Math 571 Chapter 3 The Prime Number Theorem

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- Although we will use some of these results we will not need to be familiar with the techniques for establishing them.
- As I mentioned earlier, Gauss had suggested that

$$\mathrm{li}(x) = \int_2^\infty \frac{d\alpha}{\log \alpha}$$

should be a good approximation

$$\pi(x) = \sum_{p \le x} 1$$

and we saw a table of values out to 10^{22} which illustrated this.

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- In fact this had first been studied by Euler.

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- This latter expression exists for all $z \neq 1$.
- Moreover this is differentiable when $z \neq 1$.
- Thus this latter expression gives an "analytic continuation" to $\mathbb{C} \setminus \{1\}$.

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- The variant for $\psi(x)$ of the formula that Riemann discovered is

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{1}{2}\log(1 - x^{-2}) - \frac{\zeta'(0)}{\zeta(0)}.$$

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- The formula holds for all $x \geq 2$ which are not the power of a prime.
- When $x = p^k$ for some p and k the left hand side has to be replaced by

$$\psi(x) - \frac{1}{2}\log p.$$

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- The computations have been extended considerably. Platt and Trudgian (2020) have shown that there are $12,363,153,437,138$ zeros ρ with

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and they all have $\Re\rho = \frac{1}{2}$.
- We now know that for any $T > 2$ the total number $N(T)$ of ρ with $0 < \Im\rho \le T$ is approximately

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T)$$

and that at least 40% of them have $\Re\rho = \frac{1}{2}$.

- We also know that the assertion that for every $\theta > \frac{1}{2}$

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  is equivalent to the RH,
- and that this in turn is equivalent to

$$\pi(x) - \mathrm{li}(x) \ll x^{\theta}.$$

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- More precisely de la Vallée Poussin showed that
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for some constant c.
- A proof of this is usually given in Math 568.

- The strongest result we now can prove is due to Korobov and I. M. Vinogradov (1958)

$$\pi(x) - \mathrm{li}(x) \ll x \exp\left(-\frac{c(\log x)^{3/5}}{(\log\log x)^{1/5}}\right)$$

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- and the best value for c that we have is $c = 0.2098$ due to Kevin Ford (2002).

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- The values of $L(1;\chi)$ play an important rôle in algebraic number theory.
- Also there is a Riemann Hypothesis for each one (GRH)
- and essentially all of the techniques that have been developed for treating $\zeta(s)$ can be ported over to $L(s;\chi)$.

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- Let
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- Then
$$\psi(x;q,a)=\frac{1}{\phi(q)}\sum_{\chi \pmod q}\overline{\chi}(a)\psi(x;\chi).$$

- Now GRH holds for $L(s;\chi)$ when $\chi \neq \chi_0$ if and only if for every $\theta > \frac{1}{2}$
$$\psi(x;\chi) \ll x^\theta$$
holds for all $x \geq 2$.

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- Here the current state of play is the Siegel-Walfisz theorem (1936) which states that there is a positive constant c such that if A is any fixed positive number, $x \geq 2$, $q \leq (\log x)^A$ and χ is any non-principal character modulo q, then
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- Applied to $\psi(x;q,a)$ this gives, under the same hypothesis on c, A, x, q that when $(q,a) = 1$,
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- In other words, with some constraint on q we have the analogue of de la Vallée Poussin's theorem.

- In 1965 Bombieri and A. I. Vinogradov showed, in some sense, that GRH holds on average, and this is good enough to be used as a replacement for GRH in many applications, and has been behind much of the remarkable progress in analytic number theory in recent years.

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- Equally remarkably we now have proofs of Bombieri-Vinogradov which are elementary apart from the input of the Siegel-Walfisz theorem.