Math 571, Spring 2025, Density and Sum Sets

Robert C. Vaughan

March 28, 2025

- This is an exposition of two of the standard theorems on density and sum sets, namely the Cauchy–Davenport–Chowla theorem and Mann's theorem.

- This is an exposition of two of the standard theorems on density and sum sets, namely the Cauchy–Davenport–Chowla theorem and Mann's theorem.
- There are various proofs of these theorems and their many variations in the literature.

- This is an exposition of two of the standard theorems on density and sum sets, namely the Cauchy–Davenport–Chowla theorem and Mann's theorem.
- There are various proofs of these theorems and their many variations in the literature.
- The purpose here is to give short and simple proofs of both theorems the core of which are based on a common and generic idea.

- This is an exposition of two of the standard theorems on density and sum sets, namely the Cauchy–Davenport–Chowla theorem and Mann's theorem.
- There are various proofs of these theorems and their many variations in the literature.
- The purpose here is to give short and simple proofs of both theorems the core of which are based on a common and generic idea.
- This is that one or more elements can be removed from one of the sets and their translates added to the other in such a way that the original sum set is contained in the new sum set, and so an induction on the number of elements of one of the sets can be established.

- Given a positive integer q and a collection $\mathcal{A}$ of residue classes modulo q, its local density $\rho = \rho(\mathcal{A})$ modulo q is defined by

$$\rho = q^{-1}\,\mathrm{card}(\mathcal{A}).$$

- Given a positive integer q and a collection $\mathcal{A}$ of residue classes modulo q, its local density $\rho = \rho(\mathcal{A})$ modulo q is defined by
$$\rho = q^{-1}\,\mathrm{card}(\mathcal{A}).$$
- **Theorem [Cauchy–Davenport–Chowla].** *Suppose that q is a positive integer, that $\mathcal{A}$ and $\mathcal{B}$ are sets of residue classes modulo q of local density modulo q, α and β respectively, that $0 \in \mathcal{B}$ and that every non–zero residue class in $\mathcal{B}$ is a reduced residue class modulo q. Then*
$$\rho(\mathcal{A} + \mathcal{B}) \geq \min(1, \alpha + \beta - 1/q).$$

- Given a positive integer q and a collection $\mathcal{A}$ of residue classes modulo q, its local density $\rho = \rho(\mathcal{A})$ modulo q is defined by
$$\rho = q^{-1}\,\mathrm{card}(\mathcal{A}).$$
- **Theorem [Cauchy–Davenport–Chowla].** *Suppose that q is a positive integer, that $\mathcal{A}$ and $\mathcal{B}$ are sets of residue classes modulo q of local density modulo q, α and β respectively, that $0 \in \mathcal{B}$ and that every non–zero residue class in $\mathcal{B}$ is a reduced residue class modulo q. Then*
$$\rho(\mathcal{A}+\mathcal{B}) \geq \min(1, \alpha+\beta-1/q).$$
- This is best possible, as is seen by the example
$$\mathcal{A} = \{0,1,\ldots,r-1\},\ \mathcal{B} = \{0,1,\ldots,s-1\},$$
$$\mathcal{A}+\mathcal{B} = \{0,1,\ldots,r+s-2\} \text{ when } r+s-1 \leq q.$$

- **Proof.** If $\alpha = 1$, then the conclusion is trivial.

- **Proof.** If $\alpha = 1$, then the conclusion is trivial.
- Thus we may suppose that $r = q\alpha = \text{card}(\mathcal{A}) < q$.

- **Proof.** If $\alpha = 1$, then the conclusion is trivial.
- Thus we may suppose that $r = q\alpha = \text{card}(\mathcal{A}) < q$.
- We now proceed by induction on $s = q\beta = \text{card}(\mathcal{B})$.

- **Proof.** If $\alpha = 1$, then the conclusion is trivial.
- Thus we may suppose that $r = q\alpha = \text{card}(\mathcal{A}) < q$.
- We now proceed by induction on $s = q\beta = \text{card}(\mathcal{B})$.
- When $s = 1$ the conclusion is immediate.

- **Proof.** If $\alpha = 1$, then the conclusion is trivial.
- Thus we may suppose that $r = q\alpha = \text{card}(\mathcal{A}) < q$.
- We now proceed by induction on $s = q\beta = \text{card}(\mathcal{B})$.
- When $s = 1$ the conclusion is immediate.
- Thus it remains to consider the case $s > 1$ (and $\alpha < 1$), and we may assume the conclusion holds for *all* α when $card \mathcal{B} < s$.

- When $b \in \mathcal{B}\backslash\{0\}$ we cannot have $a + b \in \mathcal{A}$ for every $a \in \mathcal{A}$, for otherwise

$$\sum_{a\in\mathcal{A}} a + br \equiv \sum_{a'\in\mathcal{A}} a' \pmod{q}$$

whence $br \equiv 0 \pmod{q}$ and then $(b, q) > 1$.

- When $b \in \mathcal{B}\backslash\{0\}$ we cannot have $a+b \in \mathcal{A}$ for every $a \in \mathcal{A}$, for otherwise

$$\sum_{a\in\mathcal{A}} a + br \equiv \sum_{a'\in\mathcal{A}} a' \pmod{q}$$

  whence $br \equiv 0 \pmod{q}$ and then $(b,q) > 1$.
- Hence there are $a_0 \in \mathcal{A}$, $b_0 \in \mathcal{B}$ such that $a_0 + b_0 \notin \mathcal{A}$.

- When $b \in \mathcal{B}\backslash\{0\}$ we cannot have $a + b \in \mathcal{A}$ for every $a \in \mathcal{A}$, for otherwise

$$\sum_{a\in\mathcal{A}} a + br \equiv \sum_{a'\in\mathcal{A}} a' \pmod q$$

whence $br \equiv 0 \pmod q$ and then $(b, q) > 1$.

- Hence there are $a_0 \in \mathcal{A}$, $b_0 \in \mathcal{B}$ such that $a_0 + b_0 \notin \mathcal{A}$.
- Let $\mathcal{A}' = \mathcal{A} \cup \{a_0 + b : b \in \mathcal{B}, a_0 + b \notin \mathcal{A}\}$ and $\mathcal{B}' = \{b : b \in \mathcal{B}, a_0 + b \in \mathcal{A}\}$.

- When $b \in \mathcal{B}\backslash\{0\}$ we cannot have $a + b \in \mathcal{A}$ for every $a \in \mathcal{A}$, for otherwise

$$\sum_{a \in \mathcal{A}} a + br \equiv \sum_{a' \in \mathcal{A}} a' \pmod q$$

  whence $br \equiv 0 \pmod q$ and then $(b, q) > 1$.
- Hence there are $a_0 \in \mathcal{A}$, $b_0 \in \mathcal{B}$ such that $a_0 + b_0 \notin \mathcal{A}$.
- Let $\mathcal{A}' = \mathcal{A} \cup \{a_0 + b : b \in \mathcal{B}, a_0 + b \notin \mathcal{A}\}$ and $\mathcal{B}' = \{b : b \in \mathcal{B}, a_0 + b \in \mathcal{A}\}$.
- Then $\operatorname{card}(\mathcal{A}') + \operatorname{card}(\mathcal{B}') = \operatorname{card}(\mathcal{A}) + \operatorname{card}(\mathcal{B}) = r + s$ and $1 \le \operatorname{card}(\mathcal{B}') \le s - 1$. Hence, by the inductive hypothesis $\rho(\mathcal{A}' + \mathcal{B}') \ge$ $\min\big(1, \rho(\mathcal{A}') + \rho(\mathcal{B}') - 1/q\big) = \min(1, \alpha + \beta - 1/q)$.

- When $b \in \mathcal{B}\backslash\{0\}$ we cannot have $a+b \in \mathcal{A}$ for every $a \in \mathcal{A}$, for otherwise

$$\sum_{a \in \mathcal{A}} a + br \equiv \sum_{a' \in \mathcal{A}} a' \pmod{q}$$

whence $br \equiv 0 \pmod{q}$ and then $(b,q) > 1$.

- Hence there are $a_0 \in \mathcal{A}$, $b_0 \in \mathcal{B}$ such that $a_0 + b_0 \notin \mathcal{A}$.
- Let $\mathcal{A}' = \mathcal{A} \cup \{a_0 + b : b \in \mathcal{B}, a_0 + b \notin \mathcal{A}\}$ and $\mathcal{B}' = \{b : b \in \mathcal{B}, a_0 + b \in \mathcal{A}\}$.
- Then $\operatorname{card}(\mathcal{A}') + \operatorname{card}(\mathcal{B}') = \operatorname{card}(\mathcal{A}) + \operatorname{card}(\mathcal{B}) = r + s$ and $1 \le \operatorname{card}(\mathcal{B}') \le s - 1$. Hence, by the inductive hypothesis $\rho(\mathcal{A}' + \mathcal{B}') \ge \min\left(1, \rho(\mathcal{A}') + \rho(\mathcal{B}') - 1/q\right) = \min(1, \alpha + \beta - 1/q)$.
- Suppose that $a' \in \mathcal{A}'$ and $b' \in \mathcal{B}'$. When $a' \in \mathcal{A}$ we have $a' + b' \in \mathcal{A} + \mathcal{B}$. When $a' \notin \mathcal{A}$ there is a $b'' \in \mathcal{B}$ such that $a' = a_0 + b''$ and so $a' + b' = a_0 + b'' + b' = a_0 + b' + b''$. Moreover $a_0 + b' \in \mathcal{A}$, so $a' + b' \in \mathcal{A} + \mathcal{B}$ in this case also. Hence $\mathcal{A}' + \mathcal{B}' \subset \mathcal{A} + \mathcal{B}$ and the theorem follows.

- For convenience, given $\mathcal{A} \subset \mathbb{Z}$ we define

$$A(n) = \text{card}\{a \in \mathcal{A} : 1 \le a \le n\}.$$

- For convenience, given $\mathcal{A} \subset \mathbb{Z}$ we define

$$A(n) = \mathrm{card}\{a \in \mathcal{A} : 1 \le a \le n\}.$$

- Then the Schnirel'man density $\sigma(\mathcal{A})$ of a set of integers $\mathcal{A}$ is given by $\sigma(\mathcal{A}) = \inf_{n \ge 1} n^{-1} A(n)$.

- For convenience, given $\mathcal{A} \subset \mathbb{Z}$ we define

$$A(n) = \operatorname{card}\{a \in \mathcal{A} : 1 \le a \le n\}.$$

- Then the Schnirel'man density $\sigma(\mathcal{A})$ of a set of integers $\mathcal{A}$ is given by $\sigma(\mathcal{A}) = \inf_{n\ge 1} n^{-1} A(n)$.
- **Theorem [Mann].** *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are sets of integers of Schnirel'man density α and β respectively and that $0 \in \mathcal{A} \cap \mathcal{B}$. Then*

$$\sigma(\mathcal{A} + \mathcal{B}) \ge \min(1, \alpha + \beta).$$

- For convenience, given $\mathcal{A} \subset \mathbb{Z}$ we define

$$A(n) = \text{card}\{a \in \mathcal{A} : 1 \le a \le n\}.$$

- Then the Schnirel'man density $\sigma(\mathcal{A})$ of a set of integers $\mathcal{A}$ is given by $\sigma(\mathcal{A}) = \inf_{n \ge 1} n^{-1} A(n)$.
- **Theorem [Mann].** *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are sets of integers of Schnirel'man density α and β respectively and that $0 \in \mathcal{A} \cap \mathcal{B}$. Then*

$$\sigma(\mathcal{A} + \mathcal{B}) \ge \min(1, \alpha + \beta).$$

- **Proof.** The case $\alpha + \beta \ge 1$ can be disposed of by a box argument.

- For convenience, given $\mathcal{A} \subset \mathbb{Z}$ we define

$$A(n) = \mathrm{card}\{a \in \mathcal{A} : 1 \le a \le n\}.$$

- Then the Schnirel'man density $\sigma(\mathcal{A})$ of a set of integers $\mathcal{A}$ is given by $\sigma(\mathcal{A}) = \inf_{n\ge 1} n^{-1}A(n)$.
- **Theorem [Mann].** *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are sets of integers of Schnirel'man density α and β respectively and that $0 \in \mathcal{A} \cap \mathcal{B}$. Then*

$$\sigma(\mathcal{A} + \mathcal{B}) \ge \min(1, \alpha + \beta).$$

- **Proof.** The case $\alpha + \beta \ge 1$ can be disposed of by a box argument.
- For a given $n \in \mathbb{N}$ consider the $A(n) + B(n) + 2$ numbers (objects) a with $0 \le a \le n$ and $a \in \mathcal{A}$ and $n - b$ with $0 \le b \le n$ and $b \in \mathcal{B}$.

- For convenience, given $\mathcal{A} \subset \mathbb{Z}$ we define

$$A(n) = \mathrm{card}\{a \in \mathcal{A} : 1 \le a \le n\}.$$

- Then the Schnirel'man density $\sigma(\mathcal{A})$ of a set of integers $\mathcal{A}$ is given by $\sigma(\mathcal{A}) = \inf_{n\ge 1} n^{-1}A(n)$.
- **Theorem [Mann].** *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are sets of integers of Schnirel'man density α and β respectively and that $0 \in \mathcal{A} \cap \mathcal{B}$. Then*

$$\sigma(\mathcal{A}+\mathcal{B}) \ge \min(1, \alpha+\beta).$$

- **Proof.** The case $\alpha+\beta \ge 1$ can be disposed of by a box argument.
- For a given $n \in \mathbb{N}$ consider the $A(n)+B(n)+2$ numbers (objects) a with $0 \le a \le n$ and $a \in \mathcal{A}$ and $n-b$ with $0 \le b \le n$ and $b \in \mathcal{B}$.
- Since $A(n)+B(n)+2 \ge \alpha n + \beta n + 2 \ge n+2$, one of the $n+1$ numbers m with $0 \le m \le n$ must be both an a and an $n-b$.

- For convenience, given $\mathcal{A} \subset \mathbb{Z}$ we define
$$A(n) = \text{card}\{a \in \mathcal{A} : 1 \le a \le n\}.$$
- Then the Schnirel'man density $\sigma(\mathcal{A})$ of a set of integers $\mathcal{A}$ is given by $\sigma(\mathcal{A}) = \inf_{n\ge1} n^{-1}A(n)$.
- **Theorem [Mann].** *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are sets of integers of Schnirel'man density α and β respectively and that $0 \in \mathcal{A} \cap \mathcal{B}$. Then*
$$\sigma(\mathcal{A} + \mathcal{B}) \ge \min(1, \alpha + \beta).$$
- **Proof.** The case $\alpha + \beta \ge 1$ can be disposed of by a box argument.
- For a given $n \in \mathbb{N}$ consider the $A(n) + B(n) + 2$ numbers (objects) a with $0 \le a \le n$ and $a \in \mathcal{A}$ and $n - b$ with $0 \le b \le n$ and $b \in \mathcal{B}$.
- Since $A(n) + B(n) + 2 \ge \alpha n + \beta n + 2 \ge n + 2$, one of the $n + 1$ numbers m with $0 \le m \le n$ must be both an a and an $n - b$.
- Hence $n = a + b$, $\mathbb{N} \subset \mathcal{A} + \mathcal{B}$ and the theorem follows.

- Henceforward we suppose that $\alpha + \beta < 1$

- Henceforward we suppose that $\alpha + \beta < 1$
- If $\alpha\beta = 0$, then the conclusion follows from the observation that $\mathcal{A} \subset \mathcal{A} + \mathcal{B}$ and $\mathcal{B} \subset \mathcal{A} + \mathcal{B}$.

- Henceforward we suppose that $\alpha + \beta < 1$
- If $\alpha\beta = 0$, then the conclusion follows from the observation that $\mathcal{A} \subset \mathcal{A} + \mathcal{B}$ and $\mathcal{B} \subset \mathcal{A} + \mathcal{B}$.
- Hence we may suppose that $\alpha\beta > 0$ and therefore

$$1 \in \mathcal{A} \cap \mathcal{B}. \tag{1}$$

- Henceforward we suppose that $\alpha + \beta < 1$
- If $\alpha\beta = 0$, then the conclusion follows from the observation that $\mathcal{A} \subset \mathcal{A} + \mathcal{B}$ and $\mathcal{B} \subset \mathcal{A} + \mathcal{B}$.
- Hence we may suppose that $\alpha\beta > 0$ and therefore

$$1 \in \mathcal{A} \cap \mathcal{B}. \tag{1}$$

- It suffices now to prove that for $n \in \mathbb{N}$ we have $\text{card}\{m : 1 \le m \le n, m \in \mathcal{A} + \mathcal{B}\} \ge A(n) + B(n)$.

- Henceforward we suppose that $\alpha + \beta < 1$
- If $\alpha\beta = 0$, then the conclusion follows from the observation that $\mathcal{A} \subset \mathcal{A} + \mathcal{B}$ and $\mathcal{B} \subset \mathcal{A} + \mathcal{B}$.
- Hence we may suppose that $\alpha\beta > 0$ and therefore

$$1 \in \mathcal{A} \cap \mathcal{B}. \tag{1}$$

- It suffices now to prove that for $n \in \mathbb{N}$ we have $\operatorname{card}\{m : 1 \le m \le n, m \in \mathcal{A} + \mathcal{B}\} \ge A(n) + B(n)$.
- Suppose first that $a + b \in \mathcal{A}$ whenever $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $a + b \le n$.

- Henceforward we suppose that $\alpha + \beta < 1$
- If $\alpha\beta = 0$, then the conclusion follows from the observation that $\mathcal{A} \subset \mathcal{A} + \mathcal{B}$ and $\mathcal{B} \subset \mathcal{A} + \mathcal{B}$.
- Hence we may suppose that $\alpha\beta > 0$ and therefore

$$1 \in \mathcal{A} \cap \mathcal{B}. \tag{1}$$

- It suffices now to prove that for $n \in \mathbb{N}$ we have $\text{card}\{m : 1 \le m \le n, m \in \mathcal{A} + \mathcal{B}\} \ge A(n) + B(n)$.
- Suppose first that $a + b \in \mathcal{A}$ whenever $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $a + b \le n$.
- Then for $m = 1, 2, \ldots, n - 1$ we have $1 \in \mathcal{A}$ and, if $m \in \mathcal{A}$, then $m + 1 \in \mathcal{A}$.

- Henceforward we suppose that $\alpha + \beta < 1$
- If $\alpha\beta = 0$, then the conclusion follows from the observation that $\mathcal{A} \subset \mathcal{A} + \mathcal{B}$ and $\mathcal{B} \subset \mathcal{A} + \mathcal{B}$.
- Hence we may suppose that $\alpha\beta > 0$ and therefore

$$1 \in \mathcal{A} \cap \mathcal{B}. \tag{1}$$

- It suffices now to prove that for $n \in \mathbb{N}$ we have $\mathrm{card}\{m : 1 \leq m \leq n, m \in \mathcal{A} + \mathcal{B}\} \geq A(n) + B(n)$.
- Suppose first that $a + b \in \mathcal{A}$ whenever $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $a + b \leq n$.
- Then for $m = 1, 2, \ldots, n - 1$ we have $1 \in \mathcal{A}$ and, if $m \in \mathcal{A}$, then $m + 1 \in \mathcal{A}$.
- Hence $A(n) = n$ and since $\mathcal{A} \subset \mathcal{A} + \mathcal{B}$ the desired result follows.

- Henceforward we suppose that $\alpha + \beta < 1$
- If $\alpha\beta = 0$, then the conclusion follows from the observation that $\mathcal{A} \subset \mathcal{A} + \mathcal{B}$ and $\mathcal{B} \subset \mathcal{A} + \mathcal{B}$.
- Hence we may suppose that $\alpha\beta > 0$ and therefore

$$1 \in \mathcal{A} \cap \mathcal{B}. \tag{1}$$

- It suffices now to prove that for $n \in \mathbb{N}$ we have $\mathrm{card}\{m : 1 \le m \le n, m \in \mathcal{A} + \mathcal{B}\} \ge A(n) + B(n)$.
- Suppose first that $a + b \in \mathcal{A}$ whenever $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $a + b \le n$.
- Then for $m = 1, 2, \ldots, n - 1$ we have $1 \in \mathcal{A}$ and, if $m \in \mathcal{A}$, then $m + 1 \in \mathcal{A}$.
- Hence $A(n) = n$ and since $\mathcal{A} \subset \mathcal{A} + \mathcal{B}$ the desired result follows.
- Thus we may suppose that there are $a_0 \in \mathcal{A}$, $b_0 \in \mathcal{B}$ such that $a_0 \ge 1$, $b_0 \ge 1$, $a_0 + b_0 \le n$ and $a_0 + b_0 \notin \mathcal{A}$.

- We are supposing (1) $1 \in \mathcal{A} \cap \mathcal{B}$ and that there are $a_0 \in \mathcal{A}$, $b_0 \in \mathcal{B}$ such that $a_0 \geq 1$, $b_0 \geq 1$, $a_0 + b_0 \leq n$ and $a_0 + b_0 \notin \mathcal{A}$.

- We are supposing (1) $1 \in \mathcal{A} \cap \mathcal{B}$ and that there are $a_0 \in \mathcal{A}$, $b_0 \in \mathcal{B}$ such that $a_0 \geq 1$, $b_0 \geq 1$, $a_0 + b_0 \leq n$ and $a_0 + b_0 \notin \mathcal{A}$.
- Let $s = B(n)$.

- We are supposing (1) $1 \in \mathcal{A} \cap \mathcal{B}$ and that there are $a_0 \in \mathcal{A}$, $b_0 \in \mathcal{B}$ such that $a_0 \geq 1$, $b_0 \geq 1$, $a_0 + b_0 \leq n$ and $a_0 + b_0 \notin \mathcal{A}$.
- Let $s = B(n)$.
- Then by (1) we may assume that $s \geq 1$. We proceed by induction on $s = 1, \ldots, n$.

- We are supposing (1) $1 \in \mathcal{A} \cap \mathcal{B}$ and that there are $a_0 \in \mathcal{A}$, $b_0 \in \mathcal{B}$ such that $a_0 \geq 1$, $b_0 \geq 1$, $a_0 + b_0 \leq n$ and $a_0 + b_0 \notin \mathcal{A}$.
- Let $s = B(n)$.
- Then by (1) we may assume that $s \geq 1$. We proceed by induction on $s = 1, \ldots, n$.
- Let $\mathcal{A}' = \mathcal{A} + \{a_0 + b : b \in \mathcal{B}, a_0 + b \leq n, a_0 + b \notin \mathcal{A}\}$ and $\mathcal{B}' = \mathcal{B} \backslash \{b \in \mathcal{B} : a_0 + b \leq n \text{ and } a_0 + b \notin \mathcal{A}\}$.

- We are supposing (1) $1 \in \mathcal{A} \cap \mathcal{B}$ and that there are $a_0 \in \mathcal{A}$, $b_0 \in \mathcal{B}$ such that $a_0 \geq 1$, $b_0 \geq 1$, $a_0 + b_0 \leq n$ and $a_0 + b_0 \notin \mathcal{A}$.
- Let $s = B(n)$.
- Then by (1) we may assume that $s \geq 1$. We proceed by induction on $s = 1, \ldots, n$.
- Let $\mathcal{A}' = \mathcal{A} + \{a_0 + b : b \in \mathcal{B}, a_0 + b \leq n, a_0 + b \notin \mathcal{A}\}$ and $\mathcal{B}' = \mathcal{B} \backslash \{b \in \mathcal{B} : a_0 + b \leq n \text{ and } a_0 + b \notin \mathcal{A}\}$.
- Suppose that $a' \in \mathcal{A}'$ and $b' \in \mathcal{B}'$.

- We are supposing (1) $1 \in \mathcal{A} \cap \mathcal{B}$ and that there are $a_0 \in \mathcal{A}$, $b_0 \in \mathcal{B}$ such that $a_0 \geq 1$, $b_0 \geq 1$, $a_0 + b_0 \leq n$ and $a_0 + b_0 \notin \mathcal{A}$.
- Let $s = B(n)$.
- Then by (1) we may assume that $s \geq 1$. We proceed by induction on $s = 1, \ldots, n$.
- Let $\mathcal{A}' = \mathcal{A} + \{a_0 + b : b \in \mathcal{B}, a_0 + b \leq n, a_0 + b \notin \mathcal{A}\}$ and $\mathcal{B}' = \mathcal{B} \backslash \{b \in \mathcal{B} : a_0 + b \leq n \text{ and } a_0 + b \notin \mathcal{A}\}$.
- Suppose that $a' \in \mathcal{A}'$ and $b' \in \mathcal{B}'$.
- When $a' \in \mathcal{A}$ we have $a' + b' \in \mathcal{A} + \mathcal{B}$.

- We are supposing (1) $1 \in \mathcal{A} \cap \mathcal{B}$ and that there are $a_0 \in \mathcal{A}$, $b_0 \in \mathcal{B}$ such that $a_0 \geq 1$, $b_0 \geq 1$, $a_0 + b_0 \leq n$ and $a_0 + b_0 \notin \mathcal{A}$.
- Let $s = B(n)$.
- Then by (1) we may assume that $s \geq 1$. We proceed by induction on $s = 1, \dots, n$.
- Let $\mathcal{A}' = \mathcal{A} + \{a_0 + b : b \in \mathcal{B}, a_0 + b \leq n, a_0 + b \notin \mathcal{A}\}$ and $\mathcal{B}' = \mathcal{B} \backslash \{b \in \mathcal{B} : a_0 + b \leq n \text{ and } a_0 + b \notin \mathcal{A}\}$.
- Suppose that $a' \in \mathcal{A}'$ and $b' \in \mathcal{B}'$.
- When $a' \in \mathcal{A}$ we have $a' + b' \in \mathcal{A} + \mathcal{B}$.
- When $a' \notin \mathcal{A}$ there is a $b'' \in \mathcal{B}$ such that $a' = a_0 + b''$ and so $a' + b' = a_0 + b'' + b' = a_0 + b' + b''$.

- We are supposing (1) $1 \in \mathcal{A} \cap \mathcal{B}$ and that there are $a_0 \in \mathcal{A}$, $b_0 \in \mathcal{B}$ such that $a_0 \geq 1$, $b_0 \geq 1$, $a_0 + b_0 \leq n$ and $a_0 + b_0 \notin \mathcal{A}$.
- Let $s = B(n)$.
- Then by (1) we may assume that $s \geq 1$. We proceed by induction on $s = 1, \ldots, n$.
- Let $\mathcal{A}' = \mathcal{A} + \{a_0 + b : b \in \mathcal{B}, a_0 + b \leq n, a_0 + b \notin \mathcal{A}\}$ and $\mathcal{B}' = \mathcal{B} \backslash \{b \in \mathcal{B} : a_0 + b \leq n \text{ and } a_0 + b \notin \mathcal{A}\}$.
- Suppose that $a' \in \mathcal{A}'$ and $b' \in \mathcal{B}'$.
- When $a' \in \mathcal{A}$ we have $a' + b' \in \mathcal{A} + \mathcal{B}$.
- When $a' \notin \mathcal{A}$ there is a $b'' \in \mathcal{B}$ such that $a' = a_0 + b''$ and so $a' + b' = a_0 + b'' + b' = a_0 + b' + b''$.
- Moreover $a_0 + b' \in \mathcal{A}$, so $a' + b' \in \mathcal{A} + \mathcal{B}$ in this case also.

- We are supposing (1) $1 \in \mathcal{A} \cap \mathcal{B}$ and that there are $a_0 \in \mathcal{A}$, $b_0 \in \mathcal{B}$ such that $a_0 \geq 1$, $b_0 \geq 1$, $a_0 + b_0 \leq n$ and $a_0 + b_0 \notin \mathcal{A}$.
- Let $s = B(n)$.
- Then by (1) we may assume that $s \geq 1$. We proceed by induction on $s = 1, \ldots, n$.
- Let $\mathcal{A}' = \mathcal{A} + \{a_0 + b : b \in \mathcal{B}, a_0 + b \leq n, a_0 + b \notin \mathcal{A}\}$ and $\mathcal{B}' = \mathcal{B} \backslash \{b \in \mathcal{B} : a_0 + b \leq n \text{ and } a_0 + b \notin \mathcal{A}\}$.
- Suppose that $a' \in \mathcal{A}'$ and $b' \in \mathcal{B}'$.
- When $a' \in \mathcal{A}$ we have $a' + b' \in \mathcal{A} + \mathcal{B}$.
- When $a' \notin \mathcal{A}$ there is a $b'' \in \mathcal{B}$ such that $a' = a_0 + b''$ and so $a' + b' = a_0 + b'' + b' = a_0 + b' + b''$.
- Moreover $a_0 + b' \in \mathcal{A}$, so $a' + b' \in \mathcal{A} + \mathcal{B}$ in this case also.
- Hence
$$A'(n) + B'(n) = A(n) + B(n), \tag{2}$$
$\mathcal{A}' + \mathcal{B}' \subset \mathcal{A} + \mathcal{B}$ and $\operatorname{card}\{m : 1 \leq m \leq n, m \in \mathcal{A} + \mathcal{B}\} \geq \operatorname{card}\{m : 1 \leq m \leq n, m \in \mathcal{A}' + \mathcal{B}'\}$.

- We have
$$A'(n) + B'(n) = A(n) + B(n), \quad (2)$$
$\mathcal{A}' + \mathcal{B}' \subset \mathcal{A} + \mathcal{B}$ and $\text{card}\{m : 1 \le m \le n, m \in \mathcal{A} + \mathcal{B}\} \ge \text{card}\{m : 1 \le m \le n, m \in \mathcal{A}' + \mathcal{B}'\}$

- We have
$$A'(n)+B'(n)=A(n)+B(n), \quad (2)$$
$\mathcal{A}'+\mathcal{B}'\subset\mathcal{A}+\mathcal{B}$ and $\mathrm{card}\{m:1\le m\le n, m\in\mathcal{A}+\mathcal{B}\}\ge \mathrm{card}\{m:1\le m\le n, m\in\mathcal{A}'+\mathcal{B}'\}$
- We also have $B'(n)\le s-1$.

- We have
$$A'(n) + B'(n) = A(n) + B(n), \quad (2)$$
$\mathcal{A}' + \mathcal{B}' \subset \mathcal{A} + \mathcal{B}$ and $\mathrm{card}\{m : 1 \le m \le n, m \in \mathcal{A} + \mathcal{B}\} \ge \mathrm{card}\{m : 1 \le m \le n, m \in \mathcal{A}' + \mathcal{B}'\}$
- We also have $B'(n) \le s - 1$.
- If $B'(n) = 0$ (and this includes the case $s = 1$, the initial case of the inductive argument), then
$$\begin{aligned}\mathrm{card}\{m : 1 \le m \le n, m \in \mathcal{A}' + \mathcal{B}'\}&\\ &= \mathrm{card}\{m : 1 \le m \le n, m \in \mathcal{A}'\}\\ &\ge A'(n)\\ &= A'(n) + B'(n)\end{aligned}$$
and with (2) completes the proof in this case.

- We have
$$A'(n)+B'(n)=A(n)+B(n), \quad (2)$$
$\mathcal{A}'+\mathcal{B}'\subset\mathcal{A}+\mathcal{B}$ and $\operatorname{card}\{m:1\le m\le n, m\in\mathcal{A}+\mathcal{B}\}\ge \operatorname{card}\{m:1\le m\le n, m\in\mathcal{A}'+\mathcal{B}'\}$
- We also have $B'(n)\le s-1$.
- If $B'(n)=0$ (and this includes the case $s=1$, the initial case of the inductive argument), then
$$\begin{aligned}\operatorname{card}\{m:1\le m\le n, m\in\mathcal{A}'+\mathcal{B}'\}\\ &=\operatorname{card}\{m:1\le m\le n, m\in\mathcal{A}'\}\\ &\ge A'(n)\\ &=A'(n)+B'(n)\end{aligned}$$
and with (2) completes the proof in this case.
- If $B'(n)\ge 1$, then on the inductive hypothesis for s we have
$$\operatorname{card}\{m:1\le m\le n, m\in\mathcal{A}'+B'\}\ge A'(n)+B'(n)$$
once more and again with (2) this completes the proof.