MATH 571, SPRING 2025, SOLUTIONS 9

Let

$$A(n) = \sum_{\substack{p_1,p_2,p_3\le n\\ p_1+p_3=2p_2}} (\log p_1)(\log p_2)(\log p_3).$$

Since we can write $p_3 - p_2 = p_2 = p_1 = d$ the p_j are three successive members of the arithmetic progression $p_1 + xd$. In fact we are counting, with weight $(\log p_1)(\log p_2)(\log p_3)$ all the triples of primes not exceeding n which are in arithmetic progression. Note that we are allowing $d < 0$ and $d = 0$, so each triple with $d \neq 0$ is being counted essentially twice. The terms with $d = 0$ only contribute $\pi(n)$. It is this which Green and Tao famously generalised in 2004 to k primes in a.p. The object of this homework is to show that for any fixed $B \ge 1$ we have

$$A(n) = \frac{1}{2}C_2 n^2 + O_B\big(n^2(\log n)^{-B}\big) \tag{1}$$

where C_2 is the twin prime constant $C_2 = 2\prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right)$. Note that most authors call $C_2' = \prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)$ the twin prime constant, but then write $\pi_2(x) \sim 2C_2'x(\log x)^{-2}$(!).

It is easily deduced from (1) that $a(n) = \text{card}\{p_1 < p_2 < p_3 \le n : p_1 + p_2 = 2p_3\} \sim \frac{1}{4}C_2 n^2(\log n)^{-3}$.

1. In the notation of Theorem 7.6, show that $\int_{\mathfrak{m}} S(\alpha)^2 S(-2\alpha)d\alpha \ll n^2(\log n)^{(7-A)/2}$.

By Schwarz

$$\int_0^1 |S(\alpha)S(2\alpha)|d\alpha \le \left(\int_0^1 |S(\alpha)|^2 d\alpha\right)^{\frac{1}{2}}\left(\int_0^1 |S(2\alpha)|^2 d\alpha\right)^{\frac{1}{2}} = \sum_{p\le n}(\log p)^2 \ll n\log n$$

and, by HW10, $\sup_{\mathfrak{m}} |S(\alpha)| \ll n(\log n)^{(5-A)/2}$.

2. Show that $\int_{\mathfrak{M}} S(\alpha)^2 S(-2\alpha)d\alpha = C_2 J(n) + O\big(n^2(\log n)^{1-A}\big)$ where

$$J(n) = \int_{-(\log n)^A n^{-1}}^{(\log n)^A n^{-1}} T(\beta)^2 T(-2\beta)d\beta.$$

It is possible to replace the exponent $1-A$ by $-A$, but that requires a lot more work and this suffices for our purposes. Let $S_1 = S(\alpha)$, $S_2(\alpha) = S(-2\alpha)$, $V_1 = \mu(q)T(\beta)/\phi(q)$, $V_2 =$

$\mu(q/(q,2))T(-2\beta)/\phi(q/(q,2))$. Note that if $\alpha = a/q+\beta$, then $2\alpha = (2a/(q,2))(q/(q,2)) + 2\beta$. Hence for $\alpha \in \mathfrak{M}(q,a)$ we have $S_j = V_j + E_j$ where $E_j \ll n\exp(-c\sqrt{\log n})$ and $S_1^2S_2 = (V_1+E_1)^2(V_2+E_2) = V_1^2V_2 + O\big(n^3\exp(-c\sqrt{\log n})\big)$. Integrating over $\mathfrak{M}$ gives $\mathfrak{S}\big((\log n)^A\big)J(n) + O(n^2(\log n)^{-A})$ where

$$\mathfrak{S}(Q) = \sum_{q\le Q} \frac{\mu(q)^2\mu(q/(q,2)}{\phi(q)\phi(q/(q,2))}.$$

We also have $\sum_{q>(\log n)^A} \frac{\mu(q)^2}{\phi(q)\phi(q/(q,2))} \ll \sum_{q>(\log n)^A}(\log q)q^{-2} \ll (\log n)^{1-A}$, $J(n) \ll n\int_{|\beta|\le 1/2}\min(n^2,\beta^{-2})d\beta \ll n^2$ and $\mathfrak{S}(\infty) = C_2$.

3. We have a problem in that $T(2\times 1/2) = n$. To get round this, prove that

$$\int_{-1/2}^{1/2} |T(2\beta)|^2 d\beta = \frac{1}{2}\int_{-1}^{1} |T(\beta)|^2 d\beta = \int_{-1/2}^{1/2} |T(\beta)|^2 d\beta = n$$

and

$$\int_{(\log n)^A n^{-1}\le|\beta|\le 1/2} |T(\beta)^2 T(-2\beta)|d\beta \ll n(\log n)^{-A}\int_{-1/2}^{1/2} |T(\beta)T(-2\beta)|d\beta \ll n^2(\log n)^{-A}.$$

The first equality is a simple change of variable. The second uses the periodicity of T. The third is Parseval's identity. For the last part begin by observing that $T(\beta) \ll \|\beta\|^{-1} \ll n(\log n)^{-A}$ and then apply Schwarz and the previous equalities.

4. Prove that $\int_{-1/2}^{1/2} T(\beta)^2T(-2\beta)d\beta = \operatorname{card}\{n_1,n_3 \le n : 2|n_1+n_3\} = \frac{1}{2}n^2 + O(1)$ and

$$\int_{\mathfrak{M}} S(\alpha)^2 S(-2\alpha)d\alpha = \frac{1}{2}C_2 n^2 + O\big(n^2(\log n)^{1-A})\big).$$

By the orthogonality of the additive characters the integral is the number of solutions of $n_1+n_3 = 2n_2$ with $n_j \le n$ and thus is the number of ordered pairs $n_1\le n$, $n_3 \le n$ with n_1+n_3 even. This is

$$\sum_{n_1\le n}\sum_{n_3\le n} \frac{1+(-1)^{n_1+n_3}}{2} = \frac{n^2}{2} + O(1).$$

Then, by Q3, $J(n) = \int_{-1/2}^{1/2} T(\beta)^2T(-2\beta)d\beta + O\big(n^2(\log n)^{-A}\big)$ and the last part of the question follows from Q2.

5. Deduce (1).

We have $A(n) = \int_{\mathfrak{U}} S(\alpha)^2S(-2\alpha)d\alpha$. Then (1) follows from Q1 and Q4.