and so on taking

$$x = q^A$$

with

$$A = \kappa \log(4c')$$

and $q > q_0(A)$ we have

$$\vartheta(x;q,a) > \frac{x}{2\phi(q)} > 0.$$

and this gives Theorem 28.21 in this case.

Alternatively, suppose that β_1 exists. Then, by (28.93) and Theorem 28.19,

$$\phi(q)\vartheta(x;q,a) - x + \chi_1(a)\frac{x^{\beta_1}}{\beta_1} \\ \ll (1-\beta_1)(\log x)\left(x \exp\left(-\frac{\log x}{\log q}\right) + \frac{x \log x}{q \log q}\right)$$

Since

$$1 - \beta_1 < \frac{1}{\kappa \log q} < \frac{1}{4}$$

and x is large we have

$$x - \frac{x^{\beta_1}}{\beta_1} = \int_{\beta_1}^{1} \frac{u \log x - 1}{u^2} x^u du.$$

Therefore

$$x - \frac{x^{\beta_1}}{\beta_1} \gg (1-\beta_1)(\log x) x \exp(-(1-\beta_1)\log x) \\ \gg (1-\beta_1)(\log x) x \exp\big(-(\log x)/(\kappa \log q)\big)$$

where the implicit constant is absolute. Thus for positive absolute constants c' and c'' we have

$$\phi(q)\vartheta(x;q,a)x^{-1} > \\ c'(1-\beta_1)(\log x)\exp\left(-\frac{A}{\kappa}\right) - c''(1-\beta_1)(\log x)\left(\exp(-A) + \frac{\log x}{q \log q}\right).$$

Thus if we choose

$$A = \frac{\kappa}{\kappa - 1} \log \frac{2c''}{c'},$$

then the above is

$$\geq (1-\beta_1)(\log x)\frac{c'}{2}(1-\beta_1)(\log x)\exp\big(-A/\kappa\big) - c''(1-\beta_1)(\log x)\frac{\log x}{q \log q}$$

. Then for $q > \frac{2Ac''}{c'}e^{A/\kappa}$ we have $\vartheta(x;q,a) > 0$ and that completes the proof of Theorem 28.21. □

28.8.1 Exercises

1 (Iwaniec [HI74] Iwaniec (1974)) Suppose that there is a non-negative function $f(q)$ such that $\lim_{q\to\infty} f(q) = \infty$ and

$$F(s,q) = \prod_{m|q} \prod_{\chi \bmod m}^{*} L(s,\chi)$$

contains no zero in the region

$$\sigma > 1 - \frac{f(q)}{\log\big(q(2+|t|)\big)},$$

then the least prime p with $p \equiv a \pmod q$ satisfies

$$p \ll_\varepsilon q^{\frac{12}{5}+\varepsilon}$$

2 Show that if χ is a non-principal character modulo q, then there is a constant c such that if $x > q^c$, then

$$\left|\sum_{p\leq x} \chi(p)\right| < \frac{\pi(x)}{2}.$$

3 (Fridlander [VF49] Fridlander (1949), Salié [HS49] Salié (1949), Montgomer [HM71] Monbtgomery (1971) Theorem 13.5.) Suppose that p is an odd prime and let $n_2(p)$ denote the least quadratic non-residue modulo p. Show that

$$n_2(p) = \Omega(\log p).$$

Assuming the generalized Riemann Hypothesis show that

$$n_2(p) = \Omega\big((\log p)\log\log p\big).$$

4 (Erdős [EP49] Erdős (1949))

(a) Let $q = \prod_{p \le y} p$. Show that for any $C > 0$ and any $y > y_0(c)$ there is an m such that $(m, q) - 1$ but $(m + n, q) > 1$ for $1 \le |n| \le cy$. (Hint: Use the method employed in proving Lemma 7.13)

(b) Let $d_n = p_{n+1} - p_n$ where p_n denotes the n-th prime. Show that

$$\limsup_{n\to\infty} \frac{\min(d_n, d_{n+1})}{\log p_n} = \infty.$$

28.9 Maier's theorem on irregularity of primes in short intervals

S:Maier

The theorems we have established on the distribution of primes in short intervals can be used to show that the distribution of primes in very short intervals is more irregular than is predicted by the simple, classical, probabilistic model of the primes proposed by Cramér. This states that when $n \ge 2$ the probability that n is prime is taken to be

$$\frac{1}{\log n}.$$

Thus the expected number of primes not exceeding x would be

$$\sum_{2\le n\le x} \frac{1}{\log n} = \mathrm{li}(x) + O(1).$$

which fits well with what we know of the prime number theorem. It does so also with the theorems on primes in short intervals contained in section 28.5. However this model would also predict that when $\lambda > 1$ the expected number of primes p with

$$x < p \le x + (\log x)^\lambda$$

is

$$\sum_{x<n\le x+(\log x)^\lambda} \frac{1}{\log n} \sim (\log x)^{\lambda-1}$$

The theorem below contradicts this prediction. In retrospect this is perhaps not so surprising since probability models have some difficulty distinguishing between reduced residue classes and non-reduced residue classes.

In addition to Theorem 28.20, crucial rôles are played in the proof by Buchstab's Theorem 7.11 and the Maier matrix method. For use in the proof we remind ourselves of the properties of the Buchstab function $\omega(u)$ defined in section 7.2, (7.37) and (7.38) by the equations

$$\omega(u) = \frac{1}{u} \quad (1 < u \le 2),$$

$$(u\omega(u))' = \omega(u-1) \quad (u > 2)$$

and continuity at $u = 2$. Thus

$$\omega(u) = \frac{1 + \log(u-1)}{u} \quad (2 < u \le 3).$$

Also, when $2 < u \le 3$ we have

$$\omega'(u) = \frac{1 - (u-1)\log(u-1)}{u^2(u-1)}$$

so that $\omega'(2) = \frac{1}{4}$ and $\omega'(3) = \frac{1-2\log 2}{18} < 0$. Thus for some $u \in (2,3)$, $\omega'(u) = 0$ and $\omega(u)$ has a local minimum. We also have for $u > 2$

$$u\omega'(u) = \omega(u-1) - \omega(u) = -\int_{u-1}^{u} \omega'(v)dv$$

so $\omega'(u)$ changes sign in every interval of length 1.

By considering the behaviour of the Laplace transform in Theorem 7.12 as $s \to 0+$ it follows that

$$\lim_{u\to\infty} \omega(u) = e^{-C_0}$$

where C_0 is Euler's constant.

The following lemma is particularly useful in the proof of Maier's theorem.

T:debruijn **Lemma 28.22** *The function $\omega(u) - e^{-C_0}$ changes sign in every interval $[t-1, t]$ with $t \ge 2$.*

Proof We define the auxiliary function $\xi(u)$ for $u > -1$ by

$$\xi(u) = \int_0^\infty \exp\left(-ux - x + \int_0^x \frac{e^{-y} - 1}{y} dy\right) dx.$$

The function is differentiable for $u > -1$ and by integration by parts we have

$$u\xi'(u-1) + \xi(u) = 0. \qquad (28.94)$$ E:defxi

We also have

$$\frac{1}{u+2} < \xi(u) < \frac{1}{u+1}.$$

For $t \geq 2$, let

$$\eta(t) = \int_{t-1}^{t} \omega(u)\xi(u)du + t\omega(t)\xi(t-1).$$

Then

$$\eta'(t) = 0 \quad (t > 2)$$

and

$$\lim_{t\to\infty} \eta(t) = e^{-C_0}.$$

Hence

$$\int_{t-1}^{t} \omega(u)\xi(u)du + t\omega(t)\xi(t-1) = e^{-C_0} \quad (t \geq 2).$$

Let

$$\nu(t) = \int_{t-1}^{t} \xi(u)du + t\xi(t-1).$$

Then, by (28.94), $\nu'(t) = 0$ $(t > 0)$. We also have $\nu(t) \to 1$ as $t \to \infty$. Therefore

$$\int_{t-1}^{t} \xi(u)du + t\xi(t-1) = 1(t > 0)$$

and so

$$\int_{t-1}^{t} \big(\omega(u) - e^{-C_0}\big)\eta(u)du + t\big(\omega(t) - e^{-C_0}\big)\eta(t-1) = 0.$$

Since $\eta(t) > 0$ and $\omega(u)$ is not a constant when $2 \leq u \leq 3$, then the lemma follows. □

T:Maier **Theorem 28.23** (Maier) *Let* $\lambda > 1$,

$$\Omega^-(\lambda) = \inf\{\omega(\tau) : \tau > \lambda\},\ \Omega^+(\lambda) = \sup\{\omega(\tau) : \tau > \lambda\}.$$

Then

$$\limsup_{x\to\infty} \frac{\pi\big(x + (\log x)^\lambda\big) - \pi(x)}{(\log x)^{\lambda-1}} \geq e^{C_0}\Omega^+(\lambda)$$

and

$$\liminf_{x\to\infty} \frac{\pi\big(x + (\log x)^\lambda\big) - \pi(x)}{(\log x)^{\lambda-1}} \leq e^{C_0}\Omega^-(\lambda).$$

Moreover $\Omega^-(\lambda) < e^{-C_0} < \Omega^+(\lambda)$ *and*

$$\Omega^+(\lambda) = \frac{1}{\lambda} \quad (1 < \lambda \leq 2).$$

Let

$$P(z) = \prod_{p \le z} p.$$

T:PXGspecial **Lemma 28.24** *Suppose c and κ_0 are as in Theorem 28.20 and that $A > \max(2, 6c)$. Then there are $\kappa \in [\kappa_0, 2\kappa_0]$ and arbitrarily large $z > z_0(A, \kappa)$ such that whenever $(a, P(z)) = 1$ we have*

$$\pi\big(2P(z)^A, P(z), a\big) - \pi\big(P(z)^A, P(z), a\big) - \frac{P(z)^A}{\phi\big(P(z)\big)\big(A \log P(z)\big)} \ll \frac{P(z)^A \exp(-A/\kappa)}{\phi\big(P(z)\big)\big(A \log P(z)\big)}.$$

Proof Let p_n denote the n-th prime in order of magnitude and suppose that n is large. Then consider Theorem 28.20 with $q = P(p_n)$. If there is no exceptional zero β_1 of

$$F_n(s) = \prod_{m | P(p_n)} \prod_{\chi \bmod m}^{*} L(s, \chi)$$

with

$$\beta_1 > 1 - \frac{1}{\kappa_0 \log P(p_n)}, \tag{28.95}$$ E:exccon

then we have, provided p_n is large enough in terms of A and κ_0.

$$\vartheta\big(2P(p_n)^A, P(p_n), a\big) - \vartheta\big(P(p_n)^A, P(p_n), a\big) - \frac{P(p_n)^A}{\phi\big(P(p_n)\big)} \ll \frac{P(p_n)^A \exp(-A/\kappa_0)}{\phi\big(P(p_n)\big)\big)}.$$

Moreover, for $P(p_n)^A < p \le 2P(p_n)^A$, we have

$$\log p = A \log P(p_n) + O(1)$$

and the desired conclusion follows with $z = p_n$.

Now suppose that there is an exceptional zero β_1 satisfying (28.95) of

$$F(s) = \prod_{m | P(p_n)} \prod_{\chi \bmod m}^{*} L(x, \chi)$$

and let q_1 be the corresponding conductor. Since

$$\log P(p_n) \ll \frac{1}{1 - \beta_1} \ll q_1^{1/2} \log q_1^2$$

q_1 is large in terms of n. Now choose l minimally so that $q_1 | P(p_l)$ and consider $P(p_{l-1})$. The l will also be large in terms of n, and β_1 will satisfy (28.95) with n replaced by l, so will be exceptional for $F_l(s)$. Suppose that $F_{l-1}(s)$ has an exceptional zero β_2, so that

$$\beta_2 > 1 - \frac{1}{\kappa_0 \log P(p_{l-1})}.$$

Then the associated conductor will divide $P(p_l)$ but by the minimality of l the exceptional conductor will differ from q_1. But there cannot be a second exceptional zero of $F_l(s)$, so

$$\begin{aligned}\beta_2 \le 1 - \frac{1}{\kappa_0 \log P(p_l)} &= 1 - \frac{1}{\kappa_0\big(\log P(p_{l-1}) + \log p_l}\big) \\ &< 1 - \frac{1}{2\kappa_0 \log P(p_{l-1})}.\end{aligned}$$

Thus there are no exceptional zeros of the kind

$$\beta_1 > 1 - \frac{1}{2\kappa_0 \log P(p_{l-1})}$$

associated with $P(p_{l-1})$ and we can proceed as in the first part of the proof. □

Let $\tau > \lambda$ and consider the array

$$\mathfrak{M} = (a_{uv}) \quad \big(P(z)^{A-1} < u \le 2P(z)^{A-1},\ 1 \le v \le (A \log P(z))^\tau,\ \big(v, P(z)\big) = 1\big)$$

where $a_{uv} = 1$ when $uP(z) + v$ is prime and 0 otherwise.

By Theorem 28.20, the number of non-zero entries in the v-th column is

$$\begin{aligned}&\pi\big(2P(z)^A + v, P(z), v\big) - \pi\big(P(z)^A + v, P(z), v\big) \\ &\qquad = \frac{P(z)^A}{\phi(P(z)) A \log P(z)}\big(1 + O\big(\exp(-A/\kappa)\big)\big) \qquad (28.96)\end{aligned}$$

and so, by Theorem 7.11, the total number in the array is

$$\begin{aligned}&\frac{P(z)^A \Phi\big((A\log P(z))^\tau, z\big)}{\phi\big(P(z)\big) A \log P(z)}\big(1 + O\big(\exp(-A/\kappa)\big)\big) = \\ &\quad \frac{P(z)^{A-1}\big(A \log(P(z))\big)^\tau \omega\left(\frac{\tau \log\left(A \log P(z)\right)}{\log z}\right)}{A \log P(z) \prod_{p \le z}(1 - 1/p)}\left(1 + O\left(e^{-A/\kappa}\right)\right).\end{aligned}$$

Moroever, by the standard form of the prime number theorem, Theorem

6.9,

$$\log P(z) = \vartheta(z) = z + O(z/\log z),$$

$$\log(A \log P(z)) = \log z + \log A + O\left(\frac{1}{\log z}\right)$$

$$= (\log z)\left(1 + O\left(\frac{\log A}{\log z}\right)\right)$$

and by Merten's theorem, Theorem 2.7,

$$\prod_{p \le z}(1 - 1/p) = \frac{e^{-C_0}}{\log z}\left(1 + O\left(\frac{1}{\log z}\right)\right).$$

Hence

$$\omega(\tau \log(A \log P(z))/\log z) = \omega(\tau)(1 + O(\log A/\log z))).$$

and the total number of non-zero entries in the array is

$$\frac{P(z)^{A-1}(A \log(P(z)))^{\tau} e^{C_0} \omega(\tau)}{A \log P(z)}\big(1 + O\big(\exp(-A/\kappa)\big)\big).$$

The total number of rows is $P(z)^{A-1}$. Hence there are rows with at least

$$\frac{(A \log(P(z)))^{\tau} e^{C_0} \omega(\tau)}{A \log P(z)}\big(1 + O\big(\exp(-A/\kappa)\big)\big)$$

non-zero entries. By dividing the primes counted in these rows into N subintervals of length

$$\frac{(A \log(P(z)))^{\tau}}{N}$$

where

$$N = \left\lceil (A \log(P(z)))^{\tau-\lambda} \right\rceil$$

we find that there are intervals

$$\left(X, X + \frac{(A \log(P(z)))^{\tau}}{N}\right]$$

containing at least

$$\frac{(A \log(P(z)))^{\lambda} e^{C_0} \omega(\tau)}{A \log P(z)}\big(1 + O\big(\exp(-A/\kappa)\big)\big)$$

primes where

$$P(z)^A \le X \le 2P(z)^A + \big(A \log P(z)\big)^{\tau}.$$

The length of such intervals is as most

$$\left(A \log P(z)\right)^\lambda \le (\log X)^\lambda.$$

Moreover

$$A \log P(z) = \log X + O(1)$$

Thus it follows that there are arbitrarily large X such that

$$\pi(X + (\log X)^\lambda) - \pi(X) \ge e^{C_0} \omega(\tau) (\log X)^{\lambda - 1} \big(1 + O\big(\exp(-A/\kappa)\big)\big).$$

In the opposite direction, there are rows with at most

$$\frac{(A \log(P(z)))^\tau e^{C_0} \omega(\tau)}{A \log P(z)} \big(1 + O\big(\exp(-A/\kappa)\big)\big)$$

non-zero entries. The choice

$$N = \left\lfloor \frac{(A \log(P(z)))^\tau}{\left(\log\left(2P(z)^A + \left(A \log P(z)\right)^\tau\right)\right)^\lambda} \right\rfloor$$

produces intervals

$$\left(X, X + \frac{(A \log(P(z)))^\tau}{N}\right]$$

of length at least

$$\left(\log\left(2P(z)^A + (A \log P(z))^\tau\right)\right)^\lambda \ge (\log X)^\lambda$$

containing at most

$$e^{C_0} \omega(\tau) (\log X)^{\lambda - 1} (1 + O(\exp(-A/\kappa)))$$

primes. Thus it follows that there are arbitrarily large X such that

$$\pi(X + (\log X)^\lambda) - \pi(X) \le e^{C_0} \omega(\tau) (\log X)^{\lambda - 1} (1 + O(\exp(-A/\kappa))).$$

The theorem now follows.

S:ZDT Notes

28.10 Notes

§1. The Euler product is a crucial ingredient in establishing that Dirichlet L functions have relatively few zeros, if any, off the critical line. There is an extensive literature showing that Dirichlet series without that feature can have $\asymp T$ zeros $\rho = \beta + i\gamma$ with $|\gamma| \le T$ and $\beta > \frac{1}{2}$. See, for example, [BS07] Balanzario & Sanchez-Ortiz (2007), [BG11] Bombieri & Ghosh

(2011), Bombieri & Hejhal (1995), Cassels (1961), Davenport & Heilbronn Davenport (1977), Davenport & Heilbronn (1936a), Davenport & Heilbronn (1936b), Dubickas, Garunkštis, Steuding, Steuding (2013), Kaczorowski & Kulas (2007), Karatsuba (1989), Laurinĉikas (1986), Potter & Titchmarsh (1935), Saias & Weingartner (2009), Spira (1969a,b); Spira (1994), Vaughan (2015), and Voronin (1985).

§28.4. This subject was initiated by Bohr & Landau (1914a) (see also Littlewood (1924)) who showed that if $\sigma > 1/2$, then $N(\sigma,T) \ll_\sigma T$, and then improved this to $N(\sigma,T) = o_\sigma(T)$, Bohr & Landau (1914b). The first improvement in the exponent of T is due to Carlson (1920) who obtained

$$N(\sigma,T) \ll T^{4\sigma(1-\sigma)+\varepsilon}.$$

See also Landau (1922)) and Hoheisel (1930a,b) who replaced the T^ε by a powers of logarithms. Titchmarsh (1928, 1929) improved the exponent to $4(1-\sigma)/(3-2\sigma)+\varepsilon$ and Ingham (1937) obtained bound

$$N(\sigma,T) \ll T^{\frac{3(1-\sigma)}{2-\sigma}}(\log T)^5.$$

Selberg (1946) obtained a sharper bound when

$$\sigma - \frac{1}{2} \ll \frac{\log\log T}{\log T}$$

and Jutila (1982) sharpened this further and obtained

$$N(\sigma,T) \ll T^{1-(1-\delta)(\sigma-1/2)}\log T$$

for any $\delta \in (0, 3/4)$.

Turán in Turán (1941), Turán (1943), Turán (1951), Turán (1954a), Turán (1954b), Turán (1956), Turán (1958) made a number of improvements when σ is close to 1. Iglina Iglina (1966) has similar improvements near the 1-line.

Various authors, including Linnik Linnik (1946a), Linnik (1946b), Chudakov Chudakov (1947), Chudakov (1948), Haslegrove Haselgrove (1951) have discussed bounds for $N(\sigma,\chi,T)$, N_1 and N_2, and Rényi Rényi (1948) had shown that early forms of the large sieve already gave interesting consequences for the distribution of zeros of L-functions. A. I. Vinogradov Vinogradov (1965), Vinogradov (1966) and Bombieri Bombieri (1965) used the large sieve to give significant bounds for N_2 which lead to important new results on the distribution of primes in arithmetic progressions and Halász and Turán Halász, Turán (1969) introduced ideas which resulted in significant improvemens when σ is cloe to 1. Montgomery [HM71] Montgomery (1971) gives a systematic account of these developments. This lead to intense activity over the next twenty years by Huxley, Jutila, Ivić and others. However, the generalization of Ingham's bound to $N_1(\sigma, q, T)$ and $N_2(\sigma, Q, T)$ when $\frac{1}{2} < \sigma < \frac{3}{4}$ has apparently not been significantly improved.

It is crucial in applications that the bounds are uniform in σ which the stated bounds are. Jutila [MJ77] Jutila (1977) extended the range for the density hypothesis to $\frac{11}{14} \le \sigma \le 1$ and Bourgain [JB00aa] Bourgain (2000) pushed this to $\frac{25}{32} < \sigma < 1$. This is an area which continues to be active. See Kerr [BK19] Kerr (2019).

The use of the Mollifier M already occurs in Carlson [FC20] Carlson (1920), but prior to Montgomery [HM69] Montgomery (1969), it was used *via* variants of Littlewood's Lemma. Montgomery introduced the transform (28.39) to count zeros and this and its variants have been at the core of later developments. Montgomery obtained

$$N_1(\sigma, q, T) \ll (qT)^{\frac{2(1-\sigma)}{\sigma}} (\log qT)^{14}$$

and

$$N_2(\sigma, Q, T) \ll (Q^2T)^{\frac{2(1-\sigma)}{\sigma}} (\log Q^2T)^{14}$$

which already improves upon Ingham when $\sigma > \frac{4}{5}$. Huxley [MH72a] Huxley (1972a) obtained

$$N(\sigma, T) \ll T^{(5\sigma-3)(1-\sigma)/(\sigma^2+\sigma-1)} (\log T)^9$$

which gives Corollary 28.7 with $Q = 1$, and then established (28.29) and (28.31) in [MH72b] Huxley (1972b).

For the Riemann zeta function at least, there are numerous small improvements in the range $3/4 < \sigma < 1$. See Forti & Viola [FV73] Forti & Viola (1973), Heath-Brown [HB79] Heath-Brown (1979), Huxley [MH73a] Huxley (1973), [MH75a] Huxley (1975a), [MH75b] Huxley (1975b), Ivić [AI79b] Ivić (1979), [AI80b] Ivić (1980), [AI84b] Ivić (1984), Jutila [MJ72] Jutila (1972), [MJ77] Jutila (1977), Ramachandra [KR75] Ramachandra (1975). Ivić [AI03, Chapter 11] Ivić (2003); **?** gives a comprehensive overview of the state of play in 1985.

§28.5 Results of the kind contained in this section are intimately connected with zero density estimates. The first theorem like Theorem 28.8 was established by Hoheisel [GH30b] Hoheisel (1930b), but with $\frac{7}{12}$ replaced by

$$1 - \frac{1}{33000}.$$

Hoheisel was constrained by only having the weaker Littlewood zero-free region available, and so his exponent depended on the constant in that as well as the exponent in the zero-density estimate. Heilbronn, by working hard with the constants replaced that by

$$1 - \frac{1}{250}.$$

Chudakov obtained an improved zero free region for the Riemann-zeta function ? from the Vinogradov mean value theorem which immediately reduced the exponent to $\frac{3}{4}$ Chudakov (1936a), and Ingham used his zero density estimate, combined with the Chudakov zero-free region, to obtain the exponent $\frac{5}{8}$ Ingham (1937).

In more recent times the methods displayed here have been combined with sieve methods to show that when h is somewhat smaller than $x^{\frac{7}{12}}$ there is a prime in $(x, x+h]$. The current record, due to Baker, Harman & Pintz Baker, Harman & Pintz (2001), is that this holds for $h = x^c$ with any $c > \frac{21}{40}$.

The main idea of the proof of Theorem 28.9 is taken from the proof of Lemma 6 of Saffari & Vaughan Saffari & Vaughan (1977). The second part of Exercise 1 (b) is essentially due to Selberg Selberg (1943). It differs in that the bound in the exercise is uniform for u close to 1, whereas Selberg apparently requires $u \ll x^{-\varepsilon}$.

§28.6 This section has its origins in the seminal paper of Hal'asz and Turán Halász, Turán (1969).

§28.7 Theorem 28.14 is useful when the presence of an exception zero is a nuisance, and shows that it may not be such a big nuisance. The first results of this kind were obtained by Linnik Linnik (1944a), Linnik (1944b), Linnik (1944c) in his fundamental work on the least prime in an arithmetic progression. In addition to the original application it has also found use in work on the exceptional set in Goldbach's Problem (see Montgomery & Vaughan Montgomery & Vaughan (1975)). Our exposition is inspired by Gallagher Gallagher (1970) and Bombieri in §6 of Bombieri (1974), the latter of which has the merit of giving the Deuring-Heilbronn, Corollary 28.15, for free. Again, stimulated by the question of the least prime in an arithmetic progression, there is a considerable history, with papers by Rodosskiĭ Rodosskiĭ (1954), Turán Turán (1961), Fogels Fogels (1965), Jutila Jutila (1969), Jutila (1970), Selberg Selberg (1972), Motohashi Motohashi (1976). Selberg's method gives

$$N_1(\sigma, q, T) \ll (qT)^{(3+\varepsilon)(1-\sigma)}$$

and

$$N_2(\sigma, Q, T) \ll (Q^5T^3)^{(1+\varepsilon)(1-\sigma)}.$$

§28.8 Below is a, not necessarily complete, history of bounds for the exponents A in Theorem 28.21

A	Year	
10000	1957	Pan [CP57] Pan (1957)
777	1965	Chen [JC65] Chen (1965)
550	1970	Jutila [MJ70] Jutila (1970)
168	1977	Chen [JC77] Chen (1977)
80	1977	Jutila [MJ77] Jutila (1977)
36	1977	Graham [SG77] Graham (1977)
20	1981	Graham [SG81] Graham (1981) (submitted before Chen's 1979 paper)
17	1979	Chen [JC79] Chen (1979)
16	1986	Wang [WW86] Wang (1986)
13.5	1989	Chen & Liu [CL89a] Chen & Liu (1989a), [CL89b] Chen & Liu (1989b), [CL91] Chen & Liu (1991)
8	1990	Wang [WW91] Wang (1991)
5.5	1992	Heath-Brown [HB92] Heath-Brown (1992)
5.18	2011	Xylouris [TX11a] Xylouris (2011a)
5	2011	Xylouris [TX11b] Xylouris (2011b)

§28.9 Maier's theorem [HM85] Maier (1985) came as something of a surprise, and caused some anxious rethinking of how we might model the primes in short intervals. Hildebrandt & Maier [HM89] Hildebrandt & Maier (1989) extended the method to show that even in somewhat longer intervals there are still greater biases in the distribution than had hitherto been believed. In a different direction the method has been adapted by Friedlander & Granville [FG89] Friedlander& Granville (1989), [FG91] Friedlander & Granville (1991), [FG92] Friedlander & Granville (1992) and by Friedlander, Granville, Hildebrandt and Maier [FGHM] Friedlander, Granville, Hildebrandt & Maier (1991) to show that the distribution of primes in arithmetic progressions, when the modulus is quite close to the size of the primes, is not as good as had been anticipated.

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29
The pair correlation of zeta zeros

C:PairCorr

As usual, we let $N(T)$ denote the number of zeros $\rho = \beta + i\gamma$ of the Riemann zeta function with $0 < \gamma \le T$. Multiple zeros are counted according to their multiplicity. From Theorem 10.13 we know that

$$N(T+1) - N(T) \ll \log T \tag{29.1}$$ E:DiffN(T)

for $T \ge 2$, and in Corollary 14.3 we found that

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)\,. \tag{29.2}$$ E:N(T)Est

Thus we see that the average spacing between the consecutive γ at height T is approximately $2\pi/\log T$. Our object in this Chapter (which will only be partially achieved) is to determine the distribution of the differences $\gamma - \gamma'$ between the ordinates on the scale of $2\pi/\log T$. We shall assume the Riemann Hypothesis throughout this Chapter.

Our approach to the distribution of the numbers $\gamma - \gamma'$ is to try to determine the Fourier Transform of this distribution, which is to say the asymptotic size of sums of $X^{i(\gamma-\gamma')}$. In summing this quantity we introduce a weighting $w(\gamma-\gamma')$, so that pairs of zeros that are far apart receive little weight. Specifically, for $X > 0$ and $T \ge 2$ we set

$$F(X,T) = \sum_{\substack{0<\gamma\le T\\ 0<\gamma'\le T}} X^{i(\gamma-\gamma')}w(\gamma-\gamma') \tag{29.3}$$ E:DefF

where $w(u) = 4/(4+u^2)$. Since w is an even function, on exchanging the roles of γ and γ' in the above it follows that $F(X,T)$ is real for all X and T. The weight $w(u)$ arises naturally in our analysis in the equation

(29.9), from which it follows that

$$\frac{4}{T\log\frac{T}{2\pi}}\int_{-\infty}^{\infty}\Bigl|\sum_{0<\gamma\le T}\frac{X^{i\gamma}}{1+(t-\gamma)^2}\Bigr|^2\,dt. \tag{29.4}$$ E:Fform1

From this formula we see that $F(X,T)\ge 0$ for all X and T.

To derive an asymptotic formula for $F(X,T)$ we start from a convenient explicit formula (proved below) which asserts that if $X\ge 1$, then

$$\begin{aligned}2\sum_{\gamma}&\frac{X^{i\gamma}}{1+(t-\gamma)^2}\\ &=-X^{-1/2}\Bigl(\sum_{n\le X}\Lambda(n)(X/n)^{-1/2+it}+\sum_{n>X}\Lambda(n)(X/n)^{3/2+it}\Bigr)\\ &\quad+X^{-1+it}\log\tau+O\bigl(X^{-1}\bigr)+O\bigl(X^{1/2}\tau^{-2}\bigr)\end{aligned} \tag{29.5}$$ E:ExpEq1

where $\tau=|t|+4$. We write the above briefly as $L(X,T)=R(X,T)$. In Section 29.1 we shall show that $\int_0^T|L(X,T)|^2dt$ is approximately $F(X,T)$, while the mean value theorems of Chapter 27 will be used to estimate $\int|R(X,T)|^2dt$. Unfortunately, in this latter endeavour the main term is larger than the error term only when $X=o\bigl(T(\log T)/\log\log T\bigr)$.

To see how we might derive useful information concerning the zeros from asymptotic information concerning $F(X,T)$, we observe that if $R(\alpha)\in L^1(\mathbb{R})$, then

$$\int_{-\infty}^{\infty}R(\alpha)F\bigl(T^\alpha,T\bigr)\,d\alpha=\sum_{\substack{0<\gamma\le T\\0<\gamma'\le T}}w(\gamma-\gamma')\widehat{R}\Bigl(\frac{(\gamma-\gamma')\log T}{2\pi}\Bigr). \tag{29.6}$$ E:intRF

In view of the remarks we have already made, we see that we can determine the asymptotic size of left hand side of the above only when $\operatorname{supp}R\subseteq[-1,1]$, but it is still the case that

$$\int_{-\infty}^{\infty}R(\alpha)F\bigl(T^\alpha,T\bigr)\,d\alpha\ge\int_{-1}^{1}R(\alpha)F\bigl(T^\alpha,T\bigr)\,d\alpha \tag{29.7}$$ E:intRFineq

for R with arbitrary support, provided that $R(\alpha)\ge 0$ whenever $|\alpha|\ge 1$. (The above inequality is reversed in case $R(\alpha)\le 0$ when $|\alpha|\ge 1$.)

29.1 The basic asymptotic estimate and conjectures

S:PCEst

We now prove the approximate identity (29.5). To this end, let $K(w)=\frac{2}{1-w^2}=\frac{1}{1-w}+\frac{1}{1+w}$. Thus $K(w)$ has simple poles at ±1, and $K(iv)=$

$\frac{1}{1+v^2}$. By the calculus of residues we see that

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} K(w)x^w\,dw = \begin{cases} \frac{1}{x} - x & (x \ge 1), \\ 0 & (0 \le \le 1). \end{cases}$$

Hence

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{\zeta'}{\zeta}\big(w + \tfrac{1}{2} + it\big)K(w)X^w\,dw = \sum_{n\le X}\frac{\Lambda(n)}{n^{\frac{1}{2}+it}}\Big(\frac{X}{n} - \frac{n}{X}\Big)$$

for $a > 1$. Now $\frac{\zeta'}{\zeta}\big(w+\frac{1}{2}+it\big)$ has a simple pole at $\frac{1}{2} - it$ with residue -1, simple poles at $-2n - \frac{1}{2} - it$ with residue 1 for $n = 1, 2, \ldots$, and simple poles at $i(\gamma - t)$ with residue 1. The convention here is that if a zero $\rho = \frac{1}{2} + i\gamma$ has multiplicity m_ρ, then in a sum over zeros a summand corresponding to ρ is repeated m_ρ times. Write $w = u + iv$. On moving the contour from the abscissa $u = a > 1$ to $u = -\infty$, we see that the above is

$$\begin{aligned} &= -\frac{\zeta'}{\zeta}\big(\tfrac{3}{2} + it\big)X + \frac{\zeta'}{\zeta}\big(-\tfrac{1}{2} + it\big)X^{-1} - K\big(\tfrac{1}{2} - it\big)X^{\frac{1}{2}-it} \\ &\quad + 2\sum_\gamma \frac{X^{i(\gamma-t)}}{1 + (\gamma - t)^2} + \sum_{n=1}^{\infty} K\big(-2n - \tfrac{1}{2} - it\big)X^{-2n-\frac{1}{2}-it}. \end{aligned}$$

That the contour may be moved with this result is justified in the same way that it was justified in §12.1 when we discussed the classical explicit formulæ. We note that $-\frac{\zeta'}{\zeta}\big(\frac{3}{2}+it\big) = \sum_n \Lambda(n)n^{-\frac{3}{2}-it}$, we multiply both sides of the above equation by X^{it}, and rearrange to see that

$$\begin{aligned} 2\sum_\gamma \frac{X^{i\gamma}}{1 + (\gamma - t)^2} = \frac{-1}{X^{1/2}}&\Big(\sum_{n\le X}\Lambda(n)\Big(\frac{X}{n}\Big)^{-\frac{1}{2}+it} + \sum_{n>X}\Lambda(n)\Big(\frac{X}{n}\Big)^{\frac{3}{2}+it}\Big) \\ &- \frac{\zeta'}{\zeta}\big(-\tfrac{1}{2} + it\big)X^{-1+it} + K\big(\tfrac{1}{2} - it\big)X^{\frac{1}{2}} \\ &- \sum_{n=1}^{\infty} K\big(-2n - \tfrac{1}{2} - it\big)X^{-2n-\frac{1}{2}}. \end{aligned}$$

This is an exact equation; to complete the proof of (29.5) we now estimate the last three terms above. In §10.2 we took the logarithmic derivative of the asymmetric form of the functional equation to show that

$$\frac{\zeta'}{\zeta}(s) = -\frac{\zeta'}{\zeta}(1 - s) + \log 2\pi - \frac{\Gamma'}{\Gamma}(1 - s) + \frac{\pi}{2}\cot\frac{\pi s}{2}.$$

Also, Theorem C.1 asserts that

$$\frac{\Gamma'}{\Gamma}(s) = \log s + O(1/|s|)$$

provided that $|s| \ge \delta$ and $|\arg s| < \pi - \delta$. Hence

$$\frac{\zeta'}{\zeta}\big(-\tfrac{1}{2} + it\big) = -\frac{\zeta'}{\zeta}\big(\tfrac{3}{2} - it\big) - \frac{\Gamma'}{\Gamma}\big(\tfrac{3}{2} - it\big) + O(1) = -\log \tau + O(1).$$

For the second term it suffices to observe that $K\big(\tfrac{1}{2} - it\big) \ll \tau^{-2}$. Since $X \ge 1$, the sum over n is $\ll X^{-\frac{5}{2}} \sum_n (n^2 + t^2)^{-1} \ll X^{-\frac{5}{2}}\tau^{-1} \ll X^{-1}$. Thus the proof of (29.5) is complete.

We now estimate the integral of the square of the modulus of the respective sides of (29.5) to establish

T:PC1 **Theorem 29.1** *(Assume RH.) Let $F(X,T)$ be defined as in (29.3). Then $F(1/X,T) = F(X,T)$ for $X > 0$, and*

$$F(X,T) = \frac{\log \frac{T}{2\pi} - 2}{X^2} + \frac{\log X}{\log \frac{T}{2\pi}} + O\Big(\frac{(\log 2X)^3}{X^{1/2}\log T}\Big) + O\Big(\frac{X \log\log 4X}{T \log T}\Big) \qquad (29.8)$$ E:PC1

for $1 \le X \le T$.

In the notation $X = \big(\frac{T}{2\pi}\big)^\alpha$, the main term above takes the shape

$$\Big(\frac{T}{2\pi}\Big)^{2\alpha}\Big(\log \frac{T}{2\pi} - 2\Big) + \alpha.$$

Here the first term behaves in the limit as a Dirac delta with mass 1 at the origin, since

$$\int_{-\infty}^{\infty} \Big(\frac{T}{2\pi}\Big)^{2\alpha} \log \frac{T}{2\pi}\, d\alpha = 1.$$

Proof Let $L(X,t)$ denote the left hand side of (29.5). Then

$$\int_0^T |L(X,t)|^2\, dt = 4\sum_{\gamma,\gamma'} X^{i(\gamma-\gamma')} \int_0^T \frac{dt}{\big(1 + (t-\gamma)^2\big)\big(1 + (t-\gamma')^2\big)}.$$

From (29.1) we see that if $0 \le t \le T$, then

$$\sum_{\gamma \notin [0,T]} \frac{1}{1 + (t-\gamma)^2} \ll \Big(\frac{1}{t+1} + \frac{1}{T-t+1}\Big)\log T,$$

and

$$\sum_{\gamma'} \frac{1}{1+(t-\gamma')^2} \ll \log T.$$

Hence

$$\sum_{\gamma\notin[0,T]} \sum_{\gamma'} \int_0^T \frac{dt}{\big(1+(t-\gamma)^2\big)\big(1+(t-\gamma')^2\big)} \ll (\log T)^3,$$

and so

$$\int_0^T |L(X,t)|^2\,dt = 4 \sum_{\substack{0<\gamma\le T\\ 0<\gamma'\le T}} X^{i(\gamma-\gamma')} \int_0^T \frac{dt}{\big(1+(t-\gamma)^2\big)\big(1+(t-\gamma')^2\big)} + O\big(\log T)^3\big).$$

If $t \ge T$, then

$$\sum_{0<\gamma\le T} \frac{1}{1+(t-\gamma)^2} \ll \frac{\log T}{t-T+1},$$

and similarly for γ', so

$$\sum_{\substack{0<\gamma\le T\\ 0<\gamma'\le T}} \int_T^\infty \frac{dt}{\big(1+(t-\gamma)^2\big)\big(1+(t-\gamma')^2\big)} \ll (\log T)^2 \int_T^\infty \frac{dt}{(t-T+1)^2} \ll (\log T)^3.$$

The estimate is the same if we integrate instead over the interval $-\infty < t \le 0$. Finally, it is easy to see that by the calculus of residues that

$$\int_{-\infty}^\infty \frac{dt}{\big(1+(t-a)^2\big)\big(1+(t-b)^2\big)} = \frac{2\pi}{(a-b)^2+4} = \frac{\pi}{2} w(a-b). \quad (29.9)$$ E:Def2w

Thus we conclude that

$$\int_0^T |L(X,t)|^2\,dt = F(X,T)T\log\frac{T}{2\pi} + O\big((\log T)^3\big)$$

uniformly for $X > 0$.

The error terms on the right hand side of (29.5) are due mainly to the pole and trivial zeros of the zeta function, but they are troublesome only when t is small. To avoid considering them for small t we let V be a parameter to be chosen laeter, and employ a crude method for the range $0 \le t \le V$. From (29.1) we see that

$$|R(X,t)| = |L(X,t)| \le L(1,t) \ll \log\tau$$

for all X. Hence

$$\int_0^V |R(X,t)|^2\,dt \ll V(\log V)^2$$

for $V \ge 2$.

We now consider the contributions of the main terms in $R(X,t)$ when $V \le t \le T$. We first observe that from Corollaries 26.5 and 26.6 it follows that

$$\begin{aligned}\int_V^T \Big|\sum_{n=1}^\infty \frac{\Lambda(n)}{n^{1/2}} W(n/X)n^{-it} + \frac{1}{X}\sum_{n=1}^\infty \frac{\Lambda(n)}{n^{1/2}} n^{it}\Big|^2\,dt \qquad (29.10)\\ = (T-V)\Big(\sum_{n=1}^\infty \frac{\Lambda(n)^2}{n} W(n/X)^2 + \frac{1}{X^2}\sum_{n=1}^\infty \frac{\Lambda(n)^2}{n^3}\Big)\\ + O\Big(\sum_{n=1}^\infty \frac{\Lambda(n)^2}{d_n} W(n/X)^2 + \frac{1}{X^2}\sum_{n=1}^\infty \frac{\Lambda(n)^2}{n^2}\Big)\end{aligned}$$

E:Term1

where d_n denotes the distance from n to the nearest other primepower. From the estimate $\psi(x) = x + O\big(x^{1/2}(\log 2x)^2\big)$ of Theorem 13.1 we see by integration by parts that

$$\begin{aligned}\sum_{n\le x}\Lambda(n)^2 &= \sum_{n\le x}\Lambda(n)\log n \;+\; O\big((\psi(x)-\vartheta(x))\log x\big)\\ &= x\log x - x + O\big(x^{1/2}(\log 2x)^3\big).\end{aligned}$$

Write this as $S(x) = x\log x - x + R(x)$. Then by a further integration by parts we see that

$$\begin{aligned}\sum_{n=1}^\infty \frac{\Lambda(n)^2}{n} W(n/X)^2 &= \int_0^\infty \frac{1}{u} W\big(u^2/X^2\big)\,dS(u)\\ &= \int_0^\infty \frac{\log u}{u} W\big(u^2/X^2\big)\,du + \Big[\frac{R(u)}{u} W\big(u^2/X^2\big)\Big|_0^\infty\\ &\qquad - \frac{1}{X^2}\int_0^X R(u)\,du + 3X^2\int_X^\infty \frac{R(u)}{u^4}\,du\\ &= \log X + O\big(X^{-1/2}(\log X)^3\big).\end{aligned}$$

By Theorem 26.7, the error terms in (29.10) are $\ll X\log\log 4X$. Thus the expression in (29.10) is

$$= (T-V)\Big(\log X + O\big(X^{-1/2}(\log X)^3\big)\Big) + O(X\log\log 4X). \qquad (29.11)$$

E:Term1again

The above accounts for the contributions of two of the main terms on the right hand side of (29.5). The mean square of the third main term is

$$X^{-2}\int_V^T \Big(\log\frac{\tau}{2\pi}\Big)^2\,dt = X^{-2}T\Big(\log\frac{T}{2\pi}\Big)^2 - 2X^{-2}T\log\frac{T}{2\pi} \qquad (29.12)$$
$$= \quad O\big(X^{-2}T\big) + O\big(X^{-2}V(\log V)^2\big).$$

E:Term2

The mean square size of the main terms in (29.5) are determined in (29.11) and (29.12), but it remains to show that the correlations between these two main terms is small. The first main term, times the complex conjugate of the second one, integrated, is

$$\frac{1}{X}\int_V^T \Big(\sum_{n=1}^\infty \frac{\Lambda(n)}{n^{1/2}}W(n/X)n^{-it} + \frac{1}{X}\sum_{n=1}^\infty \frac{\Lambda(n)}{n^{3/2}}n^{it}\Big)\log\frac{\tau}{2\pi}\,dt.$$

If $t > 0$, then $\tau = t + 4$, by definition. Thus if $0 < a < b$ and c is a nonzero real number, then $\int_a^b e^{ict}\log\tau\,dt \ll \frac{\log(b+4)}{|c|}$. This can be shown by integration by parts, or by appealing to Theorem **??**. Thus the above is

$$\ll \frac{\log T}{X}\Big(\sum_{n=1}^\infty \frac{\Lambda(n)}{n^{1/2}\log n}W(n/X) + \frac{1}{X}\sum_{n=1}^\infty \frac{\Lambda(n)}{n^{3/2}\log n}\Big)$$
$$\ll \frac{\log T}{X^{1/2}\log 2X}. \qquad (29.13)$$

E:Term12

On combining these estimates we deduce that

$$\int_V^T \Big|\sum_{n=1}^\infty \frac{\Lambda(n)}{n^{1/2}}W(n/X)n^{-it} - \frac{1}{X}\Big(\log\frac{\tau}{2\pi} + \frac{\zeta'}{\zeta}\big(\tfrac{3}{2} - it\big)\Big)\Big|^2\,dt$$
$$= T\log X + TX^{-2}\Big(\log\frac{T}{2\pi}\Big)^2 - 2T\log\frac{T}{2\pi} \qquad (29.14)$$
$$+ O\big(TX^{-1/2}(\log X)^3\big) + O\big(X\log\log 4X\big)$$
$$+ O(V\log X) + O\big(VX^{-2}(\log V)^2\big).$$

E:M1Est

The integral of the square of th sum of the error terms in (29.5) is

$$\int_V^T \big|X^{-1}\tau^{-1} + X^{1/2}\tau^{-2}\big|^2\,dt \ll X^{-2}V^{-1} + XV^{-3}. \qquad (29.15)$$

E:M2Est

To ensure that the last two error terms in (29.14) are majorized by the other error terms in that formula, we take $V = TX^{-1/2}(\log T)^{-2}$. Suppose that f_1 and f_2 are measurable functions, that $M_i = \int_a^b |f_i(t)|^2\,dt$, and that $M_1 \ge M_2$. Then

$$\int_a^b |f_1(t) + f_2(t)|^2\,dt = M_1 + O\big(\sqrt{M_1M_2}\big)$$

by the Cauchy–Schwarz inequality. Let M_1 Denote the integral in (29.14) and M_2 the integral in (29.15). In order that M_2 should make no contribution to our final result, we need to know that not only is M_2 majorized by the first two error terms in (29.14), but also that the larger quantity $\sqrt{M_1 M_2}$ is also majorized by these error terms. Since $M_1 \ll T(\log T)^2$ in all cases, this will be the case provided that $M_2 \ll TX^{-1}(\log T)^{-2}$. It is now easy to verify that this is the case (by a wide margin) when our choice of V is substituted into (29.15). □

If we could derive (assuming RH) an asymptotic formula for $F(X,T)$ when $T < X < T^A$ for any fixed $A > 1$, then by Fourier inversion we could determine (still assuming RH) the distribution of the differences $\gamma - \gamma'$ relative to the average spacing. The main issue would be to derive an asymptotic estimate for

$$\int_V^T \Big| \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{1/2}} W(n/X) n^{-it} \Big|^2 \, dt.$$

To assess the difficulty here, consider the simple formula

$$\int_{-T}^T \Big| \sum_{n \le N} a_n n^{-it} \Big|^2 \, dt = 2T \sum_{n \le N} |a_n|^2 + 2T \sum_{\substack{m,n \le N \\ m \ne n}} a_m \overline{a_n} \frac{\sin(T(\log m/n))}{T(\log m/n)}.$$

Let $h = n - m$. If $0 < |h| \le m/T$, then

$$\frac{\sin(T(\log m/n))}{T(\log m/n)} \asymp 1,$$

so such nondiagonal terms carry a weight comparable to that of a diagonal term. Of course, such nondiagonal terms exist only when N is larger than T. The function $\frac{\sin u}{u}$ is not in $L^1(\mathbb{R})$, nor is it of bounded variation. Thus it is not easy to determine the contribution of nondiagonal terms directly from the above. The situation is improved if we average. For example,

$$\int_{-T}^T \Big(1 - \frac{|t|}{T}\Big) \Big| \sum_{n \le N} a_n n^{-it} \Big|^2 \, dt = T \sum_{n \le N} |a_n|^2 + T \sum_{\substack{m,n \le N \\ m \ne n}} a_m \overline{a_n} \Big(\frac{\sin \frac{1}{2} T(\log n/m)}{\frac{1}{2} T(\log n/m)} \Big)^2.$$

The function $\big(\frac{\sin u}{u}\big)^2$ is in $L^1(\mathbb{R})$ and is also of bounded variation, and we could proceed from the above, but our work is made easier if we instead employ the smoothing found in Corollary 26.11.

22.1.1 Exercises

1. From (29.1) it follows that $\sum_\gamma \big(1+(t-\gamma)^2\big)^{-1} \ll \log\tau$. Use (29.5) to show that

$$\sum_\gamma \frac{1}{1+(t-\gamma)^2} = \log\frac{\tau}{2\pi} + 2\,\mathrm{Re}\,\frac{\zeta'}{\zeta}\big(\tfrac{3}{2}+it\big) + O(1/\tau).$$

Exer:Lambda^2 2. For $\sigma > 1$ set

$$f(s) = \sum_{n=1}^\infty \frac{\Lambda(n)^2}{n^s} = \sum_p \sum_{k=1}^\infty \frac{(\log p)^2}{p^{ks}},$$

$$g(s) = \sum_{n=1}^\infty \frac{\Lambda(n)\log n}{n^s} = \sum_p \sum_{k=1}^\infty \frac{k(\log p)^2}{p^{ks}} = \Big(\frac{\zeta'}{\zeta}(s)\Big)',$$

and put $F(x) = \sum_{n\le x}\Lambda(n)^2$, $G(x) = \sum_{n\le x}\Lambda(n)\log n$.

(a) Show that $\sum_{r=1}^\infty \varphi(r) f(rs) = g(s)$.

(b) Show that

$$\sum_{r=1}^\infty a_r g(rs) = f(s) \qquad (29.16)$$ E:fginverse

for $\sigma > 1$ if and only if

$$\sum_{\substack{r,k\\ rk=m}} a_r \frac{m}{r} = 1$$

for all m.

(c) Show that the identity immediately above is equivalent to asserting that

$$\zeta(s-1)\sum_{r=1}^\infty \frac{a_r}{r^s} = \zeta(s)$$

for $\sigma > 2$.

(d) Deduce that for $\sigma > 2$,

$$\sum_{r=1}^\infty \frac{a_r}{r^s} = \prod_p \Big(1 + \frac{1-p}{p^s} + \frac{1-p}{p^{2s}} + \cdots\Big).$$

(e) Conclude that (29.16) holds if

$$a_r = \prod_{p|r}(1-p).$$

3. Let $U = T/(2\pi)$. Determine the size of $U^{-2\alpha}\log U$ for the following values of α:

   (a) $\alpha = 1/\log U$;
   (b) $\alpha = (\log\log U)/\log U$;
   (c) $\alpha = \big(\log\log U - \frac{1}{2}\log\log\log U\big)/\log U$.

Exer:w,whatinvert

4. For $a > 0$ let

$$I(a,u) = 2\pi\int_{-\infty}^{\infty} e^{-2\pi a|x|}e(ux)\,dx, \qquad J(a,x) = \int_{-\infty}^{\infty}\frac{2a}{a^2+u^2}e(xu)\,du.$$

   (a) Show that

$$I(a,u) = 2\pi\int_0^{\infty} e^{-2\pi ax}\big(e(ux)+e(-ux)\big)\,dx.$$

   (b) Show that the above is

$$= \left[\frac{e^{-2\pi(a+iu)x}}{a+iu}\right]_0^{\infty} + \left[\frac{e^{-2\pi(a-iu)x}}{a-iu}\right]_0^{\infty}.$$

   (c) Deduce that

$$I(a,u) = \frac{2a}{a^2+u^2}.$$

   (d) Deduce that

$$w(u) = 2\pi\int_{-\infty}^{\infty} e^{-4\pi|x|}e(ux)\,dx. \qquad (29.17)$$

E:wexpand

   (e) Show that $J(a,-x) = J(a,x)$ for all x.
   (f) Replace the real variable u in the definition of $J(a,x)$ by the complex variable $w = u+iv$, consider the integral to be a contour integral in the complex plane, assume that $x \ge 0$, form a semicircular path in the upper halfplane, and calculate the residue at the pole at $w = ia$ to show that

$$J(a,x) = 2\pi e^{-2\pi a|x|}.$$

(g) Deduce that

$$\widehat{w}(x) = \int_{-\infty}^{\infty} w(u)e(-xu)\,du = 2\pi e^{-4\pi|x|}. \qquad (29.18) \quad \boxed{\texttt{E:whatexpand}}$$

Thus (29.17) asserts that $w(u) = \int_{\mathbb{R}} \widehat{w}(x)e(ux)\,dx$ is a valid Fourier expansion.

5. With $\widehat{w}(x)$ defined as above, show that

$$F(X, T2\pi \log T\Big)^{-1} \int_{-\infty}^{\infty} \widehat{w}(x)\Big|\sum_{0<\gamma\le T} X^{i\gamma}e(\gamma x)\Big|^2\,dx. \qquad (29.19) \quad \boxed{\texttt{E:Fform2}}$$

6. Let $a_1, a_2, a_3, \ldots$ be real numbers.

(a) Show that

$$\int_{-\infty}^{\infty}\Big(\sum_{i=1}^{n}\frac{1}{1+(x-a_i)^2}\Big)^2\,dx = 2\pi\sum_{1\le i,j\le n}\frac{1}{4+(a_i-a_j)^2}$$
$$\asymp \sum_{1\le i,j\le n}\frac{1}{1+(a_i-a_j)^2}.$$

(b) By Cauchy's inequality, or otherwise, show that

$$n^4 \le \Big(\sum_{1\le i,j\le n}\frac{1}{1+(a_i-a_j)^2}\Big)\Big(\sum_{1\le i,j\le n}(1+(a_i-a_j)^2)\Big).$$

(c) (Putnam 2011 B5) Show that if there is a constant $A > 0$ such that

$$\int_{-\infty}^{\infty}\Big(\sum_{i=1}^{n}\frac{1}{1+(x-a_i)^2}\Big)^2\,dx \le An$$

for all n, then there is a constant B such that

$$\sum_{1\le i,j\le n}(1+(a_i-a_j)^2) \ge Bn^3$$

for all n.

29.2 Applications

S:PCapps

22.3.1 Exercises

29.3 An arithmetic equivalent of the Pair Correlation Conjecture

S:PCArithEquiv

22.4.1 Exercises

29.4 Notes

S:NotesPairCorr

[MVB88] Berry, (1988) used physical reasoning to conjecture that

$$\int_0^T (S(t+\delta) - S(t))^2 \, dd = f(T)T + o(T)$$

where $f(T) =$. [AF90] Fujii (1990) used a formula for $S(t)$ due to [DAG87] Goldston (1987) that is similar to, but more useful than a similar formula of [AS46] Selberg (1946) to show that ... assuming RH. If, in addition the Strong Pair Correlation Conjecture is assumed, then Berry's Conjecture follows.

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Appendix I
The Weak Distribution of Measures

C:LimDist

I.1 Basic theory

S:BasicThy

We now develop an analogue for the real line $\mathbb{R}$ of uniform distribution in the circle group $\mathbb{T}$. This is useful in discussing the limiting distribution of the error term in the Prime Number Theorem, and in Chapter 23 the limiting distribution of additive functions.

Let μ be a measure on the real line. We say that μ is a *probability measure* if it is nonnegative and if its total mass is 1. That is, $\mu(\mathcal{S}) \ge 0$ for all measurable sets $\mathcal{S}$ and $\mu(\mathbb{R}) = 1$. The *distribution function* of μ is the function $F(x) = \mu((-\infty, x])$. Clearly μ is a probability measure if and only if F is increasing and

$$\lim_{x\to-\infty} F(x) = 0, \qquad \lim_{x\to+\infty} F(x) = 1. \tag{I.1}$$

E:limF(x)

Conversely, if F is increasing, right-continuous (i.e., $F(x^+) = F(x)$ for all real x), and satisfies (I.1), then there is a probability measure μ of which F is the distribution function. If μ is a measure, we say that a sequence of probability measures μ_N *tends weakly* to μ if $\lim_{N\to\infty} F_N(x) = F(x)$ whenever x is a point of continuity of F. We note that the distribution function F of a nonnegative measure can have at most countably many jump discontinuities. As a first observation about sequences of measures, we have

T:HellyThm

Theorem I.1 (Helly's theorem) *Let μ_1, μ_2, $\mu_3, \ldots$ be a sequence of probability measures. Then there is a strictly increasing sequence n_k of positive integers such that the subsequence μ_{n_k} is weakly convergent.*

Proof We proceed by a Cantor diagonal process. Let $x_1, x_2, \ldots$ be distinct and dense in $\mathbb{R}$. The numbers $F_N(x_1)$ have a limit point. Thus we may choose indices $N(j,1)$ so that $F_{N(j,1)}(x_1)$ converges. From the indices $N(j,1)$ we choose a subsequence $N(j,2)$ so that $F_{N(j,2)}(x_2)$ converges. We continue in this manner, and find that the sequence $F_{N(j,j)}(x_k)$ converges for all k. Hence there is a unique right-continuous function F such that $F_{N(j,j)}(x)$ tends to $F(x)$ for all points of continuity x of F. □

Consider now the particular case in which μ is the probability measure that attaches weight 1 to the integer N. The sequence μ_N converges to the measure that is identically 0. Thus a limit of probability measures need not be a probability measure. This can be explained by noting that (I.1) need not hold uniformly for all measures μ_N in a sequence. We say, however, that a collection of measures is *tight* if (I.1) holds uniformly for all measures μ of the family. Our first observation concerning tight families of measures is obvious.

T:ConvTightFam **Theorem I.2** *If $\mu_1, \mu_2, \dots$ is a sequence of probability measures converging weakly to a measure μ, and if the sequence is tight, then μ is a probability measure.*

If μ is a probability measure and f is bounded and continuous, then the integral $\int_{-\infty}^{+\infty} f(x)\,d\mu(x)$ is well-defined. In particular, for real t we let $\widehat{\mu}(t)$ denote the Fourier transform of the measure,

$$\widehat{\mu}(t) = \int_{-\infty}^{\infty} e(-tx)\,d\mu(x)\,.$$

We note that $\widehat{\mu}(0)$ is the total mass of a measure, so that $\widehat{\mu}(0) = 1$ for a probability measure.

L:PropMuHat **Lemma I.3** *Let μ be a probability measure. Then $|\widehat{\mu}(t)| \le 1$ for all t, and $\widehat{\mu}(t)$ is continuous. Let $\mathcal{S}$ be a tight family of probability measures. Then $\widehat{\mu}(t)$ is uniformly continuous in t, and uniformly so for $\mu \in \mathcal{S}$.*

Proof By the triangle inequality, $|\widehat{\mu}(t)| \le \int_{-\infty}^{\infty} 1\,d\mu = 1$. Suppose that $\varepsilon > 0$ is given, and that A is so large that $\mu([-A, A]) > 1 - \varepsilon$. Then for any real t,

$$\left| \int_{-A}^{A} e(-tx)\,d\mu(x) - \widehat{\mu}(t) \right| < \varepsilon\,.$$

Hence by the triangle inequality,

$$|\widehat{\mu}(t_1) - \widehat{\mu}(t_2)| < 2\varepsilon + \int_{-A}^{A} |e(-t_1 x) - e(-t_2 x)|\,d\mu(x)\,.$$

Here the integrand is $2|\sin \pi(t_1 - t_2)x| \le 2\pi|t_1 - t_2|A$. Thus if $|t_1 - t_2| <$

ε/A, then the integrand is uniformly $< 2\pi\varepsilon$. Hence the integral is $< 2\pi\varepsilon$, so that $|\widehat{\mu}(t_1) - \widehat{\mu}(t_2)| < 9\varepsilon$. Thus $\widehat{\mu}$ is uniformly continuous. If $\mathcal{S}$ is a tight family of measures, then the choice of A depends on ε and $\mathcal{S}$, but is independent of the individual $\mu \in \mathcal{S}$. □

L:Intfdmu **Lemma I.4** *If $f(t) \in L^1(\mathbb{R})$ and μ is a probability measure, then*

$$\int_{-\infty}^{\infty} \widehat{f}(x)\, d\mu(x) = \int_{-\infty}^{\infty} f(t)\widehat{\mu}(t)\, dt\,.$$

We note that $\widehat{f}$ is bounded and continuous, so that the integral on the left can be considered to be a Riemann–Stieltjes integral $\int \widehat{f}\, dF$. As $|f\widehat{\mu}| \le |f|$, the integral on the right is a Lebesgue integral for $f \in L^1(\mathbb{R})$.

Proof By the definitions of $\widehat{\mu}$ and $\widehat{f}$, the above asserts that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)e(-tx)\, dt\, d\mu(x) = \int_{-\infty}^{\infty} f(t) \int_{-\infty}^{\infty} e(-tx)\, d\mu(x)\, dt\,.$$

The interchange of integrals is justified by Fubini's theorem, in view of the joint integrability of $f(t)e(-tx)$. □

Co:FourierTransUnique **Corollary I.5** *If μ_1 and μ_2 are probability measures such that $\widehat{\mu}_1(t) = \widehat{\mu}_2(t)$ for all real t, then $\mu_1 = \mu_2$.*

Proof Let $I = [a, b]$ be a given interval, and let $f(x)$ be determined so that $\widehat{f}(t)$ is the piecewise linear function whose graph passes through the points $(-\infty, 0)$, $(a - \delta, 0)$, $(a, 1)$, $(b, 1)$, $(b + \delta, 0)$, $(\infty, 0)$. Then $f(x) \ll_\delta x^{-2}$, and hence $f \in L^1(\mathbb{R})$. By Lemma I.4 we see that

$$\int_{-\infty}^{\infty} \widehat{f}\, d\mu_1 = \int_{-\infty}^{\infty} \widehat{f}\, d\mu_2\,.$$

On the other hand, as δ tends to 0, the respective sides of this decrease to $\mu_i([a, b])$. Hence $\mu_1(I) = \mu_2(I)$ for all closed bounded intervals I, and it follows that $\mu_2 = \mu_2$. □

We now characterize weak convergence in a number of useful ways.

T:CharactWeakConv **Theorem I.6** *If μ, μ_1, $\mu_2, \ldots$ are probability measures, then the following are equivalent:*

(a) *The μ_N tend weakly to μ;*
(b) *For every bounded continuous function f, the integral $\int_{-\infty}^{\infty} f\, d\mu_N$ tends to $\int_{-\infty}^{\infty} f\, d\mu$;*
(c) *For each real number t, $\widehat{\mu}_N(t)$ tends to $\widehat{\mu}(t)$;*

(d) *For each real number* $T > 0$, $\widehat{\mu}_N(t)$ *tends to* $\widehat{\mu}(t)$ *uniformly for* $|t| \le T$.

Proof We first demonstrate that any one of the above conditions implies that the μ_N are tight. Suppose that (a) holds. There is an A such that $\mu([-A, A]) > 1 - \varepsilon$. Hence $\mu_N([-A, A]) > 1 - 2\varepsilon$ for all $N > N_0$, and consequently there is a B such that $\mu_N([-B, B]) > 1 - 2\varepsilon$ for all N. Since both (b) and (d) include (c), we now suppose that (c) holds. Take $f(t) = \chi_{[-\delta,\delta]}(t)/(2\delta)$ in Lemma I.4. Then

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} \widehat{\mu}_N(t)\,dt = \int_{-\infty}^{\infty} \frac{\sin 2\pi\delta x}{2\pi\delta x}\,d\mu_N(x)\,. \tag{I.2}$$

E:IntmuN

By the principle of dominated convergence, the left hand side tends to $\frac{1}{2\delta}\int_{-\delta}^{\delta}\widehat{\mu}(t)\,dt$ as N tends to infinity. But $\widehat{\mu}(0) = 1$ and $\widehat{\mu}$ is continuous at 0, so if δ is small, then

$$\left|\frac{1}{2\delta}\int_{-\delta}^{\delta} \widehat{\mu}(t)\,dt - 1\right| < \varepsilon\,.$$

Hence

$$\int_{-\infty}^{\infty} \frac{\sin 2\pi\delta x}{2\pi\delta x}\,d\mu_N(x) > 1 - 2\varepsilon$$

for all $N > N_0$. Here the integrand is ≤ 1 for all real x, and is $\le 1/2$ when $|x| \ge B = 1/\delta$, so that this integral is

$$\le \mu_N([-B, B]) + \frac{1}{2}\mu_N\{|x| > B\} = \frac{1}{2} + \frac{1}{2}\mu_N([-B, B])\,.$$

Hence $\mu_N([-B, B]) > 1 - \varepsilon$ for $N > N_0$, and therefore the μ_N are tight.

We now derive (b) from (a). Suppose that $|f(x)| \le M$ for all x, that $F(x)$ is the distribution function of μ, and that F_N is the distribution function of μ_N. Since F is increasing, the set of points of discontinuity of F is at most countable. The same is true of each F_N, and hence there a set $\mathcal{D}$ that is at most countably infinite, such that if $x \notin \mathcal{D}$, then F and each of the functions F_N is continuous at x. Suppose that $\varepsilon > 0$ is given. Choose $A \notin \mathcal{C}$ so large that $\mu\{|x| > A\} < \varepsilon$ and $\mu_N\{|x| > A\} < \varepsilon$ for all N. Such an A exists because the family is tight. Hence

$$\left|\int_{-\infty}^{\infty} f(x)\,d\mu_N - \int_{-\infty}^{\infty} f(x)\,d\mu\right| \le \left|\int_{-A}^{A} f(x)\,d\mu_N - \int_{-A}^{A} f(x)\,d\mu\right| + 2M\varepsilon.$$

Since $f(x)$ is uniformly continuous on $[-A, A]$, we may choose numbers x_k, $-A = x_0 < x_1 < \ldots < x_k = A$, so that $|f(x) - f(x')| < \varepsilon$ if

$x_{k-1} \le x \le x' \le x_k$ for some k. Moreover, these x_k may be chosen so that none of them is a member of $\mathcal{D}$. Then

$$\left| \int_{x_{k-1}}^{x_k} f(x)\, d\mu - f(x_k)\big(F(x_k) - F(x_{k-1})\big) \right| \le \varepsilon\big(F(x_k) - F(x_{k-1})\big)\,.$$

This also holds with μ replaced by μ_N. On summing over k and using the triangle inequality, we find that the difference in question has absolute value not exceeding

$$M \sum_{k=1}^{K} |\mu_N([x_{k-1}, x_k]) - \mu([x_{k-1}, x_k])| + 2(M+1)\varepsilon\,.$$

From the hypothesis (a) it follows that this first term tends to 0 as N tends to infinity. Since ε is arbitrarily small, this gives (b).

We note that (c) is the special case $f(x) = e(-tx)$ of (b), and hence (b) implies (c).

Suppose that (c) holds. Since the μ_N are tight, by Lemma I.3 it follows that the $\mu_N(t)$ are uniformly continuous both in t and in N. Hence (d) follows from (c).

To complete the proof it remains to show that (d) implies (a). Suppose that $f \in L^1(\mathbb{R})$. Then by the principle of dominated convergence

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} f(t)\widehat{\mu}_N(t)\, dt = \int_{-\infty}^{\infty} f(t)\widehat{\mu}(t)\, dt,$$

and hence by Lemma I.4,

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} \widehat{f}(x)\, d\mu_N(x) = \int_{-\infty}^{\infty} \widehat{f}(x)\, d\mu(x).$$

Let f be chosen as in the proof of Corollary I.5, so that $f \in L^1(\mathbb{R})$ and f is a piecewise-linear majorant of $\chi_{[a,b]}$. Then

$$\mu_N([a,b]) \le \int_{-\infty}^{\infty} \widehat{f}(x)\, d\mu_N(x) \longrightarrow \int_{-\infty}^{\infty} \widehat{f}(x)\, d\mu(x) \le \mu([a-\delta, b+\delta])$$

as $N \to \infty$. Here the last member tends to $\mu([a,b])$ as δ tends to 0, and thus we conclude that $\limsup_{N\to\infty} \mu_N([a,b]) \le \mu([a,b])$. Supposing that $a < b$, we may construct $f \in L^1(\mathbb{R})$ so that $\widehat{f}$ forms a piecewise-linear minorant of $\chi_{[a,b]}$, and thus similarly show that $\liminf_{N\to\infty} \mu_N([a,b]) \ge \lim_{\delta\to 0^+} \mu([a+\delta, b-\delta])$. This last expression is equal to $\mu([a,b])$ provided that $\mu(\{a\}) = \mu(\{b\}) = 0$, i.e. if the distribution function F is continuous at a and at b. Thus we have shown that

$$\lim_{N\to\infty} F_N(b) - F_N(a) = F(b) - F(a)$$

whenever a and b are points of continuity of F. Since the family μ_N is tight, the numbers $F_N(a)$ and $F(a)$ are near 0 if a is large and negative. Hence $\lim_{N\to\infty} F_N(b) = F(b)$, which is (a). □

We now sharpen Theorem I.6 by showing that the limiting measure need not be given in advance.

T:muNConv

Theorem I.7 *Let $\mu_1, \mu_2, \ldots$ be probability measures, and suppose that for each t the sequence $\widehat{\mu}_1(t), \widehat{\mu}_2(t), \ldots$ converges. Call the limit $r(t)$. If r is continuous at 0, then the μ_N are weakly convergent to a probability measure μ, and $r(t) = \widehat{\mu}(t)$.*

Proof We note that $\widehat{\mu}_N(0) = 1$ for all N, so that $r(0) = 1$. In proving Theorem I.6 we showed that condition (c) implies that the μ_N are tight. The only properties of $\widehat{\mu}(t)$ used in that argument were that $\widehat{\mu}(0) = 1$ and that $\widehat{\mu}(t)$ is continuous at 0. Hence by the same method we may show that the μ_N are tight. By Theorem I.1 there is a subsequence $\mu_{N(k)}$ that is weakly convergent to a measure μ. By Theorem I.2 we see that μ is a probability measure. By Theorem I.6 applied to the subsequence $\mu_{N(k)}$, we deduce that $\widehat{\mu}_{N(k)}(t) \to \widehat{\mu}(t)$ as $k \to \infty$, for all real t. Hence $\widehat{\mu} = r$, and $\widehat{\mu}_N(t) \to \widehat{\mu}(t)$. Then by Theorem I.6 again we conclude that the μ_N tend weakly to μ. □

We now characterize continuous measures in terms of the behaviour of $\widehat{\mu}$.

T:CharContMeas

Theorem I.8 *Let μ be a probability measure. Then μ is continuous if and only if*

$$\int_{-T}^{T} |\widehat{\mu}(t)|^2 \, dt = o(T)$$

as T tends to ∞.

Proof The left hand side is

$$\int_{-T}^{T} \int_{-\infty}^{\infty} e(-tx)\, d\mu(x) \int_{-\infty}^{\infty} e(ty)\, d\mu(y)\, dt$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-T}^{T} e(t(y-x))\, dt\, d\mu(x)\, d\mu(y)$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\sin 2\pi T(y-x)}{\pi(y-x)}\, d\mu(x)\, d\mu(y)\,.$$

If we divide by $2T$, then by the principle of dominated convergence this latter integral tends to

$$\iint_{x=y} d\mu(x)\, d\mu(y) = \sum_i d_i^2$$

as T tends to infinity, where the d_i are the heights of the jump discontinuities of μ. But μ is continuous if and only if $\sum d_i^2 = 0$, so we have the result. □

??.1 Exercises

1. Suppose that $\mu_1, \mu_2, \ldots$ are probability measures such that for each convergent subsequence the limit is a probability measure. Show that the sequence is tight.
2. Construct an example to show that in the situation of Theorem I.6, the convergence of the $\widehat{\mu}_N(t)$ to $\widehat{\mu}(t)$ need not be uniform in t.
3. Suppose that μ_N is a sequence of probability measures on $\mathbb{T}$, and that μ is also a probability measure on $\mathbb{T}$. We say that the μ_N tend weakly to μ if $\mu_N([\alpha, \beta]) \to \mu([\alpha, \beta])$ whenever $0 \le \beta - \alpha \le 1$ and $\mu(\{\alpha\}) = \mu(\{\beta\}) = 0$. Show that the following are equivalent:
   (a) The μ_N tend weakly to μ;
   (b) For each integer k, $\widehat{\mu}_N(k)$ tends to $\widehat{\mu}(k)$ as N tends to infinity;
   (c) If f is a continuous function defined on $\mathbb{T}$, then $\int_{\mathbb{T}} f(x)\, d\mu_N(x)$ tends to $\int_{\mathbb{T}} f(x)\, d\mu$ as N tends to infinity.
4. Let μ be the measure on $\mathbb{R}$ defined by $d\mu = (2\pi)^{-1/2} e^{-x^2/2}\, d\lambda$ where λ denotes Lebesgue measure. Show that μ is a probability measure. (This is called emphnormal distribution with mean 0 and variance 1.) Show that $\int_{\mathbb{R}} x\, d\mu(x) = 0$, that $\int x^2\, d\mu(x) = 1$, and that $\widehat{\mu}(t) = e^{-2\pi^2 t^2}$.
5. Let p be a fixed real number, $0 < p < 1$. For each positive integer N let μ_N be the discrete measure that has a point mass of weight $\binom{N,n}{p}^n (1-p)^{N-n}$ at $(n - Np)/\sigma$ for $n = 0, 1, 2, \ldots, N$ where $\sigma = \sqrt{np(1-p)}$. Show that μ_N is a probability measure. Show that $\widehat{\mu}_N(t) = \big(pe(-t(1-p)/\sigma) + (1-p)e(-tp/\sigma)\big)^N$. Show that if T is given, then $\widehat{\mu}_N(t) = e^{-2\pi^2 t^2} + O(n^{-1/2})$ as N tends to infinity, uniformly for $|t| \le T$. Conclude that the measures μ tend weakly to the measure μ of the preceding exercise. (This is a special case of the Central Limit Theorem of probability theory.)

I.2 Notes

S:NotesLimDist

Section **??**. The result of Exercise **??** is due to Bohl (1906) (see p. 279 of his paper). Later, Bohr (1925) created an extensive theory of almost periodic functions, and in the course of this demonstrated (cf pp. 119–121) that Bohl's Theorem is equivalent to the localized form of Kronecker's Theorem, i.e., to our Corollary **??**.

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Appendix J
Topics in harmonic analysis III

C:AppJHarmAnalIII

J.1 A majorant inequality

S:magorant

T:L2Majorant **Theorem J.1** *For positive integers n let λ_n be real, suppose that $|a_n| \le b_n$ for all n, and that $\sum_{n=1}^\infty b_n < \infty$. Then for any real T_0 and any $T > 0$,*

$$\int_{T_0-T}^{T_0+T} \Big| \sum_{n=1}^\infty a_n e(\lambda_n t)\Big|^2 \, dt \le 3 \int_{-T}^{T} \Big| \sum_{n=1}^\infty b_n e(\lambda_n t)\Big|^2 \, dt \,.$$

Proof It suffices to prove the inequality when $T_0 = 0$, for once this is done, the general case follows by replacing a_n by $a_n e(\lambda_n T_0)$. Let $K(t) = \max(1 - |t|/T, 0)$. Then

$$\widehat{K}(u) = \frac{1}{T}\Big(\frac{\sin \pi T u}{\pi u}\Big)^2 \ge 0,$$

so

$$\begin{aligned} \int_{-\infty}^{\infty} K(t)\Big| \sum_{n=1}^\infty a_n e(\lambda_n t)\Big|^2 \, dt &= \sum_{m,n} a_m \overline{a_n} \widehat{K}(\lambda_n - \lambda_m) \\ &\le \sum_{m,n} b_m b_n \widehat{K}(\lambda_n - \lambda_m) \\ &= \int_{-\infty}^{\infty} K(t)\Big| \sum_{n=1}^\infty b_n e(\lambda_n t)\Big|^2 \, dt \,. \end{aligned}$$

By replacing a_n by $a_n e(\lambda_n T_0)$, we see more generally that

$$\int_{-\infty}^{\infty} K(t - T_0)\Big| \sum_{n=1}^\infty a_n e(\lambda_n t)\Big|^2 \, dt \le \int_{-\infty}^{\infty} K(t)\Big| \sum_{n=1}^\infty b_n e(\lambda_n t)\Big|^2 \, dt \,. \quad \text{(J.1)}$$ E:WeightedL2Est

But $\chi_{[-T,T]}(t) \le K(t+T) + K(t) + K(t-T)$, so we apply (J.1) three times, with $T_0 = -T$, $T_0 = 0$, and $T_0 = T$, and sum to find that

$$\int_{-T}^{T} \Big| \sum_{n=1}^{\infty} a_n e(\lambda_n t) \Big|^2 \, dt \le 3 \int_{-\infty}^{\infty} K(t) \Big| \sum_{n=1}^{\infty} b_n e(\lambda_n t) \Big|^2 \, dt \, .$$

This gives the stated result, since $K(t) \le \chi_{[-T,T]}(t)$. $\square$

Appendix K
Notes on Turán's Power sum method

C:Turan

K.1 Turán's First Main Theorem

S:Turan1

Let a_n, $z_n \in \mathbb{C}$ and consider the power sum

$$S_h = \sum_{n=1}^{N} a_n z_n^h \quad (h = 0, 1, 2. \ldots). \tag{K.1}$$ E:PowSum

It is sometimes useful to know that h can be chosen so that the majority of the terms $a_n z_n^h$ point in approximately the same direction, And to have some control over the size or range of h for which this occurs. If the numbers $\arg z_n$ are linearly independent over $\mathbb{Q}$, then it follows by Kronecker's theorem that given $\varepsilon > 0$ there are h such that

$$|S_h| \geq (1 - \varepsilon) \sum_{n=1}^{N} |a_n z_n^h|. \tag{K.2}$$ E:PowKron

The difficulty with this in applications is that we do not have any control over the size of h in terms of ε.

Any lower bound we obtain for $|S_h|$ will depend on the nature of the a_n. The simplest general comparison, therefore is with S_0.

T:Tur1 **Theorem K.1** (Turán's First Main Theorem) *Suppose that $|z_n| \geq 1$ ($n \in \mathbb{N}$). Then for any $M \geq 0$ and $N \geq 1$ there is an integer $h \in [M + 1, M + N]$ such that*

$$|S_h| \geq \lambda(M, N)|S_0| \tag{K.3}$$ E:Turlb1

where

$$\lambda(M, N) = \frac{1}{\sum_{k=0}^{N-1} \binom{M+k}{k} 2^k}. \tag{K.4}$$ E:Tur1lam

Moreover

$$\lambda(M,N) \ge \left(\frac{N}{2e(M+N)}\right)^{N-1} \tag{K.5}$$ E:Tur2lam

T:Tur1Cor **Corollary K.2** *For any $M \ge 0$ there is an integer $h \in [M+1, M+N]$ such that*

$$|S_h| \ge \big(\min_{1\le n\le N} |z_n|\big)^h \left(\frac{N}{2e(M+N)}\right)^{N-1} |S_0|. \tag{K.6}$$ E:Turlb2

Proof It suffices to prove the theorem when the z_n are distinct, for then the case when two or more of the z_n are identical follows by combining them and appealing to the case of smaller N. Note that, for a given M, $\lambda(M,N)$ is an increasing function of N.

Let $b_0, \dots, b_{N-1}$ be complex numbers at our disposal. Then

$$\left|\sum_{j=0}^{N-1} b_j S_{M+1+j}\right| \le \sum_{j=0}^{N-1} |b_j| \max_{M+1\le h\le M+N} |S_h|. \tag{K.7}$$ E:blincom

Inserting the definition (K.1) of S_h into the left hand side and interchanging the order of summation gives

$$\sum_{n=1}^{N} a_n z_n^{M+1} \sum_{j=0}^{N-1} b_j z_n^j = \sum_{n=1}^{N} a_n z_n^{M+1} \mathcal{P}(z_n) \tag{K.8}$$ E:blinpol

where

$$\mathcal{P}(z) = \sum_{j=0}^{N-1} b_j z^j.$$

Consider the system of N linear equations in N variables $b_0, \dots, b_{N-1}$

$$\mathcal{P}(z_n) = z_n^{-M-1} \quad (n = 1, \dots, N). \tag{K.9}$$ E:blineq

The coefficient matrix is a Vandermonde matrix whose determinant

$$\prod_{1\le i<j\le N} (z_j - z_i)$$

is non-zero. Thus (K.9) determines the b_j uniquely and, by (K.7) and (K.8),

$$|S_0| \le \sum_{j=0}^{N-1} |b_j| \max_{M+1\le h\le M+N} |S_h|. \tag{K.10}$$ E:blub

Let $Q_0(z) = 1$ and

$$Q_k(z) = \prod_{n=1}^{k} (z - z_n).$$

Then there are c_k such that

$$\mathcal{P}(z) = \sum_{k=0}^{N-1} c_k Q_k(z),$$

since the system of linear equations connecting thec_j and the b_j is triangular with 1s on diagonal.

Now consider for $k \geq 0$ the integral

$$\frac{1}{2\pi i} \int_{\mathcal{C}_R} \frac{\mathcal{P}(z)}{Q_{k+1}(z)} dz$$

where $\mathcal{C}_R$ denotes the circle centered at 0 and of radius R, described in the positive sense, and such that

$$R > |z_n| \quad (n = 1, \ldots, N).$$

The function

$$Q_{k+1}^{-1}(z) \sum_{j=k+1}^{N-1} c_j Q_j(z)$$

has only removable singularities. Hence

$$\frac{1}{2\pi i} \int_{\mathcal{C}_R} \frac{\sum_{j=k+1}^{N-1} c_j Q_j(z)}{Q_{k+1}(z)} dz = 0$$

Moreover when $j \leq k - 1$

$$\frac{Q_j(z)}{Q_{k+1}} \ll R^{-2}$$

so, by Cauchy's theorem, letting $R \to \infty$ shows that

$$\frac{1}{2\pi i} \int_{\mathcal{C}_R} \frac{\sum_{j=0}^{k-1} c_j Q_j(z)}{Q_{k+1}(z)} dz = 0.$$

Therefore

$$\frac{1}{2\pi i} \int_{\mathcal{C}_R} \frac{\mathcal{P}(z)}{Q_{k+1}(z)} dz = c_k.$$

We also have

$$\frac{z^{-M-1}}{Q_{k+1}(z)} \ll R^{-2}.$$

Hence

$$c_k = \frac{1}{2\pi i}\int_{\mathcal{C}_R} \frac{\mathcal{P}(z) - z^{-M-1}}{\mathcal{Q}_{k+1}(z)}dz$$

By (K.9), $z^{M+1}\mathcal{P}(z) - 1$ has a zero at z_n $(n = 1, \dots, N)$. Thus the integrand is analytic for $z \neq 0$ and we can replace $\mathcal{C}_R$ by $\mathcal{C}_r$ where $0 < r < 1$. But now $\mathcal{P}(z)/\mathcal{Q}_{k+1}(z)$ is analytic for $|z| \le r$. Therefore

$$\begin{aligned} c_k &= \frac{1}{2\pi i}\int_{\mathcal{C}_r} \frac{-z^{-M-1}}{\mathcal{Q}_{k+1}(z)}dz \\ &= \frac{(-1)^k \lambda_k(M)}{z_1 \dots z_{k+1}} \end{aligned}$$

where

$$\lambda_k(M) = \frac{1}{2\pi i}\int_{\mathcal{C}_r} z^{-M-1}\prod_{n=1}^{k+1}(1 - z/z_n)^{-1}dz.$$

When $|w| < r$ we have

$$\begin{aligned} \sum_{M=0}^{\infty}\lambda_k(M)w^M &= \frac{1}{2\pi i}\int_{\mathcal{C}_r} \frac{\prod_{n=1}^{k+1}(1 - z/z_n)^{-1}}{z - w}dz. \\ &= \prod_{n=1}^{k+1}(1 - w/z_n)^{-1}. \end{aligned}$$

Since

$$(1 - w/z_n)^{-1} = \sum_{M=0}^{\infty} z_n^{-M}w^M$$

has coefficients of modulus $|z_n|^{-M} \le 1$ it follows that $|\lambda_k(M)|$ does not exceed the coefficient of w^M in $(1 - w)^{-k-1}$ and this is

$$(-1)^M\binom{-k-1}{M} = \binom{M+k}{k}.$$

Thus

$$|c_k| \le \binom{M+k}{k}/|z_1 \dots z_{k+1}|.$$

Now we compare

$$\sum_{k=0}^{N-1} c_k\mathcal{Q}_k(z) = \mathcal{P}(z) = \sum_{h=0}^{N-1} b_h z^h$$

with

$$\sum_{k=0}^{N-1}\binom{M+k}{k}(z+1)^k = \mathcal{Q}(z) = \sum_{h=0}^{N-1} B_h z^h.$$

The coefficients of the polynomial $\mathcal{Q}(z)$ majorise those of $\mathcal{P}(z)$. Thus

$$\sum_{h=0}^{N-1}|b_h| \le \sum_{h=0}^{N-1} B_h = \mathcal{Q}(1) = \sum_{k=0}^{N-1}\binom{M+k}{k}2^k,$$

and, by (K.10), gives (K.3) with (K.4). To obtain (K.5) observe that

$$\sum_{k=0}^{N-1}\binom{M+k}{k}2^k < 2^{N-1}\sum_{k=0}^{N-1}\binom{M+k}{k}$$

and here the sum is the coefficient of z^M in

$$\sum_{k=0}^{N-1}(1+z)^{M+k} = z^{-1}(1+z)^M\big((1+z)^N - 1\big)$$

and this is

$$\binom{N+M}{M+1} \le \frac{(M+N)^{N-1}}{(N-1)!} \le \left(\frac{eM+eN}{N}\right)^{N-1}$$

which completes the proof of the theorem. □

I.1 Exercises

1. Prove that the constant $\lambda(M,N)$ in Theorem K.1 cannot be made any smaller.

2. Prove that for any $d \in \mathbb{N}$ there is a polynomial $\mathcal{Q}(z)$ of degree d such that $\mathcal{Q}(0) = \mathcal{Q}(1) = 1$ and

$$\max_{|z|\le 1}|\mathcal{Q}(z)| \le 1 + 2/d.$$

3. Suppose that $|z_n| \ge 1$ $(1 \le n \le N)$.

   (a) Prove that if $N \le H \le N^2$, then

$$\sum_{h=1}^{H}|S_h|^2 \ge e^{-8N^2/H}|S_0|^2.$$

(b) Prove that if $N \le H \le N^2$, then

$$\sum_{h=1}^{H} |S_h|^2 \ge e^{-8N^2/H} |S_0|^2.$$

(c) Prove that if $H \ge N^2$, then

$$\sum_{h=1}^{H} |S_h|^2 \gg HN^{-2} |S_0|^2.$$

K.2 Turán's Second main Theorem

S:Turan2

In §K.1 lower bounds were obtained for S_h in terms of S_0 and $(\min_n |z_n|)^h$. For some questions it is necessary to have bounds in term of S_0 and $(\max_n |z_n|)^h$. However if we take ε to be a small positive number and define

$$z_1 = 1,\ z_n = \varepsilon\,(n > 1),\ b_1 = \varepsilon,\ b_n = \frac{1}{N-1}\,(n > 1)$$

it follows that

$$S_0 = 1 + \varepsilon,\ |S_h| = \varepsilon^h + \varepsilon \le 2\varepsilon |S_0|.$$

Thus it is necessary to have extra constraints in order to deal with the more demanding requirements.

T:Tur2 **Theorem K.3** (Turán's Second Main Theorem) *Suppose that*

$$1 = |z_1| \ge |z_2| \ge \ldots \ge |z_N|.$$

Then for any $M \ge 0$ and $N \ge 1$ there is an h such that $M + 1 \le h \le M + N$ and

$$|S_h| \ge 2 \left(\frac{N}{8e(M+N)} \right)^N \min_{1 \le J \le N} \left| \sum_{n=1}^{J} a_n \right|.$$

T:Tur2cor **Corollary K.4** *Suppose that $|z_1| \ge |z_2| \ge \ldots \ge |z_N|$. Then for any $M \ge 0$ and $N \ge 1$ there is an h such that $M + 1 \le h \le M + N$ and*

$$|S_h| \ge 2\big(\max_{1 \le n \le N} |z_n| \big)^h \left(\frac{N}{8e(M+N)} \right)^N \min_{1 \le J \le N} \left| \sum_{n=1}^{J} a_n \right|.$$

In applications it is useful to have a more relaxed range for h.

T:Tur2corcor **Corollary K.5** *Suppose that $|z_1| \ge |z_2| \ge \ldots \ge |z_N|$. Then for any $L \ge N$ and $N \ge 1$ there is an h such that $L+1 \le h \le 2L$ and*

$$|S_h| \ge 2(16e)^{-L}\big(\max_{1\le n\le N}|z_n|\big)^h \min_{1\le J\le N}\left|\sum_{n=1}^{J} a_n\right|.$$

To deduce this corollary let $M = L$ in Corollary K.4 and obseerve that the inequality

$$1 + y/2 \le (16e)^y$$

holds for all $y \ge 0$. Then with $y = -1 + L/N$ it follows that

$$8e(N+L)/N \le (16e)^{L/N}, \ \left(\frac{N}{8e(L+N)}\right)^N \ge (16e)^{-L}.$$

Proof To prove Theorem K.3 we may suppose as in the proof of Theorem K.1 that the z_n are distinct.

A theorem of Chebyshev (see Exercise K.2.1) states that if $f(x)$ is a monic polynomial of degree d and I is an interval of length l, then

$$\max_{x\in I}|f(x)| \ge 2(l/4)^d.$$

We apply this to

$$f(x) = \prod_{n=1}^{N}(x - |z_n|)$$

and $I = [a,1]$ where

$$a = \frac{M}{N+M}.$$

Then there is an $r \in [a,1]$ such that

$$|f(r)| \ge 2\left(\frac{1-a}{4}\right)^N$$

and so

$$r^M \prod_{n=1}^{N} |r - |z_n|| \ge 2(1+N/M)^{-M}\left(\frac{N}{4M+4N}\right)^N > 2\left(\frac{N}{4e(M+N)}\right)^N. \quad \text{(K.11)}$$ E:auxbd

Note that $r \ne |z_n|$ $(1 \le n \le N)$. Now we choose $J \ge 1$ maximally so that $|z_J| > r$, and then for $n > J$ we have $|z_n| < r$.

We follow essentially the same idea as in Theorem K.1. We now choose the coefficients b_j of $\mathcal{P}$ so that

$$\mathcal{P}(z_n) = \begin{cases} z_n^{-M-1} & (1 \le n \le J), \\ 0 & (J < n \le N). \end{cases}$$

and recall that they are uniquely determined. Then in place of (K.10) we obtain

$$\left| \sum_{n=1}^{J} a_n \right| \le \sum_{j=0}^{N-1} |b_j| \max_{M+1 \le h \le M+N} |S_h|.$$

Again, as before,

$$\frac{1}{2\pi i} \int_{\mathcal{C}_R} \frac{\mathcal{P}(z) - z^{-M-1}}{\mathcal{Q}_{k+1}(z)} dz = c_k.$$

Also $z^{M+1}\mathcal{P}(z) - 1$ has a zero at z_n when $1 \le n \le J$. Hence the integrand has only removeable singularities for $|z| \ge r$ and so

$$c_k = \frac{1}{2\pi i} \int_{\mathcal{C}_r} \frac{\mathcal{P}(z) - z^{-M-1}}{\mathcal{Q}_{k+1}(z)} dz.$$

The zeros of $\mathcal{Q}_{k+1}(z)$ with $|z| < r$ are of the form $z = z_n$ with $n > J$. But such z_n are also zeros of $\mathcal{P}(z)$. Hence $\mathcal{P}(z)/\mathcal{Q}_{k+1}(z)$ only has removeable singularities on and inside $\mathcal{C}_r$. Therefore

$$c_k = \frac{1}{2\pi i} \int_{\mathcal{C}_r} \frac{-1}{z^{M+1}\mathcal{Q}_{k+1}(z)} dz.$$

and so

$$|c_k| \le \frac{1}{r^M \prod_{n=1}^{k+1} |r - |z_n||}.$$

Therefore, by (K.11),

$$|c_k| \le \lambda$$

where

$$\lambda = \frac{1}{2} \left(\frac{4e(M+N)}{N} \right)^N.$$

Now we compare

$$\sum_{k=0}^{N-1} c_k \mathcal{Q}_k(z) = \mathcal{P}(z) = \sum_{h=0}^{N-1} b_h z^h$$

with

$$\sum_{k=0}^{n-1} \lambda(z+1)^k = \mathcal{Q}(z) = \sum_{h=0}^{N-1} B_h z^h.$$

The coefficients of the polynomial $\mathcal{Q}(z)$ majorise those of $\mathcal{P}(z)$. Thus

$$\sum_{h=0}^{N-1} |b_h| \leq \sum_{h=0}^{N-1} B_h = \mathcal{Q}(1) = \sum_{k=0}^{N-1} \lambda 2^k < \lambda 2^N,$$

The theorem now follows. □

I.2 Exercises

1. The n-th Chebyshev polynomial $T_n(x)$ ($x \in \mathbb{R}$) is defined for $x \in [-1,1]$ by $T_n(x) = \cos(n \cos^{-1} x)$.

   (a) Prove that $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$ and

   $$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

   (b) Prove that

   $$\max_{x \in [-1,1]} |T_n(x)| = 1$$

   and that the maximum is attained precisely when $x = x_k = \cos \frac{\pi k}{n}$ $(0 \leq k \leq n)$ and then $T_n(x_k) = (-1)^k$.

   (c) Prove that if $f(x)$ is a monic polynomial of degree n, then

   $$\max_{x \in [-1,1]} |f(x)| \geq \frac{1}{2^{n-1}}.$$

   Hint: Show that if the maximum is smaller than 2^{n-1}, then $f(x) - 2^{1-n}T_n(x)$ changes sign at least n times.

   (d) Prove that if $f(x)$ is a monic polynomial of degree n, and I is an arbitrary interval of length l, then

   $$\max_{x \in I} |f(x)| \geq \frac{1}{2} \left(\frac{l}{4} \right)^n.$$

2. Suppose that $b_n = 1$ $(n = 1, \ldots, N)$

(a) Prove that if $|z_n| \geq 1$ $(n = 1, \ldots, N)$, then there is an h with $1 \leq h \leq N$ such that $|S_h| \geq 1$, and show that this is best possible.

(b) Prove that if S_h is real for all positive integers hm then there is an h with $1 \leq h \leq N + 1$ such that $S_h \geq 0$.

(c) Suppose that $\max_{1 \leq n \leq N} |z_n| = 1$. Prove that, for some h with $1 \leq h \leq 2N - 1$, $|S_h| \geq 1$.