Math 571, Spring 2025, Solutions 7

1. Suppose α, x, $y \in \mathbb{R}$ with $x \ge 1$, $y \ge 1$ and that there is an $A \in \mathbb{R}$ such that for $m \le y$ the complex numbers a_m satisfy $|a_m| \le A$. $\|\alpha\| = \min_{n\in\mathbb{Z}} |\alpha - n|$. (i). Prove the triangle inequality $\|\alpha+\beta\| \le \|\alpha\| + \|\beta\|$. (ii) Prove that $\sum_{m\le y} a_m \sum_{n\le x/m} e(\alpha mn) \ll A\sum_{m\le y} \min\left(\frac{x}{m}, \frac{1}{\|\alpha m\|}\right)$. (iii) Suppose $h = 0$. Prove that $\|\alpha m\| \ge \|aj/q\| - \frac{1}{2q} \ge \frac12 \|aj/q\|$. Deduce that $\sum_{m\le \min(y,q/2)} \min\left(\frac{x}{m}, \frac{1}{\|\alpha m\|}\right) \ll \sum_{1\le j\le \min(y,q/2)} \|aj/q\|^{-1} \ll \sum_{l\le \min(y,q)} \frac{q}{l} \ll q\log 2y$. (iv) Suppose $h > 0$. Put $\beta = q^2(\alpha - a/q)$ and let k be a nearest integer to βh. Prove that if $\left\|\frac{ja+k}{q}\right\| \ge \frac{2}{q}$, then $\|\alpha m\| \ge \frac12 \left\|\frac{ja+k}{q}\right\|$. Deduce that $\sum_{hq-q/2<m\le hq+q/2} \min\left(\frac{x}{m}, \frac{1}{\|\alpha m\|}\right) \ll \frac{x}{hq} + \sum_{l=1}^{q-1} \frac{q}{l}$. (v) Prove that
$\sum_{m\le y} a_m \sum_{n\le x/m} e(\alpha mn) \ll \left(\frac{x}{q} + y + q\right) A\log 2x$.

(i) For some integers m and n we have $\|\alpha\| + \|\beta\| = |\alpha - m| + |\beta - n| \ge |\alpha + \beta - m - n| \ge \|\alpha+\beta\|$. (ii)

If $\alpha m \notin \mathbb{Z}$, then the inner sum is a sum of terms of a g.p. with c.r. $e(\alpha m)$. Hence it is $\ll 1/|\sin \pi\alpha m| \ll 1/\|\alpha m\|$. Also, for any m the sum is $\ll x/m$. Suppose further that there are $q \in \mathbb{N}$, $a \in \mathbb{Z}$ with $(q,a) = 1$ such that $|\alpha - a/q| \le q^{-2}$. When $1 \le m \le y$ put $m = hq + j$ where $-q/2 < j \le q/2$ so that $0 \le h \le \frac{y}{q} + \frac12$ and $j > 0$ when $h = 0$. (iii)

$1 \le m = j \le q/2, \le y$. $\|\alpha m\| = \|aj/q + (\alpha - a/q)j\| \ge \|aj/q\| - |\alpha - a/q|j \ge \|aj/q\| - 1/(2q) \ge \|aj/q\| - \frac12\|aj/q\|$ since $q \nmid aj$. (iv) $\|\alpha m\| = \|a(hq+j)/q + \beta q^{-2}(hq+j)\| = \|(aj+k)/q + \beta q^{-2} j\| \ge \|(aj+k)/q\| - |\beta h - h|/q - |\beta q^{-2}||j| \ge \|(aj+k)/q\| - 1/q$, etc. Given h there are at most three values of j for which $\|(aj+h)/q\| < 2/q$. For these j we have $x/m \le x/((h-1/2)q) \ll x/(hq)$. (v) By (i), (ii), (iii) LHS $\ll \sum_{l\le q} Aq/l + \sum_{h\le y/q+1/2} A(x/(hq) + \sum_{l\le q} q/l) \ll (x/q + y + q)A\log(2qxy)$. There can be no terms with $m > x$, so we may suppose $y \le x$. Also trivially the sum is $\ll Ax\log(2x)$ so one can suppose that $q \le x$.

2. Suppose that $a_1, a_2, \dots, b_1, b_2, \dots$ are complex numbers and $\alpha \in \mathbb{R}$, and given $M \le X$ write $N = \lfloor X/M \rfloor$ define $T = \displaystyle\sum_{M<m\le 2M} \sum_{n\le X/m} a_m b_n e(\alpha mn)$.

(i) Prove $|T|^2 \le \left(\displaystyle\sum_{M<m\le 2M} |a_m|^2\right) \sum_{n_1\le N} \sum_{n_2\le N} b_{n_1}\overline{b_{n_2}} \sum_{M<m\le \min(2M, \frac{X}{n_1}, \frac{X}{n_2})} e\big(\alpha m(n_1 - n_2)\big)$

$$\ll \left(\sum_{M<m\le 2M} |a_m|^2 \sum_{n=1}^{N} |b_n|^2\right)\left(M + \sum_{h=1}^{N} \min\left(\frac{X}{h}, \frac{1}{\|\alpha h\|}\right)\right).$$

(ii) Again suppose $(q,a) = 1$ and $|\alpha - a/q| \le q^{-2}$. Prove that

$$|T|^2 \ll \left(\sum_{M<m\le 2M} |a_m|^2 \sum_{n\le X/M} |b_n|^2\right)\left(Xq^{-1} + M + X/M + q\right)\log(2X).$$

(i) Write $T = \sum_m a_m(\sum_n b_n \dots)$ and apply Cauchy-Schwarz. Then multiply out $|\sum_n b_n \dots|^2$ and interchange the order of summation. The conditions between m, n_1, n_2 carry through to the innermost sum. Then use $|b_{n_1} b_{n_2}| \le \frac12(|b_{n_1}|^2 + |b_{n_2}|)$ and note that the symmetry between n_1 and n_2 means that one can write the double sum over n_1 and n_2 as two copies of $\frac12 \sum_{n_1} |b_{n_1}|^2 \sum_{n_2} |\dots|$ and this is at most $\left(\sum_{n_1} |b_{n_1}|^2\right) \max_{n_1} \sum_{n_2} |\dots|$.

Summing the g.p. gives the next line. Then ignore the factors X/n_1 and X/n_2. The contribution from terms with $n_2 = n_1$ is $\ll M$. For the remaining terms put $|n_1 - n_2| = h$. Then $1 \le h \le N$ and for each n_1 there are at most two n_2 such that $|n_1 - n_2| = h$. Whenever $h \le N$ we have $h \le X/M$ and so $M \le X/h$. (ii) The bound for $\sum_{h=1}^{N} \min\left(\frac{X}{h}, \frac{1}{\|\alpha h\|}\right)$, which was not stated explicitly in question 1, but was at the core of the final estimate in 1(v) (see 1(ii)) is $\ll (Xq^{-1} + N + q)\log X$.

3. Suppose below that α is a real number, $a \in \mathbb{Z}$, $q \in \mathbb{N}$ with $(a,q) = 1$ and $|\alpha - a/q| \le q^{-2}$, $X \ge 2$ and $1 < u < \sqrt{X}$.
(i) Prove that $S = S(\alpha) = \sum\limits_{n \le X} \Lambda(n) e(\alpha n) = S_1 + S_2 - S_3 + S_4$, where

$$S_1 = \sum_{m>u} \sum_{u<n\le X/m} a_m \mu(n) e(\alpha m n), \quad S_2 = \sum_{m \le u} \mu(m) \sum_{n \le X/m} (\log n) e(\alpha m n),$$

$$S_3 = \sum_{m \le u^2} c_m \sum_{n \le X/m} e(\alpha m n), \quad S_4 = \sum_{n \le u} \Lambda(n) e(\alpha n),$$

$$a_m = \sum_{\substack{k|m \\ k>u}} \Lambda(k), \quad c_m = \sum_{k \le u} \sum_{\substack{l \le u \\ kl=m}} \Lambda(k)\mu(l).$$

(ii) Prove that $0 \le a_m \le \log m$ and $|c_m| \le \log m$.
(iii) Let $\mathcal{M} = \{2^j u : 0 \le j,\ 2^j \le Xu^{-2}\}$ and write $S_1 = \sum_{M \in \mathcal{M}} T(M)$ where

$$T(M) = \sum_{M<m\le 2M} \sum_{u<n\le X/m} a_m \mu(n) e(\alpha m n).$$

Prove that (question 2 is useful here)

$$\begin{aligned} S_1 &\ll \sum_{M \in \mathcal{M}} \left(M(\log X)^2\right)^{\frac12} (X/M)^{\frac12} \left(Xq^{-1} + M + X/M + q\right)^{\frac12} (\log X)^{1/2} \\ &\ll (Xq^{-1/2} + Xu^{-1/2} + X^{1/2}q^{1/2})(\log X)^{5/2}. \end{aligned}$$

(iv) Prove that $S_2 = \displaystyle\int_1^X \sum_{m \le \min(u, X/v)} \mu(m) \sum_{v<n\le X/m} e(\alpha m n) \frac{dv}{v}$ and hence that

$$S_2 \ll (Xq^{-1} + u + q)(\log X)^2.$$

The results of question 1 are useful here and in the next question.
(v) Prove that $S_3 \ll (Xq^{-1} + u^2 + q)(\log X)^2$.
(vi) Prove that $S \ll (Xq^{-1/2} + X^{4/5} + X^{1/2}q^{1/2})(\log X)^{5/2}$.

(i) As in class but with $e(\alpha m)$ replacing $\chi(m)$. (ii) $0 \le a_m \le \sum_{k|m} \Lambda(k) = \log m$. $|c_m| \le \sum_{k|m} \Lambda(k)$. (iii) The first line is immediate from question 2. The number of summands is $\ll \log X$ and $u \le M \le X/u$. (iv) Write $\log n = \int_1^n \frac{dv}{v}$ and interchange the order of summation and integration. Then there is an additional constraint on n that $v < n$, and that restricts m to being $\le X/v$. Then use question 1(v) on the double sum. (v) This is immediate from question 3(ii) and question 1(v). (vi) Take $u = X^{2/5}$ which is essentially optimal. One can shave an extra $(\log X)^{9/10}$ off the $X^{4/5}$ term by being careful.