AN UPPER BOUND FOR $G(k)$

sec:intro

1. Introduction

Our aim here is to give a relatively simple proof of the following theorem.

thm:Gubound

Theorem 1.1. *There is a constant C such that whenever $k \geq 3$ we have*

$$G(k) < 3k \log k + Ck.$$

The best general upper bound we have for large k,

$$G(k) < \lfloor k(\log k + 4.20032) \rfloor,$$

due to Brüdern and Wooley BW23 [2023], uses much more sophisticated methods.

sec:Dim

2. Diminishing ranges

Given a $t \in \mathbb{N}$ and a real X which can be supposed to be sufficiently large where necessary we define a finite sequence $P_j = P_j(X)$ by

$$P_1 = X^{1/k},\ P_j = \frac{1}{2} P_{j-1}^{1-1/k}\ (2 \leq j \leq t-1),\ P_t = P_{t-1}.$$

Then we have the fundamental lemma of diminishing ranges.

lem:fund

Lemma 2.1. *Suppose that $t \geq 2$ and $a(m) = a(m;X)$ denotes the number of solutions of*

$$x_1^k + \cdots x_t^k = m$$

with $P_j < x_j \leq 2P_j$. Then

$$\sum_m a(m)^2 \ll P_1 \dots P_t X^\varepsilon \ll \sum_m a(m) X^\varepsilon \ll \left(\sum_m a(m) \right)^2 X^{-1+\Delta_t+\varepsilon}$$

where

$$\Delta_t = \left(1 - \frac{2}{k}\right)\left(1 - \frac{1}{k}\right)^{t-2}.$$

Proof. We show that if $P_j < x_j, y_j \leq P_j$ and

$$x_1^k + \cdots x_t^k = y_1^k + \cdots y_t^k,$$

then

$$x_j = y_j \text{ for } 1 \leq j \leq t-2. \tag{2.1}$$

eq:diag

Thus the equation reduces to

$$x_{t-1}^k - y_{t-1}^k = y_t^k - x_t^k$$

and the number of solutions of this is $\ll P_t^2 X^\varepsilon = P_{t-1} P_t X^\varepsilon$. Hence it suffices to prove (2.1) [eq:diag].

We argue by induction. Suppose that $x_r = y_r$ for $r \leq j-1$ (this includes the case $j = 1$) and $x_j \neq y_j$. Then

$$|x_j^k - y_j^k| = |x_j - y_j|(x_j^{k-1} + \cdots y_j^{k-1}) \geq k P_j^{k-1} = k 2^k P_j^k.$$

We have $x_{j+1}^k + \cdots x_t^k \leq 2^{k+1} P_{j+1}^k + (t-j-2) 2^k P_{j+1}^{k-1}$ and likewise for $y_{j+1}^k + \cdots y_t^k$. Therefore

$$|x_j^k - y_j^k| > |x_{j+1}^k + \cdots + x_t^k - y_{j+1}^k - \ldots - y_t^k|,$$

which is impossible. □

We apply this construction in two different ways, namely to get much improved versions of both Hua's Lemma and Weyl's inequality.

Let

$$\mathcal{A}(\alpha) = \mathcal{A}(\alpha; X, t) = \sum_m a(m) e(\alpha m) \tag{2.2}$$

eq:defA

and

$$\mathcal{B}(\alpha) = \mathcal{B}(\alpha; Y, u)) = \sum_{Y^{1/k} < p \leq 2Y^{1/k}} \mathcal{A}(\alpha p^k; Y, u). \tag{2.3}$$

eq:defB

We define the coefficients $b(m)$ by

$$\mathcal{B}(\alpha) = \sum_m b(m) e(\alpha m) \tag{2.4}$$

eq:coeffB

so that $b(m)$ is the number of solutions of

$$p^k w_1^k + \cdots p^k w_u^k = m$$

with $Y^{1/k} < p \leq 2Y^{1/k}$ and $P_j(Y) < w_j \leq 2P_j(Y)$.

The following is a good substitute for Hua's Lemma.

lem:Huasub **Lemma 2.2.** *We have*

$$\int_0^1 |\mathcal{A}(\alpha)|^2 d\alpha \ll \mathcal{A}(0)^2 X^{-1+\Delta_t+\varepsilon}$$

Proof. This is immediate by Lemma lem:fund 2.1 □

A more surprising result is the next lemma.

lem:Weylsub **Lemma 2.3.** *Suppose that $\alpha \in \mathbb{R}$, $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $q \le 2^{k+1}Y$, $(a,q) = 1$ and $|\alpha - a/q| \le 1/(2^{k+1}qY)$. If $q \ge Y^{1/k}$, then*

$$B(\alpha) \ll Y^{\varepsilon+\Delta_u/2-1/(2k)} B(0), \tag{2.5}$$ eq:Weylminor

and if $q \le Y^{1/k}$ and $\alpha \ne a/q$, then

$$B(\alpha) \ll Y^{\varepsilon+\Delta_u/2-1/(2k)} \left(1 + \frac{Y^{1/(2k)}}{Y|q\alpha - a|^{1/2}}\right) B(0) \tag{2.6}$$ eq:Weylmajor

Proof. We first eliminate the primes p dividing q. These contribute at most

$$\ll Y^{\varepsilon/2}\mathcal{A}(0; Y, u) \ll Y^{\varepsilon-1/k} B(0)$$

by Chebyshev's inequality. Thus we may suppose that $p \nmid q$. When $(m, q) = 1$ the number of solutions of $x^k \equiv m \pmod{q}$ is $\ll q^{\varepsilon}$. Thus we can partition the set $\mathcal{P}$ of primes p with $Y < p \le 2Y$ and $p \nmid q$ into $\ll Y^{\varepsilon}$ subsets $\mathcal{P}_r$ which have the property that if $p_1, p_2 \in \mathcal{P}_r$ and $p_1^k \equiv p_2^k \pmod{q}$, then $p_1 \equiv p_2 \pmod{q}$. Hence it suffices to consider

$$\mathcal{B}_r(\alpha) = \sum_{p \in \mathcal{P}_r} \mathcal{A}(\alpha p^k; Y, u).$$

By Cauchy's inequality

$$B_r(\alpha)^2 \ll Y^{1/k} \sum_{p \in \mathcal{P}_r} |\mathcal{A}(\alpha p^k; Y, u)|^2.$$

We now consider the spacing of the αp^k modulo 1. We have

$$|\alpha - a/q||p_1^k - p_2^k| \le 2^k Y/(2^{k+1}qY) \le \frac{1}{2q}.$$

If $p_1 \not\equiv p_2 \pmod q$, so that $p_1^k \not\equiv p_2^k \pmod q$, then

$$\|\alpha p_1^k - \alpha p_2^k\| \geq \|a(p_1^k - p_2^k)/q\| - \frac{1}{2q} \geq \frac{1}{2q}. \qquad (2.7)$$ eq:spacing

Moreover if $q > Y^{1/k}$, so that $q > |p_1 - p_2|$ then we do have $p_1 \not\equiv p_2$. Hence, by the large sieve,

$$B_r(\alpha)^2 \ll Y^{1/k}(Y+q)\sum_n |a(n;Y)|^2 \ll Y^{1+1/k}\sum_n |a(n;Y)|^2.$$

Hence, by Lemma lem:fund 2.1,

$$B_r(\alpha)^2 \ll B(0)^2 Y^{-\frac{1}{k}+\Delta_u+\varepsilon}.$$

Now suppose that $q \leq Y^{1/k}$ and $\alpha \neq a/q$. We have already seen that if $p_1 \not\equiv p_2 \pmod q$, then (eq:spacing 2.7) holds. Now suppose that $p_1 \equiv p_2 \pmod q$, but $p_1 \neq p_2$. Then

$$\|\alpha p_1^k - \alpha p_2^k\| = \|(\alpha - a/q)(p_1^k - p_2^k)\|.$$

Since

$$|(\alpha - a/q)(p_1^k - p_2^k)| \leq (2^{k+1}qY)^{-1}2^k Y \leq \frac{1}{2}$$

we have

$$\|\alpha p_1^k - \alpha p_2^k\| = |\alpha - a/q||p_1^k - p_2^k|.$$

Moreover

$$|p_1^k - p_2^k| = |p_1 - p_2|(p_1^{k-1} + \cdots + p_2^{k-1}) \geq qY^{1-1/k}.$$

Thus

$$\|\alpha p_1^k - \alpha p_2^k\| \geq |q\alpha - a|Y^{1-1/k}.$$

Therefore

$$\min_{p_1 \neq p_2} \|\alpha p_1^k - \alpha p_2^k\| \geq \min\left(q^{-1}, |q\alpha - a|Y^{1-1/k}\right).$$

Hence, by the large sieve once more,

$$\begin{aligned} B_r(\alpha)^2 &\ll Y^{1/k}\left(Y + q + \frac{Y^{1/k}}{Y|q\alpha - a|}\right)\sum_n |a(n;Y)|^2 \\ &\ll Y^{1/k}\left(Y + \frac{Y^{1/k}}{Y|q\alpha - a|}\right)\sum_n |a(n;Y)|^2. \end{aligned}$$

Therefore, by Lemma lem:fund 2.1

$$B_r(\alpha) \ll Y^{\varepsilon+\Delta_u/2-1/(2k)}\left(1+\frac{Y^{1/2k}}{Y|q\alpha-a|^{1/2}}\right)B_r(0)$$

□

cor:Weylsuba **Corollary 2.4.** *Suppose that $\alpha \in \mathbb{R}$, $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $q \le 2^{k+1}Y$, $(a,q) = 1$ and $|\alpha - a/q| \le 1/(2^{k+1}qY)$. If $q > Y^{1/k}$ or if $q \le Y^{1/k}$ but $|q\alpha - a|Y^2 \gg Y^{1/k}$, then*

$$B(\alpha) \ll Y^{\varepsilon+\Delta_u/2-1/(2k)}B(0),$$

3. Minor Arcs

Let n be a large positive integer, define

$$f(\alpha) = \sum_{x\le n^{1/k}} e(\alpha x^k),$$

and let $X = \delta n$ and $Y = \delta n^{1/2}$ where δ is a small positive constant. Then we now focus on

$$R(n) = \int_{\mathbb{T}} f(\alpha)^{4k}\mathcal{A}(\alpha)^2\mathcal{B}(\alpha)e(-\alpha n)d\alpha$$

so that $R(n)$ is the number of solutions of

$$x_1^k + \cdots x_{4k}^k + y_1^k + \cdots y_t^k + z_1^k + \cdots + z_t^k + p^k w_1^k + \cdots p^k w_u^k = n$$

with $x_j \le n^{1/k}$, $P_j(X) < y_j, z_j \le 2P_j(X)$, $Y^{1/k} < p \le 2Y^{1/k}$ and $P_j(Y) < w_j \le 2P_j(Y)$. The variables y_j, z_j, pw_j with $j \ge 2$ are all $o(n^{1/k})$ and $x_1^k, y_1^k \le 2^k\delta n$ and $p^k w_1^k \le 4^k\delta^2 n$, so

$$y_1^k + \cdots y_t^k + z_1^k + \cdots + z_t^k + p^k w_1^k + \cdots p^k w_u^k \le \big((2^{k+1}+1)\delta + 4^k\delta^2\big)n$$

and

$$y_1^k + \cdots y_t^k + z_1^k + \cdots + z_t^k + p^k w_1^k + \cdots p^k w_u^k < \frac{1}{2}n \qquad (3.1)$$ eq:support

on choosing δ suitably small.

Let $\tau = Y^{1/k-2}$ and define the minor arcs $\mathfrak{m}$ to be the set of $\alpha \in (\tau, 1+\tau]$ such that if $|q\alpha - a| \le Y^{-1}2^{-k-1}$, then $q > Y^{1/k}$ or $|q\alpha - a| > Y^{1/k-2}$. By Dirichlet's theorem on diophantine approximation, given any α, there are always q and a with $(a,q) = 1$ such that $q \le 2^{k+1}Y^2$

and $|q\alpha - a| \le 2^{-k-1}Y^{-2}$. Moreover, if $q > Y^{1/k}$ or $|q\alpha - a| > Y^{1/k-2}$, then by Lemma eq:Weylmajor 2.6 and Corollary cor:Weylsuba 2.4 we have

$$\int_{\mathfrak{m}} |f(\alpha)^{4k}\mathcal{A}(\alpha)^2\mathcal{B}(\alpha)|d\alpha \ll n^3\mathcal{A}(0)^2\mathcal{B}(0)n^{\Delta_t+\Delta_u/4-1/(4k)+\varepsilon}.$$

For convenience we take

$$t = u \text{ and } u = 1 + \lfloor k\log(16k/3)\rfloor. \tag{3.2}$$ eq:tandu

Then

$$u\log\frac{1}{1-\frac{1}{k}} > k^{-1}u > \log(16k/3)$$

so that

$$(1-1/k)^{-u} > 16k/3 > k\frac{16k(k-2)}{3(k-1)^2} = \frac{16}{3}k(1-2/k)(1-1/k)^{-2}.$$

Hence

$$(1-2/k)(1-1/k)^{u-2} < \frac{3}{16k}$$

and

$$\Delta_t + \Delta_u/4 - 1/(16k/3) = \frac{5}{4}\Delta_u - \frac{1}{4k} < -\frac{1}{64k}.$$

Thus

$$\int_{\mathfrak{m}} |f(\alpha)^{4k}\mathcal{A}(\alpha)^2\mathcal{B}(\alpha)|d\alpha \ll n^3\mathcal{A}(0)^2\mathcal{B}(0)n^{-\frac{1}{64k}}. \tag{3.3}$$ eq:allminor

4. The Major Arcs

We are left with the complement of $\mathfrak{m}$. We define for $1 \le a \le q \le Y^{1/k}$ and $(a,q) = 1$

$$\mathfrak{M}(q,a) = \{\alpha \in (\tau, 1+\tau] : |q\alpha - a| \le Y^{1/k-2}\}$$

and let $\mathfrak{M}$ denote their union with $1 \le a \le q \le Y^{1/k}$ and $(a,q) = 1$. It is clear from Dirichlet's theorem that $\mathfrak{m}$ and $\mathfrak{M}$ partition $(\tau, 1+\tau]$.

In the notation used in class, for $\alpha \in \mathfrak{M}(q,a)$ we have

$$f(\alpha) = q^{-1}S(q,a)v(\beta) + O(q + qn|\beta|)$$

where $\beta = \alpha - a/q$. We also know that

$$S(q,a) \ll q^{1-1/k} \quad (q,a) = 1 \tag{4.1}$$ eq:Sqabound

and

$$v(\beta) \ll \frac{n^{1/k}}{(1+n|\beta|)^{1/k}}.$$

Now on $\mathfrak{M}(q,a)$ we have

$$f(\alpha)^{4k} - q^{-4k}S(q,a)^{4k}v(\beta)^{4k} \ll \frac{n^{4-\frac{1}{k}}}{(q+qn|\beta|)^3} + (q+qn|\beta|)^{4k}$$

and so

$$\int_{\mathfrak{M}(q,a)} |f(\alpha)^{4k} - q^{-4k}S(q,a)^{4k}v(\beta)^{4k}|d\alpha$$
$$\ll q^{-3}n^{3-\frac{1}{k}} + (q+nY^{1/k-2})^{4k}q^{-1}Y^{1/k-2}.$$

Recalling that $Y = \delta n^{1/2}$, this is

$$\ll q^{-3}n^{3-\frac{1}{k}} + (n^{1/(2k)})^{4k}q^{-1}n^{1/(2k)-1} \ll q^{-3}n^{3-\frac{1}{k}} + nq^{-1}.$$

Summing this over the major arcs gives, for an arbitrary m

$$\int_{\mathfrak{M}} f(\alpha)^{4k}e(-\alpha m)d\alpha -$$
$$\sum_{q\le Y^{1/k}} \sum_{\substack{a=1\\(a,q)=1}}^{q} \frac{S(q,a)^{4k}}{q^{4k}} e\left(-\frac{am}{q}\right) \int_{-\frac{Y^{1/k}}{qY^2}}^{\frac{Y^{1/k}}{qY^2}} v(\beta)^{4k}e(-\beta m)d\beta$$
$$\ll n^{3-\frac{1}{k}}.$$

We can complete the analysis of the major arcs much as in class, but we can use the better bound (eq:Sqabound 4.1). We do need $m \gg n$ in order to get decent asymptotics for $J(m)$. Thus, when $\frac{1}{2}n \le m \le n$ we have

$$\int_{\mathfrak{M}} f(\alpha)^{4k}e(-\alpha m)d\alpha = \frac{1}{6}\Gamma\left(1+\frac{1}{k}\right)^{4k} m^3\mathfrak{S}(m) + O(n^{3-1/k}).$$

We write

$$\mathcal{A}(\alpha)^2\mathcal{B}(\alpha) = \sum_m c(m)e(\alpha m)$$

where, by (eq:support 3.1), $c(m)$ is the number of solutions of

$$y_1^k + \cdots y_t^k + z_1^k + \cdot z_t^k + p^k w_1^k + \cdots p^k w_u^k = m$$

with $P_j(X) < y_j, z_j \le 2P_j(X)$, $Y^{1/k} < p \le 2Y^{1/k}$ and $P_j(Y) < w_j \le 2P_j(Y)$ and the support of $c(m)$ lies in

$$[1, \big((2^{k+1}+1)\delta + 4^k\delta^2\big)n] \subset [1, n/2).$$

Thus

$$\int_{\mathfrak{M}} f(\alpha)^{4k}\mathcal{A}(\alpha)^2\mathcal{B}(\alpha)e(-\alpha n)d\alpha = \frac{1}{6}\Gamma\left(1+\frac{1}{k}\right)^{4k}\sum_m c(m)(n-m)^3\mathfrak{S}(n-m) + O\big(n^{3-1/k}\mathcal{A}(0)^2\mathcal{B}(0)\big). \quad (4.2)$$

eq:allmajor

5. The Endgame

By combining (3.3) and (4.2) we obtain

$$\int_0^1 f(\alpha)^{4k}\mathcal{A}(\alpha)^2\mathcal{B}(\alpha)e(-\alpha n)d\alpha = \frac{1}{6}\Gamma\left(1+\frac{1}{k}\right)^{4k}\sum_m c(m)(n-m)^3\mathfrak{S}(n-m) + O\big(n^{3-\frac{1}{64k}}\mathcal{A}(0)^2\mathcal{B}(0)\big). \quad (5.1)$$

eq:all

The $c(m)$ are non-negative and have their support in $[1, n/2)$, and we know from class (for sums of $4k$ k-th powers) that $\mathfrak{S}(n-m) \gg 1$. Hence the main term here, and so the integral on the left, is

$$\gg n^3\mathcal{A}(0)^2\mathcal{B}(0)$$

Hence every large n is the sum of at $4k+2t+u = 4k+3u$ k-th powers, and by (3.2) this is

$$4k+t+u = 4k+3+3\lfloor k\log(16k/3)\rfloor.$$

This proves Theorem 1.1 with Ck replaced by $3+4k+3k\log 4$.

References

[2023] J. Brüdern & T. D. Wooley, On Waring's problem for larger powers, J. reine angew. Math. 805(2023), 115–142. https://doi.org/10.1515/crelle-2023-0072