Math 571, Spring 2025, Solutions 8

1. Suppose that α is a real number, $1 < u < \sqrt{X}$, and $S(\alpha) = \sum_{n\le X}\mu(n)e(\alpha n)$. Prove that $S(\alpha) = S_1 - S_2 + 2S_3$ where $S_1 = \sum_{m>u}\sum_{u<n\le X/m} a_m\mu(n)e(\alpha mn)$, $S_2 = \sum_{m\le u^2} c_m \sum_{n\le X/m} e(\alpha mn)$, $S_3 = \sum_{n\le u}\mu(n)e(\alpha n)$, $a_m = \sum_{m|n, m\le u}\mu(m)$, $c_m = \sum_{k\ell=m, k\le u, \ell\le u}\mu(k)\mu(\ell)$.

This is as in the case of Λ but we instead use the identity $\zeta(s)^{-1} = \big(1-\zeta(s)F(s)\big)\big(\zeta(s)^{-1} - F(s)\big) - F^2\zeta(s) + 2F(s)$ where $F(s) = \sum_{n\le u}\mu(n)n^{-s}$. The identity theorem for Dirichlet series then gives $\mu(n) = c_1(n) - c_2(n) + c_3(n)$ where

$$c_1(n) = \sum_{\ell m=n, u<\ell, u<m}\ \sum_{k|\ell, k>u}\mu(k)\mu(m),$$

$c_2(n) = \sum_{k\ell m=n, k\le u, \ell\le u}\mu(k)\mu(\ell)$ when $n\le u$ and $c_2(n) = 0$ otherwise, and $c_3(n) = \mu(n)$. Now

$$\sum_{n\le X} c_1(n)e(\alpha n) = \sum_{\ell>u}\ \sum_{u<m\le X/\ell} a_\ell\mu(m)e(\alpha\ell m) = S_1,$$

$\sum_{n\le X} c_2(n)e(\alpha n) = \sum_{j\le u^2} c_j\sum_{m\le X/k} e(\alpha jm) = S_2$ and

$$\sum_{n\le X} c_3(n)e(\alpha n) = S_3.$$

2. Suppose that α is a real number, $a\in\mathbb{Z}$, $q\in\mathbb{N}$ with $(a,q) = 1$ and $|\alpha - a/q| \le q^{-2}$. Prove that there is a positive constant C such that $S(\alpha) \ll (Xq^{-1/2} + X^{4/5} + X^{1/2}q^{1/2})X^{\varepsilon}$.

There seems to be no easy way to replace the X^ε by $(\log X)^C$ in every term. We try to imitate Homework 7. The first difference is that now we only have $|a_m| \le d(m)$. Since $\sum_{m\le Y} d(m)^2 \ll Y(\log Y)^3$, by question 2 on Homework 7 we obtain the slightly inflated $S_1 \ll (Xq^{-1/2} + Xu^{-1/2} + X^{1/2}q^{1/2})(\log X)^3$.

Of course $S_3 \ll u$ and we will take $u = X^{2/5}$. Thus we can concentrate on S_2. We have

$$S_2 \ll \sum_{m\le u^2} d(m)\min\left(\frac{x}{m}, \frac{1}{\|\alpha m\|}\right).$$

By Cauchy's inequality this is

$$\left(\sum_{m\le u^2}\frac{d(m)X}{m}\right)^{1/2}\left(\sum_{m\le u^2}\min\left(\frac{x}{m}, \frac{1}{\|\alpha m\|}\right)\right)^{1/2}$$

We can apply the method of homework 7, Question 1 to obtain

$$\begin{aligned} S_2 &\ll \big(X(\log X)^4\big)^{1/2}\big((Xq^{-1}+X^{4/5}+q)\log(2X)\big)^{1/2} \\ &\ll (Xq^{-1/2}+X^{9/10}+X^{1/2}q^{1/2})(\log X)^3. \end{aligned}$$

This is good enough provided that $q \le X^{1/5}$ or $q > X^{4/5}$. Thus it remains to deal with the case $X^{1/5} < q \le X^{4/5}$. Again follow the arguments of homework 7, Question 1 (iii) and (iv). Following (iii), the contribution from m with $1 \le m \le q/2$, as $d(m) \ll q^{1/8} \ll X^{1/10} \ll (X/q)^{1/2}$ we obtain the bound

$$\sum_{m\le q/2} d(m)\|\alpha m\|^{-1} \ll (X/q)^{1/2} \sum_{m\le q/2} \|\alpha m\|^{-1} \ll X^{1/2}q^{1/2}\log(2q).$$

In following (iv) we obtain

$$\begin{aligned} &\sum_{hq-q/2<m\le hq+q/2} d(m)\min\left(\frac{x}{m}, \frac{1}{\|\alpha m\|}\right) \\ &\qquad\qquad \ll \sum_{m\in\mathcal{M}(h)} d(m)\frac{X}{hq} + \sum_{j\in\mathcal{J}(h)} d(hq+j)\frac{1}{\|(ja+k)/q\|} \end{aligned}$$

where now $1 \le h \le u^2/q$ and, for a given h, $\mathcal{J}(h)$ consists of those j with $-q/2 < j \le q/2$ for which $\|(ja+k)/q\| \ge 2/q$ and $\mathcal{M}(h)$ has at most three elements. In the sum over $\mathcal{M}(h)$ we have $d(m) \ll X^{1/10} \ll q^{1/2}$ and so the sum contributes $\ll Xq^{-1/2}h^{-1}$. Summing over h gives an adequate bound. For the sum over $\mathcal{J}(h)$ there seems to be nothing obviously better than replacing $d(hj+j)$ by $X^{\varepsilon/2}$. Then following on as in homework 7 we obtain for the sum over $\mathcal{J}(h)$ the bound

$$X^{\varepsilon/2} \sum_{h\le u^2/q+1/2} q\log(2q) \ll X^{4/5+\varepsilon}.$$

Thus we have obtained $S(\alpha) \ll (Xq^{-1/2}+X^{1/2}q^{1/2})(\log X)^3 + X^{\frac{4}{5}+\varepsilon}$.