Math 571 Chapter 6 The Bombieri-Vinogradov Theorem

Robert C. Vaughan

January 13, 2025

- The Bombieri-A. I. Vinogradov Theorem is concerned with the distribution of primes into arithmetic progressions.

- The Bombieri-A. I. Vinogradov Theorem is concerned with the distribution of primes into arithmetic progressions.
- Define the von Mangoldt function by

$$\Lambda(n) = \begin{cases} \log p & \text{when } n = p^k \text{ for some } p \text{ and } k \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

- The Bombieri-A. I. Vinogradov Theorem is concerned with the distribution of primes into arithmetic progressions.
- Define the von Mangoldt function by

$$\Lambda(n) = \begin{cases} \log p & \text{when } n = p^k \text{ for some } p \text{ and } k \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

- and define

$$\psi(x; q, a) = \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \Lambda(n)$$

which counts the prime powers $p^k \leq x$ with $p^k \equiv a \pmod{p}$ with weight $\log p$.

- The Bombieri-A. I. Vinogradov Theorem is concerned with the distribution of primes into arithmetic progressions.
- Define the von Mangoldt function by

$$\Lambda(n) = \begin{cases} \log p & \text{when } n = p^k \text{ for some } p \text{ and } k \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

- and define

$$\psi(x; q, a) = \sum_{\substack{n \leq x \\ n \equiv a \pmod q}} \Lambda(n)$$

which counts the prime powers $p^k \leq x$ with $p^k \equiv a \pmod p$ with weight $\log p$.
- The higher powers of primes contribute a relatively small amount, and

$$\vartheta(x; q, a) = \psi(x; q, a) + O(x^{\frac{1}{2}})$$

where $\vartheta(x; q, a) = \displaystyle\sum_{\substack{p \leq x \\ p \equiv a \pmod q}} \log p.$

- All the main theorems stated here can be restated with $\psi(x;q;a)$ replaced by $\vartheta(x;q,a)$ or

$$\pi(x;q;a)=\sum_{\substack{p\le x\\ p\equiv a \pmod q}} 1.$$

- All the main theorems stated here can be restated with $\psi(x;q;a)$ replaced by $\vartheta(x;q,a)$ or
$$\pi(x;q;a) = \sum_{\substack{p\le x\\ p\equiv a \pmod q}} 1.$$
- Note that
$$\pi(x;q,a) = \frac{\vartheta(x;q,a)}{\log x} + \int_2^x \frac{\vartheta(u;q,a)}{u\log^2 u}du$$

- All the main theorems stated here can be restated with $\psi(x;q;a)$ replaced by $\vartheta(x;q,a)$ or

$$\pi(x;q;a) = \sum_{\substack{p\le x\\ p\equiv a \pmod q}} 1.$$

- Note that

$$\pi(x;q,a) = \frac{\vartheta(x;q,a)}{\log x} + \int_2^x \frac{\vartheta(u;q,a)}{u\log^2 u}du$$

- The main reason for preferring Λ is that it arises naturally as the coefficient in the Dirichlet series expansion of the logarithmic derivative of

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

viz.

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

when $\Re s > 1$.

- The best general estimate we have for an individual pair q, a, which is uniform in q, is the

Theorem 1

Siegel [1935]–Walfisz [1936] Theorem Suppose that $A > 0$ is a fixed real number. When $(a, q) = 1$ and $q \le (\log x)^A$ we have

$$\psi(x; q, a) = \frac{x}{\phi(q)} + O_A\Big(x \exp\big(-c_1\sqrt{\log x}\big)\Big)$$

where c_1 is an absolute positive constant.

- The best general estimate we have for an individual pair q, a, which is uniform in q, is the

Theorem 1

Siegel [1935]–Walfisz [1936] Theorem Suppose that $A > 0$ is a fixed real number. When $(a,q) = 1$ and $q \le (\log x)^A$ we have

$$\psi(x;q,a) = \frac{x}{\phi(q)} + O_A\Big(x\exp\big(-c_1\sqrt{\log x}\big)\Big)$$

where c_1 is an absolute positive constant.

- It is possible to extend the range for q, but weaken the error term, and be forced to include extra terms from zeros close to 1 which could almost cancel the main term, but the above is the most convenient balanced result we have.

- Let χ denote a Dirichlet character modulo q and put

$$\psi(x;\chi) = \sum_{n\le x} \chi(n)\Lambda(n).$$

- Let χ denote a Dirichlet character modulo q and put

$$\psi(x;\chi) = \sum_{n \le x} \chi(n)\Lambda(n).$$

- Then, by orthogonality

$$\psi(x;q,a) = \frac{1}{\phi(q)} \sum_{\chi \pmod q} \overline{\chi}(a)\psi(x;\chi), \tag{1}$$

and clearly

$$\psi(x;\chi) = \sum_{a=1}^{q} \chi(a)\psi(x;q,a).$$

- The proof of the above is established by applying the following.

Theorem 2

Siegel–Walfisz Theorem variant Suppose that $A > 0$ is a fixed real number. When $q \le (\log x)^A$ and χ is a Dirichlet character modulo q we have

$$\psi(x;\chi) - \delta(\chi)x \ll_A x \exp\big(-c_1\sqrt{\log x}\big)$$

where c_1 is an absolute positive constant and $\delta(\chi)$ is 1 or 0 according as χ is principal or non-principal.

- The proof of the above is established by applying the following.

Theorem 2

Siegel–Walfisz Theorem variant Suppose that $A > 0$ is a fixed real number. When $q \le (\log x)^A$ and χ is a Dirichlet character modulo q we have

$$\psi(x;\chi) - \delta(\chi)x \ll_A x \exp\left(-c_1\sqrt{\log x}\right)$$

where c_1 is an absolute positive constant and $\delta(\chi)$ is 1 or 0 according as χ is principal or non-principal.

- Good references for these two results are Davenport [2000] or Estermann [1952] or Montgomery and Vaughan [2006].

- When $\Re s > 1$ we define

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

- When $\Re s > 1$ we define

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

- This has an analytic continuation to $\mathbb{C}$, and is entire except when χ is principal, in which case it is analytic except at 1 where it has a simple pole with residue

$$\frac{\phi(q)}{q}.$$

- When $\Re s > 1$ we define
$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$
- This has an analytic continuation to $\mathbb{C}$, and is entire except when χ is principal, in which case it is analytic except at 1 where it has a simple pole with residue
$$\frac{\phi(q)}{q}.$$
- Indeed,
$$L(s,\chi_0) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right).$$

- GRH states that $L(s,\chi) \neq 0$ when $\Re s > \frac{1}{2}$.

- GRH states that $L(s,\chi) \neq 0$ when $\Re s > \frac{1}{2}$.
- If GRH holds for $L(s,\chi)$, then we know (Titchmarsh [1930]) that

$$\psi(x;\chi) - \delta(\chi)x \ll x^{\frac{1}{2}}(\log qx)^2,$$

- GRH states that $L(s,\chi) \neq 0$ when $\Re s > \frac{1}{2}$.
- If GRH holds for $L(s,\chi)$, then we know (Titchmarsh [1930]) that

$$\psi(x;\chi) - \delta(\chi)x \ll x^{\frac{1}{2}}(\log qx)^2,$$

- and so GRH for all χ modulo q implies that uniformly for all q,

$$\psi(x;q,a) = \frac{x}{\phi(q)} + O\big(x^{\frac{1}{2}}(\log x)^2\big)$$

- GRH states that $L(s,\chi) \neq 0$ when $\Re s > \frac{1}{2}$.
- If GRH holds for $L(s,\chi)$, then we know (Titchmarsh [1930]) that

$$\psi(x;\chi) - \delta(\chi)x \ll x^{\frac{1}{2}}(\log qx)^2,$$

- and so GRH for all χ modulo q implies that uniformly for all q,

$$\psi(x;q,a) = \frac{x}{\phi(q)} + O\big(x^{\frac{1}{2}}(\log x)^2\big)$$

- We can compare this with

Theorem 3

Bombieri [1965] For any fixed positive number A,

$$\sum_{q\le Q} \max_{(a,q)=1} \sup_{y\le x} \left| \psi(y;q,a) - \frac{y}{\phi(q)} \right| \ll_A x(\log x)^{-A} + x^{1/2}Q(\log xQ)^4.$$

- Bombieri had a somewhat inflated logarithmic factor compared with the above, but in applications that is usually of no significance.

- Bombieri had a somewhat inflated logarithmic factor compared with the above, but in applications that is usually of no significance.
- Vinogradov had required $Q \le x^{\frac{1}{2}-\varepsilon}$.

- Bombieri had a somewhat inflated logarithmic factor compared with the above, but in applications that is usually of no significance.
- Vinogradov had required $Q \le x^{\frac{1}{2}-\varepsilon}$.
- We see that the above is practically as good, when we average over q, as having GRH for all χ to all moduli $q \le x^{1/2}(\log x)^{4-A}$. Consequently this theorem has many applications.

- Bombieri had a somewhat inflated logarithmic factor compared with the above, but in applications that is usually of no significance.
- Vinogradov had required $Q \le x^{\frac{1}{2}-\varepsilon}$.
- We see that the above is practically as good, when we average over q, as having GRH for all χ to all moduli $q \le x^{1/2}(\log x)^{4-A}$. Consequently this theorem has many applications.
- Also, apart from the log power there is no known way in general of improving the crucial term $x^{1/2}Q(\log xQ)^4$ even if one assumes GRH.

- Bombieri had a somewhat inflated logarithmic factor compared with the above, but in applications that is usually of no significance.
- Vinogradov had required $Q \le x^{\frac{1}{2}-\varepsilon}$.
- We see that the above is practically as good, when we average over q, as having GRH for all χ to all moduli $q \le x^{1/2}(\log x)^{4-A}$. Consequently this theorem has many applications.
- Also, apart from the log power there is no known way in general of improving the crucial term $x^{1/2}Q(\log xQ)^4$ even if one assumes GRH.
- Something can be done if one fixes a for all q, replaces y by x or does not take absolute values, but such results are of limited applicability.

- The crude estimate $(x/q+1)\log x$ for each term gives

$$x(\log xQ)^2 + Q\log x$$

which is better than the theorem when $Q > x^{\frac{1}{2}}$, so we can suppose

$$Q \le x^{\frac{1}{2}}.$$

- The crude estimate $(x/q+1)\log x$ for each term gives
$$x(\log xQ)^2 + Q\log x$$
which is better than the theorem when $Q > x^{\frac{1}{2}}$, so we can suppose
$$Q \le x^{\frac{1}{2}}.$$
- All the proofs of the above start off the same way.

- The crude estimate $(x/q+1)\log x$ for each term gives

$$x(\log xQ)^2 + Q\log x$$

which is better than the theorem when $Q > x^{\frac{1}{2}}$, so we can suppose

$$Q \le x^{\frac{1}{2}}.$$

- All the proofs of the above start off the same way.
- One observes that, by (1),

$$\left|\psi(y;q,a) - \frac{y}{\phi(q)}\right| \le \frac{1}{\phi(q)} \sum_{\chi \pmod q} |\psi(y;\chi) - \delta(\chi)y|$$

- The crude estimate $(x/q+1)\log x$ for each term gives
$$x(\log xQ)^2 + Q\log x$$
which is better than the theorem when $Q > x^{\frac{1}{2}}$, so we can suppose
$$Q \le x^{\frac{1}{2}}.$$
- All the proofs of the above start off the same way.
- One observes that, by (1),
$$\left|\psi(y;q,a) - \frac{y}{\phi(q)}\right| \le \frac{1}{\phi(q)} \sum_{\chi \pmod q} |\psi(y;\chi) - \delta(\chi)y|$$
- and so it suffices to bound
$$\sum_{q\le Q} \frac{1}{\phi(q)} \sum_{\chi \pmod q} \sup_{y\le x} |\psi(y;\chi) - \delta(\chi)y| \qquad (2)$$

- The crude estimate $(x/q+1)\log x$ for each term gives
$$x(\log xQ)^2 + Q\log x$$
which is better than the theorem when $Q > x^{\frac{1}{2}}$, so we can suppose
$$Q \le x^{\frac{1}{2}}.$$
- All the proofs of the above start off the same way.
- One observes that, by (1),
$$\left|\psi(y;q,a) - \frac{y}{\phi(q)}\right| \le \frac{1}{\phi(q)} \sum_{\chi \pmod q} |\psi(y;\chi) - \delta(\chi)y|$$
- and so it suffices to bound
$$\sum_{q\le Q} \frac{1}{\phi(q)} \sum_{\chi \pmod q} \sup_{y\le x} |\psi(y;\chi) - \delta(\chi)y| \tag{2}$$
- This already throws away some cancellation in the summation over χ. Almost certainly any improvements will have to make some use of it.

- When χ is induced by the primitive character χ^*, so that the conductor q^* divides q we have

$$\psi(y;\chi) = \psi(y;\chi^*) + O\left(\sum_{p|q, p\nmid q^*} (\log p) \sum_{k \le (\log y)/\log p} 1\right).$$

- When χ is induced by the primitive character χ^*, so that the conductor q^* divides q we have

$$\psi(y;\chi) = \psi(y;\chi^*) + O\left(\sum_{p|q, p\nmid q^*} (\log p) \sum_{k \le (\log y)/\log p} 1 \right).$$

- The error term here is $\ll (\log q)\log y$ and so (2) is

$$= \sum_{q\le Q} \frac{1}{\phi(q)} \sum_{q^*|q} \sum_{\chi^* \pmod{q^*}}^{*} \sup_{y\le x} |\psi(y;\chi^*) - \delta(\chi^*)y| + O\left(Q(\log Q)(\log x)\right)$$

where $\sum^*$ indicates restriction to primitive characters.

- When χ is induced by the primitive character χ^*, so that the conductor q^* divides q we have

$$\psi(y;\chi) = \psi(y;\chi^*) + O\left(\sum_{p|q, p\nmid q^*} (\log p) \sum_{k \le (\log y)/\log p} 1\right).$$

- The error term here is $\ll (\log q)\log y$ and so (2) is

$$= \sum_{q\le Q} \frac{1}{\phi(q)} \sum_{q^*|q} \sum_{\chi^* \pmod{q^*}}^{*} \sup_{y\le x} |\psi(y;\chi^*) - \delta(\chi^*)y| + O\left(Q(\log Q)(\log x)\right)$$

where $\sum^*$ indicates restriction to primitive characters.

- The error term here is more than acceptable, and on interchanging the order of summation and replacing q by q^*r, the main term becomes

$$\sum_{q^*\le Q} \sum_{r\le Q/q^*} \frac{1}{\phi(q^*r)} \sum_{\chi \pmod{q^*}}^{*} \sup_{y\le x} |\psi(y;\chi) - \delta(\chi)y|. \tag{3}$$

- Now
$$\frac{1}{\phi(q^*r)} \le \frac{1}{\phi(q^*)\phi(r)}$$
and
$$\sum_{q\le Q} \frac{1}{\phi(q)} \ll \log 2Q.$$

- Now
$$\frac{1}{\phi(q^*r)} \le \frac{1}{\phi(q^*)\phi(r)}$$
and
$$\sum_{q\le Q} \frac{1}{\phi(q)} \ll \log 2Q.$$
- To see this write $1/\phi(q) = \frac{1}{q}\sum_{r|q} \frac{\mu(r)^2}{\phi(r)}$, and put $q = rm$.

- Now
$$\frac{1}{\phi(q^*r)} \le \frac{1}{\phi(q^*)\phi(r)}$$
and
$$\sum_{q\le Q} \frac{1}{\phi(q)} \ll \log 2Q.$$
- To see this write $1/\phi(q) = \frac{1}{q}\sum_{r|q} \frac{\mu(r)^2}{\phi(r)}$, and put $q = rm$.
- Then the sum is $\sum_{r\le Q} \mu(r)^2 r^{-2} \sum_{m\le Q/r} \frac{1}{m}$.

- Now
$$\frac{1}{\phi(q^*r)} \le \frac{1}{\phi(q^*)\phi(r)}$$
and
$$\sum_{q\le Q} \frac{1}{\phi(q)} \ll \log 2Q.$$
- To see this write $1/\phi(q) = \frac{1}{q}\sum_{r|q}\frac{\mu(r)^2}{\phi(r)}$, and put $q = rm$.
- Then the sum is $\sum_{r\le Q}\mu(r)^2 r^{-2}\sum_{m\le Q/r}\frac{1}{m}$.
- Hence, on replacing q^* by q (9) is
$$\ll \sum_{q\le Q}\frac{\log 2Q}{\phi(q)} \sum_{\chi \pmod q}^{*} \sup_{y\le x}\left|\psi(y;\chi) - \delta(\chi)y\right|.$$
Let $R = (\log x)^{6+A}$.

- Then, by the variant Siegel–Walfisz theorem we have

$$\sum_{q\le R}\frac{\log 2Q}{\phi(q)}\sum_{\chi\pmod q}^{*}\sup_{y\le x}|\psi(y;\chi)-\delta(\chi)y| \ll_A (\log x)Rx\exp\left(-c_2\sqrt{\log x}\right)$$

where c_2 is a positive constant.

- Then, by the variant Siegel–Walfisz theorem we have

$$\sum_{q\le R}\frac{\log 2Q}{\phi(q)}\sum_{\chi \pmod q}^{*}\sup_{y\le x}|\psi(y;\chi)-\delta(\chi)y| \ll_A (\log x)Rx\exp\big(-c_2\sqrt{\log x}\big)$$

  where c_2 is a positive constant.
- We can suppose that $x > x_0(A)$.

- Then, by the variant Siegel–Walfisz theorem we have

$$\sum_{q\le R}\frac{\log 2Q}{\phi(q)}\sum_{\chi}\sideset{}{^*}\sum_{(\mathrm{mod}\ q)}\sup_{y\le x}|\psi(y;\chi)-\delta(\chi)y|$$
$$\ll_A (\log x)Rx\exp\big(-c_2\sqrt{\log x}\big)$$

where c_2 is a positive constant.

- We can suppose that $x > x_0(A)$.
- Then we distinguish two cases.

- Then, by the variant Siegel–Walfisz theorem we have

$$\sum_{q\le R}\frac{\log 2Q}{\phi(q)}\sum_{\chi \pmod q}{}^{*}\sup_{y\le x}|\psi(y;\chi)-\delta(\chi)y|$$
$$\ll_A (\log x)Rx\exp\big(-c_2\sqrt{\log x}\big)$$

  where c_2 is a positive constant.
- We can suppose that $x>x_0(A)$.
- Then we distinguish two cases.
- If $y\le\sqrt{x}$, then we get the conclusion at once.

- Then, by the variant Siegel–Walfisz theorem we have

$$\sum_{q\le R}\frac{\log 2Q}{\phi(q)}\sum_{\chi \pmod q}^{*}\sup_{y\le x}|\psi(y;\chi)-\delta(\chi)y| \ll_A (\log x)Rx\exp\big(-c_2\sqrt{\log x}\big)$$

  where c_2 is a positive constant.
- We can suppose that $x > x_0(A)$.
- Then we distinguish two cases.
- If $y \le \sqrt{x}$, then we get the conclusion at once.
- If $\sqrt{x} \le y \le x$, then the conditions of the Siegel-Walfsiz theorem are satisfied, possibly with a slightly large value of A.

- Then, by the variant Siegel–Walfisz theorem we have

$$\sum_{q\le R}\frac{\log 2Q}{\phi(q)}\sum_{\chi \pmod q}{}^{*}\sup_{y\le x}|\psi(y;\chi)-\delta(\chi)y| \ll_A (\log x)Rx\exp\left(-c_2\sqrt{\log x}\right)$$

where c_2 is a positive constant.

- We can suppose that $x>x_0(A)$.
- Then we distinguish two cases.
- If $y\le\sqrt{x}$, then we get the conclusion at once.
- If $\sqrt{x}\le y\le x$, then the conditions of the Siegel-Walfsiz theorem are satisfied, possibly with a slightly large value of A.
- Hence

$$\sum_{q\le R}\frac{\log 2Q}{\phi(q)}\sum_{\chi \pmod q}{}^{*}\sup_{y\le x}|\psi(y;\chi)-\delta(\chi)y| \ll_A x(\log x)^{-A},$$

which is acceptable.

- Everything so far is classical and could have been done in 1935.

- Everything so far is classical and could have been done in 1935.
- By definition $\delta(\chi) = 0$ for primitive characters with conductor $q > 1$. Thus it remains (!) to deal with the sum

$$(\log 2Q) \sum_{R < q \le Q} \frac{1}{\phi(q)} \sideset{}{^*}\sum_{\chi \pmod{q}} \sup_{y \le x} |\psi(y;\chi)| .$$

- The essential extra ingredient is the following

Theorem 4 (Basic Mean Value Theorem)

Let

$$T(x,Q)=\sum_{q\le Q}\frac{q}{\phi(q)}\sum_{\chi \pmod q}^{*}\sup_{y\le x}|\psi(y;\chi)|$$

where $\sum^*$ *indicates that the sum is over primitive characters modulo* q*, and suppose that* $Q\ge 1$*,* $x\ge 2$*. Then*

$$T(x,Q)\ll\left(x+x^{5/6}Q+x^{1/2}Q^2\right)(\log xQ)^3.$$

- The essential extra ingredient is the following

Theorem 4 (Basic Mean Value Theorem)

Let

$$T(x,Q)=\sum_{q\le Q}\frac{q}{\phi(q)}\sum_{\chi \pmod q}^{*}\sup_{y\le x}|\psi(y;\chi)|$$

where $\sum^*$ *indicates that the sum is over primitive characters modulo q, and suppose that* $Q\ge 1$, $x\ge 2$. *Then*

$$T(x,Q)\ll\left(x+x^{5/6}Q+x^{1/2}Q^2\right)(\log xQ)^3.$$

- We remark in passing that by working harder it is possible to replace the middle term by $x^{4/5}Q$.

- The essential extra ingredient is the following

Theorem 4 (Basic Mean Value Theorem)

Let

$$T(x,Q)=\sum_{q\le Q}\frac{q}{\phi(q)}\sum_{\chi \pmod q}^{*}\sup_{y\le x}|\psi(y;\chi)|$$

where $\sum^*$ *indicates that the sum is over primitive characters modulo* q*, and suppose that* $Q\ge 1$, $x\ge 2$*. Then*

$$T(x,Q)\ll\left(x+x^{5/6}Q+x^{1/2}Q^2\right)(\log xQ)^3.$$

- We remark in passing that by working harder it is possible to replace the middle term by $x^{4/5}Q$.
- The desired conclusion now follows from the above by partial summation.

- To see this, let

$$f(q) = \frac{1}{\phi(q)} \sum_{\chi \pmod q}^{*} \sup_{y \le x} |\psi(y;\chi)|.$$

- To see this, let
$$f(q) = \frac{1}{\phi(q)} \sum_{\chi \pmod q}^{*} \sup_{y\le x} |\psi(y;\chi)|.$$

- Then the sum in question is $(\log 2Q) \sum_{R<q\le Q} f(q) =$

$$(\log 2Q) \sum_{R<q\le Q} f(q)q \left(\frac{1}{Q} + \int_q^Q \frac{dt}{t^2} \right) =$$
$$\frac{\log 2Q}{Q} \sum_{R<q\le Q} qf(q) + (\log 2Q) \int_R^Q \sum_{R<q\le t} qf(q) \frac{dt}{t^2}$$
$$\le (\log 2Q)Q^{-1}T(x,Q) + \int_R^Q t^{-2}T(x,t)dt.$$

- We have

$$(\log 2Q) \sum_{R<q\le Q} f(q)$$

$$\le (\log 2Q) Q^{-1} T(x,Q) + \int_R^Q t^{-2} T(x,t)\,dt.$$

- We have

$$(\log 2Q) \sum_{R<q\le Q} f(q)$$

$$\le (\log 2Q) Q^{-1} T(x,Q) + \int_R^Q t^{-2} T(x,t) dt.$$

- By the Basic Mean Value Theorem this is

$$\ll Q^{-1}\left(x + x^{5/6}Q + x^{1/2}Q^2\right)(\log x)^4$$

$$+ \int_R^Q t^{-2}\left(x + x^{5/6}t + x^{1/2}t^2\right)(\log x)^4 dt$$

$$\ll \left(xR^{-1} + x^{5/6}\log(2Q/R) + x^{1/2}Q\right)(\log x)^4.$$

- We have

$$(\log 2Q) \sum_{R<q\le Q} f(q)$$
$$\le (\log 2Q) Q^{-1} T(x,Q) + \int_R^Q t^{-2} T(x,t)dt.$$

- By the Basic Mean Value Theorem this is

$$\ll Q^{-1}\left(x + x^{5/6}Q + x^{1/2}Q^2\right)(\log x)^4$$
$$+ \int_R^Q t^{-2}\left(x + x^{5/6}t + x^{1/2}t^2\right)(\log x)^4 dt$$
$$\ll \left(xR^{-1} + x^{5/6}\log(2Q/R) + x^{1/2}Q\right)(\log x)^4.$$

- We recall our choice $R = (\log x)^{6+A}$ to conclude that

$$(\log 2Q) \sum_{R<r\le Q} f(r) \ll x(\log x)^{-A} + x^{1/2}Q(\log x)^4$$

as required.

von Mangoldt

- The general philosophy is that we have good information about various kinds of bilinear forms, at least on average.

von Mangoldt

- The general philosophy is that we have good information about various kinds of bilinear forms, at least on average.
- Thus we want to convert our sums involving $\Lambda(n)$ into double sums.
- One, possibly naive, way of doing this is *via* the formula

$$\Lambda(n) = \sum_{lm=n} \mu(l) \log m$$

so that, for example,

$$\sum_{n\le x} \Lambda(n) f(n) = \sum_{l\le x} \sum_{m\le x/l} \mu(l)(\log m) f(lm)$$

and we would think of $\mu(l)$ and $\log m$ as being values of the variables in the bilinear form and f(lm) as being the coefficient of the bilinear form.

von Mangoldt

- The general philosophy is that we have good information about various kinds of bilinear forms, at least on average.
- Thus we want to convert our sums involving $\Lambda(n)$ into double sums.
- One, possibly naive, way of doing this is *via* the formula

$$\Lambda(n) = \sum_{lm=n} \mu(l) \log m$$

so that, for example,

$$\sum_{n \le x} \Lambda(n) f(n) = \sum_{l \le x} \sum_{m \le x/l} \mu(l)(\log m) f(lm)$$

and we would think of $\mu(l)$ and $\log m$ as being values of the variables in the bilinear form and f(lm) as being the coefficient of the bilinear form.
- The first person to successfully attack such a problem was I. M. Vinogradov [1937].

- Vinogradov needed to bound $\displaystyle\sum_{p\le x} e(p\alpha)$.

- Vinogradov needed to bound $\sum_{p\le x} e(p\alpha)$.
- His first step is not dissimilar to that mentioned above in the case of $\Lambda(n)$, but used the sieve of Eratosthenes.

- Vinogradov needed to bound $\displaystyle\sum_{p\le x} e(p\alpha)$.
- His first step is not dissimilar to that mentioned above in the case of $\Lambda(n)$, but used the sieve of Eratosthenes.
- It was while examining Vinogradov's methods that RCV[1977] found a way of dealing with

$$\sum_{n\le x}\Lambda(n)e(n\alpha)$$

which was intrinsically more direct, and focussed towards the available information on bilinear forms.

- In considering bilinear forms

$$\sum_m \sum_n a_m b_n c_{mn}$$

which might arise one has to have some idea of which ones can be sensibly dealt with.

- In considering bilinear forms

$$\sum_m \sum_n a_m b_n c_{mn}$$

  which might arise one has to have some idea of which ones can be sensibly dealt with.
- Here we should think of the c_{mn} as oscillating and potentially giving some cancellation.

- In considering bilinear forms

$$\sum_m \sum_n a_m b_n c_{mn}$$

which might arise one has to have some idea of which ones can be sensibly dealt with.
- Here we should think of the c_{mn} as oscillating and potentially giving some cancellation.
- Typical examples are additive or multiplicative characters.

- In considering bilinear forms

$$\sum_m \sum_n a_m b_n c_{mn}$$

which might arise one has to have some idea of which ones can be sensibly dealt with.
- Here we should think of the c_{mn} as oscillating and potentially giving some cancellation.
- Typical examples are additive or multiplicative characters.
- It is useful to divide bilinear forms into two categories.

- **Type I**. In these one of the variables is smooth, ideally always 1, such as

$$\sum_m \sum_n a_m c_{mn}$$

and it is possible to perform the summation over n with effect.

- **Type I**. In these one of the variables is smooth, ideally always 1, such as
$$\sum_m \sum_n a_m c_{mn}$$
and it is possible to perform the summation over n with effect.
- Usually the only constraint is that the sum over m should not be too long, i.e. ideally we want to ensure that the m are restricted to a fairly short interval.

- **Type I**. In these one of the variables is smooth, ideally always 1, such as
$$\sum_m \sum_n a_m c_{mn}$$
and it is possible to perform the summation over n with effect.
- Usually the only constraint is that the sum over m should not be too long, i.e. ideally we want to ensure that the m are restricted to a fairly short interval.
- **Type II**. In these we are not lucky enough to find that one of the variables is congenial. One needs to use quite general bounds, such as those provided by the large sieve.

- To illustrate this let us look at the bound provided by Lemma 5.5. For sake of argument, lets suppose that $MN \asymp x$, and

$$\sum_m |a_m|^2 \ll M, \quad \sum_n |b_n|^2 \ll N.$$

- To illustrate this let us look at the bound provided by Lemma 5.5. For sake of argument, lets suppose that $MN \asymp x$, and

$$\sum_m |a_m|^2 \ll M, \quad \sum_n |b_n|^2 \ll N.$$

- Then Lemma 5.5 gives the bound

$$\ll \sqrt{(M+Q^2)(N+Q^2)MN}$$
$$\ll x + xQM^{-\frac{1}{2}} + xQN^{-\frac{1}{2}} + x^{\frac{1}{2}}Q^2$$

- To illustrate this let us look at the bound provided by Lemma 5.5. For sake of argument, lets suppose that $MN \asymp x$, and

$$\sum_m |a_m|^2 \ll M, \quad \sum_n |b_n|^2 \ll N.$$

- Then Lemma 5.5 gives the bound

$$\ll \sqrt{(M+Q^2)(N+Q^2)MN}$$
$$\ll x + xQM^{-\frac{1}{2}} + xQN^{-\frac{1}{2}} + x^{\frac{1}{2}}Q^2$$

- and this is a good bound (cf BMVT) provided that M and N are both large (or equivalently M is large but not too close to x).

- To illustrate this let us look at the bound provided by Lemma 5.5. For sake of argument, lets suppose that $MN \asymp x$, and
$$\sum_m |a_m|^2 \ll M, \quad \sum_n |b_n|^2 \ll N.$$
- Then Lemma 5.5 gives the bound
$$\ll \sqrt{(M+Q^2)(N+Q^2)MN}$$
$$\ll x + xQM^{-\frac{1}{2}} + xQN^{-\frac{1}{2}} + x^{\frac{1}{2}}Q^2$$
- and this is a good bound (cf BMVT) provided that M and N are both large (or equivalently M is large but not too close to x).
- In effect we are saying that the rectangular coefficient matrix (c_{mn}) should not be too "thin".

- We can "partition" Λ so as to obtain "good" bilinear type I and II forms.

- We can "partition" Λ so as to obtain "good" bilinear type I and II forms.

Lemma 5

Suppose $u>0$, $v>0$, $y\ge 2$ *and* $f:\mathbb{N}\to\mathbb{C}$. *Then* $\sum_{n\le y}\Lambda(n)f(n)=S_1-S_2-S_3+S_4$ *where*

$$S_1=\sum_{m\le u}\mu(m)\sum_{n\le y/m}(\log n)f(mn),$$

$$S_2=\sum_{m\le uv}c_m\sum_{n\le y/m}f(mn) \text{ where } c_m=\sum_{\substack{k\le u,l\le v\\ kl=m}}\Lambda(k)\mu(l),$$

$$S_3=\sum_{m>u}\sum_{\substack{n>v\\ mn\le y}}\Big(\sum_{\substack{k|m\\ k>u}}\Lambda(k)\Big)\mu(n)f(mn),\quad S_4=\sum_{n\le v}\Lambda(n)f(n).$$

-

- $$S_1 = \sum_{m\le u}\mu(m)\sum_{n\le y/m}(\log n)f(mn),$$

$$S_2 = \sum_{m\le uv} c_m \sum_{n\le y/m} f(mn) \text{ where } c_m = \sum_{\substack{k\le u, l\le v\\ kl=m}} \Lambda(k)\mu(l),$$

$$S_3 = \sum_{m>u}\sum_{\substack{n>v\\ mn\le y}}\Big(\sum_{\substack{k|m\\ k>u}}\Lambda(k)\Big)\mu(n)f(mn),$$

$$S_4 = \sum_{n\le v}\Lambda(n)f(n).$$

- $$S_1 = \sum_{m\le u} \mu(m) \sum_{n\le y/m} (\log n) f(mn),$$

$$S_2 = \sum_{m\le uv} c_m \sum_{n\le y/m} f(mn) \text{ where } c_m = \sum_{\substack{k\le u, l\le v\\ kl=m}} \Lambda(k)\mu(l),$$

$$S_3 = \sum_{m>u} \sum_{\substack{n>v\\ mn\le y}} \Big(\sum_{\substack{k|m\\ k>u}} \Lambda(k)\Big)\mu(n) f(mn),$$

$$S_4 = \sum_{n\le v} \Lambda(n) f(n).$$

- One can see that if u and v are allowed to grow, but not too fast, then S_1 and S_2 will be good bilinear forms of type I and S_3 will be a good bilinear form of type II.

- $$S_1 = \sum_{m\le u} \mu(m) \sum_{n\le y/m} (\log n) f(mn),$$

$$S_2 = \sum_{m\le uv} c_m \sum_{n\le y/m} f(mn) \text{ where } c_m = \sum_{\substack{k\le u, l\le v\\ kl=m}} \Lambda(k)\mu(l),$$

$$S_3 = \sum_{m>u} \sum_{\substack{n>v\\ mn\le y}} \Big(\sum_{\substack{k|m\\ k>u}} \Lambda(k)\Big) \mu(n) f(mn),$$

$$S_4 = \sum_{n\le v} \Lambda(n) f(n).$$

- One can see that if u and v are allowed to grow, but not too fast, then S_1 and S_2 will be good bilinear forms of type I and S_3 will be a good bilinear form of type II.
- Presumably the number of terms in S_4 will be relatively small so it can be bounded trivially.

- **Proof** Consider the identity

$$-\frac{\zeta'}{\zeta}(s) = \big(-\zeta'(s)\big)G(s) - F(s)G(s)\zeta(s) - (-\zeta'(s) - F(s)\zeta(s))\left(G(s) - \frac{1}{\zeta(s)}\right) + F(s)$$

where $F(s) = \sum_{n\le u} \Lambda(n)n^{-s}, \quad G(s) = \sum_{n\le v} \mu(n)n^{-s},$

- **Proof** Consider the identity

$$-\frac{\zeta'}{\zeta}(s) = \big(-\zeta'(s)\big)G(s) - F(s)G(s)\zeta(s) - (-\zeta'(s) - F(s)\zeta(s))\left(G(s) - \frac{1}{\zeta(s)}\right) + F(s)$$

where $F(s) = \sum_{n\le u} \Lambda(n)n^{-s}, \quad G(s) = \sum_{n\le v} \mu(n)n^{-s},$

- and write this as $D_1(s) - D_2(s) - D_3(s) + D_4(s)$.

- **Proof** Consider the identity

$$-\frac{\zeta'}{\zeta}(s) = \big(-\zeta'(s)\big)G(s)$$
$$- F(s)G(s)\zeta(s)$$
$$-\big(-\zeta'(s) - F(s)\zeta(s)\big)\left(G(s) - \frac{1}{\zeta(s)}\right) + F(s)$$

where $F(s) = \sum_{n\le u} \Lambda(n)n^{-s}, \quad G(s) = \sum_{n\le v} \mu(n)n^{-s},$

- and write this as $D_1(s) - D_2(s) - D_3(s) + D_4(s)$.
- Each of the $D_j(s)$ can be written as a Dirichlet series. Let $\Lambda_j(n)$ be the coefficient of n^{-s} in $D_j(n)$.

- **Proof** Consider the identity

$$-\frac{\zeta'}{\zeta}(s) = \big(-\zeta'(s)\big)G(s) - F(s)G(s)\zeta(s) - (-\zeta'(s) - F(s)\zeta(s))\left(G(s) - \frac{1}{\zeta(s)}\right) + F(s)$$

where $F(s) = \sum_{n\le u} \Lambda(n)n^{-s}, \quad G(s) = \sum_{n\le v} \mu(n)n^{-s},$

- and write this as $D_1(s) - D_2(s) - D_3(s) + D_4(s)$.
- Each of the $D_j(s)$ can be written as a Dirichlet series. Let $\Lambda_j(n)$ be the coefficient of n^{-s} in $D_j(n)$.
- Then, by the identity theorem for Dirichlet series,

$$\Lambda(n) = \Lambda_1(n) - \Lambda_2(n) - \Lambda_3(n) + \Lambda_4(n).$$

- **Proof** Consider the identity

$$-\frac{\zeta'}{\zeta}(s) = \big(-\zeta'(s)\big)G(s) - F(s)G(s)\zeta(s) - (-\zeta'(s) - F(s)\zeta(s))\left(G(s) - \frac{1}{\zeta(s)}\right) + F(s)$$

where $F(s) = \sum_{n\le u} \Lambda(n)n^{-s}, \quad G(s) = \sum_{n\le v} \mu(n)n^{-s},$

- and write this as $D_1(s) - D_2(s) - D_3(s) + D_4(s)$.
- Each of the $D_j(s)$ can be written as a Dirichlet series. Let $\Lambda_j(n)$ be the coefficient of n^{-s} in $D_j(n)$.
- Then, by the identity theorem for Dirichlet series,

$$\Lambda(n) = \Lambda_1(n) - \Lambda_2(n) - \Lambda_3(n) + \Lambda_4(n).$$

- Multiply by $f(n)$ and sum over n. By inspection of each of the Dirichlet series $D_j(s)$ we can see that each S_j satisfies $S_j = \sum_n \Lambda_j(n)f(n)$. $\Box$

- We now return to the proof of the theorem, that is, we bound

$$T(x,Q) = \sum_{q\le Q} \frac{q}{\phi(q)} \sum_{\chi \pmod q}^{*} \sup_{y\le x} |\psi(y;\chi)|$$

- We now return to the proof of the theorem, that is, we bound

$$T(x,Q)=\sum_{q\le Q}\frac{q}{\phi(q)}\sum_{\chi \pmod q}^{*}\sup_{y\le x}|\psi(y;\chi)|$$

- It is useful to deal with some special situations first.

- We now return to the proof of the theorem, that is, we bound

$$T(x,Q)=\sum_{q\le Q}\frac{q}{\phi(q)}\sum_{\chi \pmod q}^{*}\sup_{y\le x}|\psi(y;\chi)|$$

- It is useful to deal with some special situations first.
- If $Q^2>x$, then using Lemma 5.6 directly with $M=1$, $a_1=1$, $N=\lfloor x\rfloor$, $b_n=\Lambda(n)$ gives the bound

$$\ll\left(Q^2(x+Q^2)\sum_{n\le x}\Lambda(n)^2\log x\right)^{\frac12}\ll x^{\frac12}Q^2\log Qx.$$

- We now return to the proof of the theorem, that is, we bound

$$T(x,Q)=\sum_{q\le Q}\frac{q}{\phi(q)}\sum_{\chi\pmod q}^{*}\sup_{y\le x}|\psi(y;\chi)|$$

- It is useful to deal with some special situations first.
- If $Q^2>x$, then using Lemma 5.6 directly with $M=1$, $a_1=1$, $N=\lfloor x\rfloor$, $b_n=\Lambda(n)$ gives the bound

$$\ll\left(Q^2(x+Q^2)\sum_{n\le x}\Lambda(n)^2\log x\right)^{\frac12}\ll x^{\frac12}Q^2\log Qx.$$

- Thus we can suppose that $Q^2\le x$.

- Let $u = v = \min\left(Q^2, x^{1/3}, xQ^{-2}\right)$

- Let $u = v = \min\left(Q^2, x^{1/3}, xQ^{-2}\right)$
- Then again by Lemma 5.6, when the sup is restricted to $y \le u^2$, we get

$$\sum_{q\le Q} \frac{q}{\phi(q)} \sum_{\chi}{}^{*} \sup_{y\le u^2} |\psi(y;\chi)| \ll \left(Q^2(u^2+Q^2) \sum_{n\le u^2} \Lambda(n)^2 \log x \right)^{\frac{1}{2}}$$

$$\sum_{q\le Q} \frac{q}{\phi(q)} \sum_{\chi}{}^{*} \sup_{y\le u^2} |\psi(y;\chi)| \ll (Qx^{2/3} + Q^2x^{1/3}) \log x \tag{4}$$

which is good enough.

- Let $u = v = \min\left(Q^2, x^{1/3}, xQ^{-2}\right)$
- Then again by Lemma 5.6, when the sup is restricted to $y \le u^2$, we get

$$\sum_{q\le Q} \frac{q}{\phi(q)} \sum_{\chi}{}^{*} \sup_{y\le u^2} |\psi(y;\chi)| \ll \left(Q^2(u^2+Q^2) \sum_{n\le u^2} \Lambda(n)^2 \log x \right)^{\frac12}$$

$$\sum_{q\le Q} \frac{q}{\phi(q)} \sum_{\chi}{}^{*} \sup_{y\le u^2} |\psi(y;\chi)| \ll (Qx^{2/3} + Q^2x^{1/3}) \log x \tag{4}$$

which is good enough.

- Thus it suffices to bound

$$\sum_{q\le Q} \frac{q}{\phi(q)} \sum_{\chi}{}^{*} \sup_{u^2\le y\le x} |\psi(y;\chi)|.$$

- Thus it suffices to bound

$$\sum_{q\le Q}\frac{q}{\phi(q)}\sum_{\chi}{}^{*}\sup_{u^2\le y\le x}|\psi(y;\chi)|.$$

- Thus it suffices to bound

$$\sum_{q\le Q}\frac{q}{\phi(q)}\sum_{\chi}{}^{*}\sup_{u^2\le y\le x}|\psi(y;\chi)|.$$

- In view of Lemma 5 with $f(n)=\chi(n)$ when $n\le y$ and $f(n)=0$ otherwise it then suffices to bound

$$T_j=\sum_{q\le Q}\frac{q}{\phi(q)}\sum_{\chi}{}^{*}\sup_{u^2\le y\le x}|S_j(\chi)|$$

for $j=1,2,3,4$.

- The case $j = 4$ is easy since

$$S_4(\chi) = \sum_{n \le u} \chi(n)\Lambda(n) = \psi(u;\chi)$$

and $u \le u^2$, and so we can appeal to (4).

- The expression T_1 is also fairly easy, since log is smooth and

$$S_1(\chi) = \sum_{m\le u} \mu(m)\chi(m) \sum_{n\le y/m} \chi(n) \int_1^n \frac{dt}{t}$$
$$= \int_1^y \sum_{m\le \min(u,y/t)} \mu(m)\chi(m) \sum_{t<n\le y/m} \chi(n)\frac{dt}{t}$$

- The expression T_1 is also fairly easy, since log is smooth and

$$S_1(\chi) = \sum_{m\le u} \mu(m)\chi(m) \sum_{n\le y/m} \chi(n) \int_1^n \frac{dt}{t}$$
$$= \int_1^y \sum_{m\le \min(u,y/t)} \mu(m)\chi(m) \sum_{t<n\le y/m} \chi(n)\frac{dt}{t}$$

- and so when $q>1$ the Pólya-Vinogradov [Homework 6] inequality gives the bound

$$\ll \int_1^y uq^{1/2}\log q\frac{dt}{t} \ll uq^{1/2}(\log q)(\log y).$$

- The expression T_1 is also fairly easy, since log is smooth and

$$S_1(\chi) = \sum_{m\le u} \mu(m)\chi(m) \sum_{n\le y/m} \chi(n) \int_1^n \frac{dt}{t}$$
$$= \int_1^y \sum_{m\le\min(u,y/t)} \mu(m)\chi(m) \sum_{t<n\le y/m} \chi(n)\frac{dt}{t}$$

- and so when $q>1$ the Pólya-Vinogradov [Homework 6] inequality gives the bound

$$\ll \int_1^y uq^{1/2}\log q\frac{dt}{t} \ll uq^{1/2}(\log q)(\log y).$$

- This together with the trivial bound $x(\log x)^2$ for the term $q=1$ gives

$$T_1 \ll (x+uQ^{5/2})(\log xQ)^2.$$

- We have
$$T_1 \ll (x + uQ^{5/2})(\log xQ)^2.$$

- We have
$$T_1 \ll (x + uQ^{5/2})(\log xQ)^2.$$
- Recall that $u = Q^2$, $u = x^{1/3}$, $u = xQ^{-2}$ according as $Q \le x^{1/6}$, $x^{1/6} < Q \le x^{1/3}$ and $Q > x^{1/3}$.

- We have
$$T_1 \ll (x + uQ^{5/2})(\log xQ)^2.$$
- Recall that $u = Q^2$, $u = x^{1/3}$, $u = xQ^{-2}$ according as $Q \le x^{1/6}$, $x^{1/6} < Q \le x^{1/3}$ and $Q > x^{1/3}$.
- When $Q \le x^{1/6}$, we have $uQ^{5/2} = Q^{9/2} \le x^{3/4}$.

- We have
$$T_1 \ll (x + uQ^{5/2})(\log xQ)^2.$$
- Recall that $u = Q^2$, $u = x^{1/3}$, $u = xQ^{-2}$ according as $Q \le x^{1/6}$, $x^{1/6} < Q \le x^{1/3}$ and $Q > x^{1/3}$.
- When $Q \le x^{1/6}$, we have $uQ^{5/2} = Q^{9/2} \le x^{3/4}$.
- When $x^{1/6} < Q \le x^{1/3}$, we have $uQ^{5/2} \le x^{1/3}Q^2x^{1/6} = x^{1/2}Q^2$.

- We have
$$T_1 \ll (x + uQ^{5/2})(\log xQ)^2.$$
- Recall that $u = Q^2$, $u = x^{1/3}$, $u = xQ^{-2}$ according as $Q \le x^{1/6}$, $x^{1/6} < Q \le x^{1/3}$ and $Q > x^{1/3}$.
- When $Q \le x^{1/6}$, we have $uQ^{5/2} = Q^{9/2} \le x^{3/4}$.
- When $x^{1/6} < Q \le x^{1/3}$, we have $uQ^{5/2} \le x^{1/3}Q^2x^{1/6} = x^{1/2}Q^2$.
- When $x^{1/3} < Q \le x^{1/2}$, we have $uQ^{5/2} = xQ^{-3/2}Q^2 < x^{1/2}Q^2$.

- The expression T_3 is more complicated to deal with. We want $MN \asymp x$ but both m and n have to range over more than $x^{\frac{1}{2}}$ values.

- The expression T_3 is more complicated to deal with. We want $MN \asymp x$ but both m and n have to range over more than $x^{\frac{1}{2}}$ values.
- We keep control of the overall number of pairs by splitting up the range for m dyadically.

- The expression T_3 is more complicated to deal with. We want $MN \asymp x$ but both m and n have to range over more than $x^{\frac{1}{2}}$ values.
- We keep control of the overall number of pairs by splitting up the range for m dyadically.
- Let $\mathcal{M} = \left\{ 2^k \lfloor u \rfloor : k = 0, 1, \ldots; 2^k \lfloor u \rfloor \le x/u \right\}$ so that $\mathrm{card}\mathcal{M} \ll \log x$.

- The expression T_3 is more complicated to deal with. We want $MN \asymp x$ but both m and n have to range over more than $x^{\frac{1}{2}}$ values.
- We keep control of the overall number of pairs by splitting up the range for m dyadically.
- Let $\mathcal{M} = \left\{2^k\lfloor u\rfloor : k = 0, 1, \ldots; 2^k\lfloor u\rfloor \le x/u\right\}$ so that $\mathrm{card}\mathcal{M} \ll \log x$.
- Then $S_3(\chi) \ll \sum_{M\in\mathcal{M}} |S_3(\chi; M)|$ where

$$S_3(\chi; M) = \sum_{M<m\le 2M} \sum_{\substack{u<n\le x/M \\ mn\le y}} \left(\sum_{\substack{k|m \\ k>u}} \Lambda(k) \right) \mu(n)\chi(mn).$$

- The expression T_3 is more complicated to deal with. We want $MN \asymp x$ but both m and n have to range over more than $x^{\frac{1}{2}}$ values.
- We keep control of the overall number of pairs by splitting up the range for m dyadically.
- Let $\mathcal{M} = \left\{2^k\lfloor u\rfloor : k = 0, 1, \ldots; 2^k\lfloor u\rfloor \le x/u\right\}$ so that $\mathrm{card}\mathcal{M} \ll \log x$.
- Then $S_3(\chi) \ll \sum_{M\in\mathcal{M}} |S_3(\chi; M)|$ where

$$S_3(\chi; M) = \sum_{M<m\le 2M} \sum_{\substack{u<n\le x/M \\ mn\le y}} \left(\sum_{\substack{k|m \\ k>u}} \Lambda(k)\right) \mu(n)\chi(mn).$$

- Note that here the upper limit x/M is never smaller than y/m and will only come into play after we have used Lemma 5.6 to remove the condition $mn \le y$.

- It follows now that $T_3 \le \sum_{M \in \mathcal{M}} T_3(M)$ where

$$T_3(M) = \sum_{q \le Q} \frac{q}{\phi(q)} \sum_{\chi}{}^{*} \sup_{u^2 \le y \le x} |S_3(\chi; M)|.$$

- It follows now that $T_3 \le \sum_{M \in \mathcal{M}} T_3(M)$ where

$$T_3(M) = \sum_{q \le Q} \frac{q}{\phi(q)} \sum_{\chi}{}^{*} \sup_{u^2 \le y \le x} |S_3(\chi; M)| .$$

- It is also useful to note that $\sum_{\substack{k|m \\ k>u}} \Lambda(k) \le \sum_{k|m} \Lambda(k) = \log m.$

- It follows now that $T_3 \le \sum_{M\in\mathcal{M}} T_3(M)$ where

$$T_3(M) = \sum_{q\le Q} \frac{q}{\phi(q)} \sum_{\chi}{}^{*} \sup_{u^2\le y\le x} |S_3(\chi;M)|.$$

- It is also useful to note that $\sum_{\substack{k|m\\k>u}} \Lambda(k) \le \sum_{k|m} \Lambda(k) = \log m$.
- Thus by Lemma 5.6, $T_3(M) \ll$

$$(\log x)\sqrt{(M+Q^2)\left(\frac{x}{M}+Q^2\right)\sum_{m\le 2M}(\log m)^2 \sum_{n\le x/M} \mu(n)^2}.$$

- It follows now that $T_3 \le \sum_{M\in\mathcal{M}} T_3(M)$ where

$$T_3(M) = \sum_{q\le Q} \frac{q}{\phi(q)} \sum_{\chi}{}^{*} \sup_{u^2\le y\le x} |S_3(\chi;M)|\,.$$

- It is also useful to note that $\displaystyle\sum_{\substack{k|m\\k>u}} \Lambda(k) \le \sum_{k|m} \Lambda(k) = \log m$.
- Thus by Lemma 5.6, $T_3(M) \ll$

$$(\log x)\sqrt{(M+Q^2)\left(\frac{x}{M}+Q^2\right)\sum_{m\le 2M}(\log m)^2 \sum_{n\le x/M} \mu(n)^2}.$$

- Since $\displaystyle\sum_{m\le z}(\log m)^2 \ll z(\log 2z)^2, \quad \sum_{n\le z}\mu(n)^2 \ll z$ we have

$$T_3(M) \ll (\log x)^2\left(x+\frac{x}{M^{1/2}}Q + x^{\frac12}M^{\frac12}Q + x^{\frac12}Q^2\right).$$

- We have

$$T_3(M) \ll (\log x)^2 \left(x + \frac{x}{M^{1/2}} Q + x^{\frac{1}{2}} M^{\frac{1}{2}} Q + x^{\frac{1}{2}} Q^2 \right).$$

- We have
$$T_3(M) \ll (\log x)^2 \left(x + \frac{x}{M^{1/2}} Q + x^{\frac12} M^{\frac12} Q + x^{\frac12} Q^2 \right).$$
- Now we sum over $m \in \mathcal{M}$ to obtain
$$T_3 \ll (\log x)^3 \left(x + x u^{-1/2} Q + x^{1/2} Q^2 \right).$$
- Recall that $u = Q^2$, $u = x^{1/3}$, $u = xQ^{-2}$ according as $Q \le x^{1/6}$, $x^{1/6} < Q \le x^{1/3}$ and $Q > x^{1/3}$.

- We have

$$T_3(M) \ll (\log x)^2 \left(x + \frac{x}{M^{1/2}} Q + x^{\frac{1}{2}} M^{\frac{1}{2}} Q + x^{\frac{1}{2}} Q^2 \right).$$

- Now we sum over $m \in \mathcal{M}$ to obtain

$$T_3 \ll (\log x)^3 \left(x + xu^{-1/2} Q + x^{1/2} Q^2 \right).$$

- Recall that $u = Q^2$, $u = x^{1/3}$, $u = xQ^{-2}$ according as $Q \le x^{1/6}$, $x^{1/6} < Q \le x^{1/3}$ and $Q > x^{1/3}$.
- When $Q \le x^{1/6}$, we have $xu^{-1/2} Q = x$.

- We have
$$T_3(M) \ll (\log x)^2 \left(x + \frac{x}{M^{1/2}} Q + x^{\frac{1}{2}} M^{\frac{1}{2}} Q + x^{\frac{1}{2}} Q^2 \right).$$
- Now we sum over $m \in \mathcal{M}$ to obtain
$$T_3 \ll (\log x)^3 \left(x + xu^{-1/2} Q + x^{1/2} Q^2 \right).$$
- Recall that $u = Q^2$, $u = x^{1/3}$, $u = xQ^{-2}$ according as $Q \le x^{1/6}$, $x^{1/6} < Q \le x^{1/3}$ and $Q > x^{1/3}$.
- When $Q \le x^{1/6}$, we have $xu^{-1/2} Q = x$.
- When $x^{1/6} < Q \le x^{1/3}$, we have $xu^{-1/2} Q = x^{5/6} Q$.

- We have
$$T_3(M) \ll (\log x)^2 \left(x + \frac{x}{M^{1/2}} Q + x^{\frac12} M^{\frac12} Q + x^{\frac12} Q^2 \right).$$
- Now we sum over $m \in \mathcal{M}$ to obtain
$$T_3 \ll (\log x)^3 \left(x + x u^{-1/2} Q + x^{1/2} Q^2 \right).$$
- Recall that $u = Q^2$, $u = x^{1/3}$, $u = xQ^{-2}$ according as $Q \le x^{1/6}$, $x^{1/6} < Q \le x^{1/3}$ and $Q > x^{1/3}$.
- When $Q \le x^{1/6}$, we have $xu^{-1/2}Q = x$.
- When $x^{1/6} < Q \le x^{1/3}$, we have $xu^{-1/2}Q = x^{5/6}Q$.
- When $x^{1/3} < Q \le x^{1/2}$, we have $xu^{-1/2}Q = x^{1/2}Q^2$.

- Finally T_2 is treated by a hybrid method.

- Finally T_2 is treated by a hybrid method.
- We have $S_2(\chi) = \displaystyle\sum_{m \le u^2} \sum_{n \le y/m} c_m \chi(mn)$

- Finally T_2 is treated by a hybrid method.
- We have $S_2(\chi) = \sum_{m \le u^2} \sum_{n \le y/m} c_m \chi(mn)$
- We now split this into two parts, so that

$$S_2(\chi) = S_2'(\chi) + S_2''(\chi)$$

where $S_2'(\chi)$ is over $m \le u$ and $S_2''(\chi)$ the remainder.

- Finally T_2 is treated by a hybrid method.
- We have $S_2(\chi) = \sum_{m\le u^2} \sum_{n\le y/m} c_m\chi(mn)$
- We now split this into two parts, so that

$$S_2(\chi) = S_2'(\chi) + S_2''(\chi)$$

where $S_2'(\chi)$ is over $m \le u$ and $S_2''(\chi)$ the remainder.
- The sum $T_2' = \sum_{q\le Q} \frac{q}{\phi(q)} \sum_{\chi}^{*} \sup_{u^2\le y\le x} \left|S_2'(\chi)\right|$ is then treated *via* the Pólya-Vinogradov inequality similarly to T_1

- Finally T_2 is treated by a hybrid method.
- We have $S_2(\chi) = \sum_{m\le u^2}\sum_{n\le y/m} c_m\chi(mn)$
- We now split this into two parts, so that

$$S_2(\chi) = S_2'(\chi) + S_2''(\chi)$$

where $S_2'(\chi)$ is over $m \le u$ and $S_2''(\chi)$ the remainder.
- The sum $T_2' = \sum_{q\le Q}\frac{q}{\phi(q)}\sum_{\chi}^{*}\sup_{u^2\le y\le x}\left|S_2'(\chi)\right|$ is then treated *via* the Pólya-Vinogradov inequality similarly to T_1
- and $T_2'' = \sum_{q\le Q}\frac{q}{\phi(q)}\sum_{\chi}^{*}\sup_{u^2\le y\le x}\left|S_2''(\chi)\right|$ is treated like T_3.

- Finally T_2 is treated by a hybrid method.
- We have $S_2(\chi) = \sum_{m\le u^2} \sum_{n\le y/m} c_m \chi(mn)$
- We now split this into two parts, so that

$$S_2(\chi) = S_2'(\chi) + S_2''(\chi)$$

where $S_2'(\chi)$ is over $m \le u$ and $S_2''(\chi)$ the remainder.

- The sum $T_2' = \sum_{q\le Q} \frac{q}{\phi(q)} \sum_{\chi}^{*} \sup_{u^2\le y\le x} \left|S_2'(\chi)\right|$ is then treated *via* the Pólya-Vinogradov inequality similarly to T_1
- and $T_2'' = \sum_{q\le Q} \frac{q}{\phi(q)} \sum_{\chi}^{*} \sup_{u^2\le y\le x} \left|S_2''(\chi)\right|$ is treated like T_3.
- Note that $|c_m| = \left| \sum_{\substack{k\le u, l\le u\\ kl=m}} \mu(k)\Lambda(l) \right| \le \sum_{l|m} \Lambda(l) = \log m$ and completes the proof of Bombieri's theorem.

- E. Bombieri, On the large sieve, Mathematika 12(1965),201–225
- E. Bombieri and H. Davenport, On the large sieve method, Abh. Zahlentheorie Anal. 1968, 9–22
- H. Davenport, Multiplicative number theory, Markham, Chicago 1967.
- H. Davenport, Multiplicative Number Theory, third edition Springer-Verlag, Berlin 2000.
- T. Estermann, Introduction to modern prime number theory, Cambridge University Press, Cambridge, Tract No. 41, 1952
- P. X. Gallagher, The large sieve, Mathematika 14(1967), 14–20
- P. X. Gallagher, Bombieri's mean value theorem, Mathematika 15(1968), 1–6

Yu. V. Linnik, The large sieve, C. R. (Dokl.) Acad. Sci. URSS, n. Ser. 30(1941), 292–294

Yu. V. Linnik, A remark on the least quadratic non-residue, C. R. (Dokl.) Acad. Sci. URSS, n. Ser. 36(1942), 119–120

H. L. Montgomery, A note on the large sieve, J. Lond. Math. Soc. 43(1968), 93–98

H. L. Montgomery, The analytic principle of the large sieve, Bull. Am. Math. Soc. 84(1978), 547–567

H. L. Montgomery, Ten Lectures on the Interface Between Analytic Number Theory and Harmonic Analysis (CBMS Regional Conference Series in Mathematics), Volume 84, 1994, 220 pp.

H. L. Montgomery and R. C. Vaughan, The large sieve, Mathematika 20(1973), 119–134

- H. L. Montgomery and R. C. Vaughan, Hilbert's inequality, J. Lond. Math. Soc. (2) 8(1974), 73–82
- H. L. Montgomery and R. C. Vaughan, Multiplicative Number Theory. I. Classical Theory, Cambridge University Press, Cambridge 2006
- G. Pólya, Über die Verteilung der quadratischen Reste und Nichtreste, Nachr. Akad. Wiss. Göttingen 1918, 21–29
- K. F. Roth, On the large sieves of Linnik and Renyi, Mathematatika 12(1965), 1–9
- I. Schur, Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen, J. Reine Angew. Math., 140(1911), 1—28
- I. Schur, Einige Bemerkungen zu der vorstehenden Arbeit des Herrn G. Pólya: Über die Verteilung der quadratischen Reste und Nichtreste, Nachr. Akad. Wiss. Göttingen 1918, 30–36

- A. Selberg, Collected papers. Volume II, Springer-Verlag, Berlin 1991
- C. L. Siegel,Über die Klassenzahl quadratischer Zahlkörper, Acta Arith. 1(1935), 83–86
- E. C. Titchmarsh, A divisor problem, Rend. Circ. Mat. Palermo 54(1930), 414–429
- R. C. Vaughan, Sommes trigonométriques sur les nombres premiers, C. R. Acad. Sci. Paris, Série A 285(1977), 981-983
- R. C. Vaughan, An elementary method in prime number theory, Acta Arith. 37(1980), 111–115
- A. I. Vinogradov, On the density hypothesis for Dirichlet L–series, Izv. Akad. Nauk SSSR, Ser. Mat. 29(1965), 903–934
- A. I. Vinogradov, Corrections to the work of A.I. Vinogradov 'On the density hypothesis for Dirichlet L–series', Izv. Akad. Nauk SSSR, Ser. Mat. 30(1966), 719–729

I. M. Vinogradov, Sur la distribution des résidus et des nonrésidus des puissances, J. Soc. Phys. Math. Univ. Permi 1918, 18–28

I. M. Vinogradov, Über die Verteilung der quadratischen Reste und Nichtreste, J. Soc. Phys. Math. Univ. Permi 1919, 1–14

I. M. Vinogradov, Some theorems concerning the theory of primes, Recueil Math. (2) 44(1937), 179–195

A. Walfisz, Zur additiven Zahlentheorie. II, Math. Z. 40(1936), 592–607