Math 571 Analytic Number Theory I, Spring 2025, Solutions 5

1. Let $\pi_2(X)$ denote the number of of prime numbers $p \le X$ for which $p+2$ is also prime and let $D = \sqrt{X}$ and $P = \prod_{p\le D} p$. Define a_n to be 0 unless $(n(n+2), P) = 1$ in which case take $a_n = 1$, and define $Z = \sum_{n\le N} a_n$. Prove that $\pi_2(X) \le Z + \sqrt{X}$.

If p is counted by $\pi_2(X)$, then either $p \le \sqrt{X}$ or $p(p+2)$ has no common factor with P.

2. Prove that if ω is the multiplicative function with $\omega(2) = 1$, $\omega(p) = 2$ when $2 < p \le Q$ and $\omega(p^k) = 0$ otherewise, then $\pi_2(X) \ll \frac{X}{S(D)} + R$ where $S(D) = \sum_{q\le Q} \mu(q)^2 \prod_{p|q} \frac{\omega(p)}{p-\omega(p)}$ and $R = \sum_{q\le D}\sum_{r\le D} \mu(q)^2\mu(r)^2\omega([q,r])$.

Clearly the n counted by n are odd and omit the two residue classes 0 and -2 modulo p for each odd prime $p \le Q$. The bound then follows from Selberg's theorem.

3. Prove that $S(D) = T(D) + T(D/2)$ where

$$T(Q) = \sum_{\substack{q\le Q\\ q \text{ odd}}} \mu(q)^2 \prod_{p|q} \frac{2}{p-2}.$$

The terms in $S(D)$ with q odd contribute $T(D)$. Those with q even contribute $T(D/2)$.

4. Prove that if $p > 2$, then $\dfrac{2}{p-2} = \displaystyle\sum_{k=1}^{\infty} \frac{2^k}{p^k}$ and that if g is the multiplicative function with $g(p^k) = 2^k$, then $T(Q) \ge \sum_{\substack{q\le Q\\ q \text{ odd}}} \frac{g(q)}{q}$.

Let $s(q) = \prod_{p|q} p$, the squarefree kernel of q. Then $T(Q) = \sum_{\substack{s\le Q\\ s \text{ odd}}} \mu(s)^2 \prod_{p|s}\left(\sum_{k=1}^{\infty} \frac{2^k}{p^k}\right) = \sum_{\substack{s\le Q\\ s \text{ odd}}} \mu(s)^2 \sum_{\substack{q\\ s(q)=s}} \frac{g(q)}{q} = \sum_{\substack{q \text{ odd}\\ s(q)\le Q}} \frac{g(q)}{q}$. Clearly every odd $q \le Q$ occurs.

5. Prove that $g(q) \ge d(q)$ and that $T(Q) \ge \sum_{\substack{q\le Q\\ q \text{ odd}}} \frac{d(q)}{q}$.

For every $k \in \mathbb{N}$ we have $2^k \ge k+1$.

6. Prove that if $Q \ge 2$, then $\sum_{\substack{q\le Q\\ q \text{ odd}}} \frac{d(q)}{q} \gg (\log R)^2$ and hence that if $X \ge 2$, then $\pi_2(X) \ll \frac{X}{(\log X)^2}$.

The sum in question is $\sum_{\substack{mn\le Q\\ mn \text{ odd}}} \frac{1}{mn} \ge \left(\sum_{\substack{m\le\sqrt{Q}\\ m \text{ odd}}} \frac{1}{m}\right)^2$ and for large Q, $\sum_{\substack{m\le\sqrt{Q}\\ m \text{ odd}}} \frac{1}{m} = \frac{1}{2}\log Q + O(1)$. Also $R \le \left(\sum_{q\le D} d(q)\right)^2 \ll D^2\log^2 D$. Take $D = X^{1/2}(\log X)^{-2}$ in question 2.

7. (Brun 1919) Let $\mathcal{P}_2$ denote the set of primes p for which $p+2$ is also prime. Prove that $\displaystyle\sum_{p\in\mathcal{P}_2} \frac{1}{p}$ converges.

We have $\sum_{\substack{p\le X\\ p\in\mathcal{P}_2}} \frac{1}{p} \ll \sum_{k\le 2\log 2X} \sum_{\substack{2^{k-1}<p\le 2^k\\ p\in\mathcal{P}_2}} \frac{1}{2^k}$ and by Q6 this is $\ll \sum_{k\le 2\log 2x} \frac{1}{k^2} \ll 1$.