# Math 571 Chapter 0

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January 12, 2025

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- I start by giving an overview of the syllabus and general organizational matters.

# Syllabus

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- If you need to contact me outside the class, the quickest way is via email at rcv4@psu.edu and if necessary we can arrange a suitable time to meet in person.
- If you need a refresher on basic material one possible source is the text book
  A Course of Elementary Number Theory,
  which can be downloaded from
  http://personal.psu.edu/rcv4/CENT.pdf

## Homework

- Homework is due Mondays or the first class in the week when Monday is a holiday.

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- Collaboration is allowed on homework, but only if it is described in the submission and the collaborators listed. Copying is otherwise strictly banned and will lead to penalties.

# Exams

- There will be no exams.

# Topics

- The following topics will be covered.
  An overview of elementary prime number theory and the elementary theory of arithmetical functions.
  A brief overview of the multiplicative structure of rings of residue classes and the theory of characters.
  An overview of the Siegel-Walfisz theorem.
  The Selberg and large sieves.
  Bombieri's theorem on the distribution of primes in arithmetic progressions.
  The Maynard-Tao theorem that there are infinitely many bounded gaps in the primes.
  Vinogradov's theorem that every large odd number is the sum of three primes.
  All of the above will be based on my own notes, copies of which will be available at
  http://www.personal.psu.edu/rcv4/571s25.html.

# Integrity

- All Penn State Policies regarding academic integrity apply to this course. Academic integrity is the pursuit of scholarly activity in an open, honest and responsible manner. Academic integrity is a basic guiding principle for all academic activity at The Pennsylvania State University, and all members of the University community are expected to act in accordance with this principle. Consistent with this expectation, the University's Code of Conduct states that all students should act with personal integrity, respect other students' dignity, rights and property, and help create and maintain an environment in which all can succeed through the fruits of their efforts.

# Integrity

- Academic integrity includes a commitment by all members of the University community not to engage in or tolerate acts of falsification, misrepresentation or deception. Such acts of dishonesty violate the fundamental ethical principles of the University community and compromise the worth of work completed by others.

# Disability

- Penn State welcomes students with disabilities into the University's educational programs. Every Penn State campus has an office for students with disabilities. Student Disability Resources (SDR) website provides contact information for every Penn State campus (http://equity.psu.edu/sdr/disability-coordinator). For further information, please visit Student Disability Resources website (http://equity.psu.edu/sdr/).

# Disability

- In order to receive consideration for reasonable accommodations, you must contact the appropriate disability services office at the campus where you are officially enrolled, participate in an intake interview, and provide documentation: See documentation guidelines (http://equity.psu.edu/sdr/guidelines). If the documentation supports your request for reasonable accommodations, your campus disability services office will provide you with an accommodation letter. Please share this letter with your instructors and discuss the accommodations with them as early as possible. You must follow this process for every semester that you request accommodations.

# Personal Challenges

- Many students at Penn State face personal challenges or have psychological needs that may interfere with their academic progress, social development, or emotional wellbeing. The university offers a variety of confidential services to help you through difficult times, including individual and group counseling, crisis intervention, consultations, online chats, and mental health screenings. These services are provided by staff who welcome all students and embrace a philosophy respectful of clients' cultural and religious backgrounds, and sensitive to differences in race, ability, gender identity and sexual orientation.

# Personal Challenges

- Counseling and Psychological Services at University Park (CAPS) (http://studentaffairs.psu.edu/counseling/): 814-863-0395 Counseling and Psychological Services at Commonwealth Campuses (https://senate.psu.edu/faculty/counseling-services-at-commonwealth-campuses/) Penn State Crisis Line (24 hours/7 days/week): 877-229-6400. Crisis Text Line (24 hours/7 days/week): Text LIONS to 741741

# Bias

- Consistent with University Policy AD29, students who believe they have experienced or observed a hate crime, an act of intolerance, discrimination, or harassment that occurs at Penn State are urged to report these incidents as outlined on the University's Report Bias webpage (http://equity.psu.edu/reportbias/)

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- Then, by uniqueness of factorization

$$P(x) = \prod_{p \le x} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) \ge \sum_{n \le x} \frac{1}{n}.$$

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- More precisely we have
$$\begin{aligned}\log P(x) &= -\sum_{p \le x} \log\left(1 - \frac{1}{p}\right)\\ &= \sum_{p \le x}\left(\frac{1}{p} + \frac{1}{2p^2} + \frac{1}{3p^2} + \cdots\right)\\ &= \sum_{p \le x}\frac{1}{p} + O\left(\sum_p \frac{1}{p(p-1)}\right).\end{aligned}$$

- Putting it all together what we have just proved that there is a constant $C$ such that

$$\sum_{p\le x}\frac{1}{p}\ge \log\log x - C.$$

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- Typically most latin letters will be integers or natural numbers, but $t$, $x$, $y$ may well be real numbers, according to context, and $z$, and in Dirichlet series $s$, will be complex numbers.

- Given functions $f$ and $g$ defined on some domain $\mathcal{X}$ with $g(x) \geq 0$ for all $x \in \mathcal{X}$ we write

$$f(x) = O(g(x)) \tag{1}$$

to mean that there is some constant $C$ such that

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- We also use
$$f(x) = o(g(x)$$
to mean that if there is some limiting operation, such as $x \to \infty$, then
$$\frac{f(x)}{g(x)} \to 0$$
and
$$f(x) \sim g(x)$$
to mean
$$\frac{f(x)}{g(x)} \to 1.$$

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- If $f$ is also non-negative we may use

$$g \gg f$$

to mean (2).

- Return to $S(x)$.

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- Given $f : [1,\infty) \to \mathbb{R}$ define

$$S_f(x) = \sum_{n \le x} f(n).$$

### Lemma 1

*Suppose that $f$ has a continuous monotonic derivative on $[1,\infty)$ and $f(\alpha) \to 0$ as $\alpha \to \infty$. Then*

$$S_f(x) = \int_1^x f(\alpha)d\alpha + C_f + O\big(|f(x)|\big)$$

*where $C_f$ depends only on $f$.*

- To prove the lemma we write

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- The integral here is

$$\int_1^x \alpha f'(\alpha)d\alpha - \int_1^x (\alpha - \lfloor\alpha\rfloor)f'(\alpha)d\alpha$$

- Inserting this in the expression for $S_f$ gives

$$S_f(x) = f(x)\lfloor x \rfloor - \int_1^x \alpha f'(\alpha)d\alpha + \int_1^x (\alpha - \lfloor \alpha \rfloor)f'(\alpha)d\alpha.$$

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- Integrating the first integral by parts gives

$$\int_1^x f(\alpha)d\alpha + f(1) + \int_1^x (\alpha - \lfloor\alpha\rfloor)f'(\alpha)d\alpha - (x - \lfloor x\rfloor)f(x).$$

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- The second integral here is

$$\int_1^\infty (\alpha - \lfloor\alpha\rfloor)f'(\alpha)d\alpha - \int_x^\infty (\alpha - \lfloor\alpha\rfloor)f'(\alpha)d\alpha.$$

  The convergence is guaranteed by monotonicity and the fact that $f(\alpha) \to 0$ as $\alpha \to \infty$.

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- We define

$$C_f = f(1) + \int_1^\infty (\alpha - \lfloor\alpha\rfloor)f'(\alpha)d\alpha$$

- Putting it together gives

$$S_f(x) = \int_1^x f(\alpha)d\alpha + C_f - (x - \lfloor x \rfloor)f(x) \\ - \int_x^\infty (\alpha - \lfloor \alpha \rfloor)f'(\alpha)d\alpha.$$

- Putting it together gives

$$S_f(x) = \int_1^x f(\alpha)d\alpha + C_f - (x - \lfloor x \rfloor)f(x) - \int_x^\infty (\alpha - \lfloor \alpha \rfloor)f'(\alpha)d\alpha.$$

- Finally

$$|(x - \lfloor x \rfloor)f(x)| \le |f(x)|$$

and, by monotonicity,

$$\left| \int_x^\infty (\alpha - \lfloor \alpha \rfloor)f'(\alpha)d\alpha \right| \le \left| \int_x^\infty f'(\alpha)d\alpha \right| = |f(x)|.$$