571 Analytic number Theory I, Spring Term 2025, Solutions 13

1. Suppose that $k \in \mathbb{N}$, $\delta_j > 0$ $(1 \le j \le k)$ and Γ is a set of points $\boldsymbol{\gamma} \in \mathbb{R}^k$ with the property that the open sets $\{\boldsymbol{\beta} \in \mathbb{R}^k : \|\beta_j - \gamma_j\| < \delta_j, 0 \le \beta_j < 1\}$ are pairwise disjoint. Let $N_j \in \mathbb{N}$ and for $\mathbf{n} \in \mathcal{N} = \prod_j^k [1, N_j]$ let $a(\mathbf{n})$ be arbitrary complex numbers. Define

$$S(\boldsymbol{\alpha}) = \sum_{\mathbf{n} \in \mathcal{N}} a(\mathbf{n}) e(\boldsymbol{\alpha}.\mathbf{n}).$$

Prove that $\displaystyle\sum_{\boldsymbol{\gamma} \in \Gamma} |S(\boldsymbol{\gamma})|^2 \ll \sum_{\mathbf{n} \in \mathcal{N}} |a(\mathbf{n})|^2 \prod_{j=1}^{k} (N_j + \delta_j^{-1}).$

Perhaps the easiest proof is to adapt Gallagher's method for the 1-dimensional large sieve. This starts from some version of

$$2\delta f(x) = \int_{x-\delta}^{x+\delta} \big(f(y) + (\delta - |x-y|) \operatorname{sgn}(x-y) f'(y) \big) dy$$

which is obtained by integrating

$$f(x) = f(y) - \int_x^y f'(u) du.$$

In k-dimensions one would have

$$2^k \delta_1 \dots \delta_k f(\mathbf{x}) = \int_{I(\mathbf{x}, \boldsymbol{\delta})} \prod_{j=1}^{k} \Big(1 + (\delta_j - |x_j - y_j|) \operatorname{sgn}(x_j - y_j) \frac{\partial}{\partial y_j} \Big) f(\mathbf{y}) d\mathbf{y}$$

where $I(\mathbf{x}, \boldsymbol{\delta}) = [x_1 - \delta_1, x_1 + \delta_1] \times \cdots \times [x_1 - \delta_k, x_k + \delta_k]$ and would take $f(\mathbf{y}) = S(\mathbf{y}) S(-\mathbf{y})$. An alternative method is given in HLM.

Note that the alternative condition that $\displaystyle\sum_{j=1}^{k} \|\gamma_j - \gamma_j'\| \delta_j^{-1} \gg 1$ when $\boldsymbol{\gamma} \ne \boldsymbol{\gamma}'$ is also sufficient. For if this holds but for some small constant c the sets $\{\boldsymbol{\beta} \in \mathbb{R}^k : \|\beta_j - \gamma_j\| < c\delta_j, 0 \le \beta_j < 1\}$ are not pairwise disjoint, then for some distinct pair $\boldsymbol{\gamma}$, $\boldsymbol{\gamma}'$ the corresponding boxes will have a common point and so by the triangle inequality $\|\gamma_j - \gamma_j'\| \le 2c\delta_j$ $(1 \le j \le k)$ and so the sum will be bounded by $2ck$ and for c sufficiently small this gives a contradiction.

2. The notation $\boldsymbol{\nu}(x) = \boldsymbol{\nu}_k(x)$ represents the vector $(x, x^2, \dots, x^k)$. Thus the polynomial $\alpha_1 x + \cdots + \alpha_k x^k = \boldsymbol{\alpha}.\boldsymbol{\nu}(x)$. Let

$$f(\boldsymbol{\alpha}; N) = \sum_{n=1}^{N} e(\boldsymbol{\alpha}.\boldsymbol{\nu}(n))$$

and suppose that $1 \le m \le N$.

(i) Prove that

$$f(\boldsymbol{\alpha}; N) = \sum_{n=1+m}^{N+m} e(\boldsymbol{\alpha}.\boldsymbol{\nu}(n-m)) = \int_0^1 g(\boldsymbol{\alpha}, \beta; m) \sum_{y=1+m}^{N+m} e(-y\beta) d\beta.$$

where $g(\boldsymbol{\alpha}, \beta; m) = \sum_{n=1}^{2N} e(\boldsymbol{\alpha}.\boldsymbol{\nu}(n-m) + n\beta)$.

(ii) Let $\mathcal{M} \subset (\mathbb{N} \cap [1, N])$ and $M = \mathrm{card}(\mathcal{M})$. Prove that

$$f(\boldsymbol{\alpha}) \ll M^{-1}(\log 2N) \sup_{\beta \in [0,1]} \sum_{m \in \mathcal{M}} g(\boldsymbol{\alpha}, \beta; m).$$

(i) Immediate on replacing the variable n by $n - m$, and then using orthogonality. (ii) Immediate from the triangle inequality and summing over the elements of $\mathcal{M}$.

3. Define the $k-1$ dimensional vector $\boldsymbol{\gamma}(m)$ by

$$\boldsymbol{\alpha}.\boldsymbol{\nu}(x-m) = \sum_{j=1}^{k} \alpha_j (-m)^j + \sum_{h=1}^{k-1} x^h \gamma_h(m) + \alpha_k x^k.$$

(i) Prove that $\gamma_h(m) = \sum_{j=h}^{k} \alpha_j \binom{j}{h} (-m)^{j-h}$.

(ii) Specialise to the case $k = 3$. Prove that if m, m' are two different elements of $\mathcal{M}$, then $\|3\alpha_3(m - m')\| = \|\gamma_2(m) - \gamma_2(m')\|$, $\|2\alpha_2(m - m')\| \le \|\gamma_1(m) - \gamma_1(m')\| + 2N\|\gamma_2(m) - \gamma_2(m')\|$.

(i) By the binomial theorem LHS $= \sum_{j=1}^{k} \alpha_j \sum_{h=0}^{j} \binom{j}{h} x^h (-m)^{j-h}$ and identity follows on interchanging the order of summation. (ii) When $k = 3$ and $h = 1$ we have $\gamma_1(m) = \alpha_1 - 2\alpha_2 m + 3\alpha_3 m^2$ and when $h = 2$ we have $\gamma_2(m) = \alpha_2 - 3\alpha_3 m$. Hence $3\alpha_3(m' - m) = \gamma_2(m) - \gamma_2(m')$ and $2\alpha_2(m' - m) = \gamma_1(m) - \gamma_1(m') - (m + m')\big(\gamma_2(m) - \gamma_2(m')\big)$. Therefore $\|2\alpha_2(m - m')\| \le \|\gamma_1(m) - \gamma_1(m')\| + 2N\|\gamma_2(m) - \gamma_2(m')\|$.