Math 571 Analytic Number Theory, Spring 2025, Solutions 3

1. Let f denote a polynomial with integer coefficients and $f(0)=0$ and for $q\in\mathbb{N}$ define $S(q;f)=\displaystyle\sum_{x=1}^{q} e\big(f(x)/q\big)$, $W(q;f)=\displaystyle\sum_{\substack{x=1\\(x,q)=1}}^{q} e\big(f(x)/q\big)$.

(i) Suppose that $q_1,\ q_2\in\mathbb{N}$ with $(q_1,q_2)=1$. Choose r_1,r_2 so that $r_1q_2\equiv 1 \pmod{q_1}$ and $r_2q_1\equiv 1\pmod{q_2}$. Prove that

$S(q_1q_2;f)=S(q_1;r_1f)S(q_2;r_2f)$ & $W(q_1q_2;f)=W(q_1;r_1f)W(q_2;r_2f)$

(ii) Ramanujan's sum $c_q(n)$ is W with $f(x)=nx$. Prove that if $q=p^k$ for some prime p and positive integer k, then $c_q(n)=\displaystyle\sum_{m|(q,n)} m\mu(q/m)$.

Deduce that this holds for general q.

(i) Note that $x_1q_2r_1+x_2q_1r_2$ runs over a complete or reduced set of residues modulo q_1q_2 as x_1 and x_2 do modulo q_1 and q_2 respectively. Thus for $k\ge 1$, $(x_1q_2r_1+x_2q_1r_2)^k(q_1q_2)^{-1}\equiv q_2^{k-1}r_1^kx_1^kq_1^{-1}+q_1^{k-1}r_2^kx_2^kq_2^{-1}\equiv r_1x_1^kq_1^{-1}+r_2x_2^kq_2^{-1} \pmod 1$. (ii) The simplest proof of this is to prove it directly as $c_q(n)=\displaystyle\sum_{x=1}^{q}\sum_{k|(q,x)}\mu(k)e\Big(\frac{xn}{q}\Big)$

$$=\sum_{k|q}\mu(k)\sum_{y=1}^{q/k}e\Big(\frac{ny}{q/k}\Big)=\sum_{\substack{k|q\\q/k|n}}\mu(k)\frac{q}{k}=\sum_{m|(q,n)}m\mu(q/m).$$

The method implied by the question is first to observe in the notation of (i) that r_jx_j runs over a reduced set of residues modulo q_j as x_j does, so in the case $f(x)=nx$, $W(q_1q_2;f)=W(q_1)W(q_2)$ and W is multiplicative. Moreover, then $W(p^k)=\displaystyle\sum_{\substack{x=1\\p\nmid x}}^{p^k}e(nxp^{-k})=\sum_{x=1}^{p^k}e(nxp^{-k})-\sum_{x=1}^{p^{k-1}}e(nxp^{1-k})$ and one can proceed as above in this special case. Alternatively one can deduce that $c_{p^k}(n)=\phi(p^k)\mu\big(p^k/(p^k,n)\big)/\phi\big(p^k/(p^k,n)\big)$ and conclude that $c_q(n)=\phi(q)\mu\big(q/(q,n)\big)/\phi\Big(\frac{q}{(q,n)}\Big)$.

2. By lower and upper bound sifting functions we mean functions $\lambda^\pm:\mathbb{N}\to\mathbb{R}$ such that $\sum_{m|n}\lambda_m^-\le\sum_{m|n}\mu(m)\le\sum_{m|n}\lambda_m^+$ respectively. (i) Let λ_d^+ be an upper bound sifting function such that $\lambda_d^+=0$ for all $d>z$. Show that for any q, $0\le\frac{\varphi(q)}{q}\sum_{\substack{d\\(d,q)=1}}\frac{\lambda_d^+}{d}\le\sum_d\frac{\lambda_d^+}{d}$. (ii) Let η_d

be real with $\eta_d = 0$ for $d > z$. Show for any q, $0 \le \frac{\varphi(q)}{q} \sum_{\substack{d,\,e \\ (de,q)=1}} \frac{\eta_d \eta_e}{[d,e]} \le \sum_{d,\,e} \frac{\eta_d \eta_e}{[d,e]}$. (iii) Let λ_d^- be a lower bound sifting function with $\lambda_d^- = 0$ for $d > z$. Show for any q, $\frac{\varphi(q)}{q} \sum_{\substack{d \\ (d,q)=1}} \frac{\lambda_d^-}{d} \ge \sum_d \frac{\lambda_d^-}{d}$.

(i) We suppose first that $p|q \Rightarrow p \le z$. Let $\mathcal{M}$ denote the set of positive integers m such that each prime factor p satisfies $p \le z$ and $p \nmid q$, and put $P = \prod_{p \le z} p$. Then $\frac{P\phi(q)}{\phi(P)q} = \prod_{p|P, p\nmid q} \frac{p}{p-1} = \sum_{m \in \mathcal{M}} \frac{1}{m}$. Thus $\dfrac{P\phi(q)}{\phi(P)q} \displaystyle\sum_{(d,q)=1} \lambda_d^+ d^{-1} = \sum_{m \in \mathcal{M}} m^{-1} \sum_{d \in \mathcal{M}} \lambda_d^+ d^{-1} = \sum_{n \in \mathcal{M}} n^{-1} \sum_{d|n} \lambda_d^+$. This is ≥ 0 and so the lower inequality is immediate. Moreover $\displaystyle\sum_{n \in \mathcal{M}} n^{-1} \sum_{d|n} \lambda_d^+ \le \sum_{n \in \mathcal{N}} n^{-1} \sum_{d|n} \lambda_d^+$ where $\mathcal{N}$ is the set of $m > 0$ such that each prime factor p satisfies $p \le z$. But this sum is $\sum_{m \in \mathcal{N}} m^{-1} \sum_d \lambda_d^+ d^{-1} = \frac{P}{\phi(P)} \sum_d \frac{\lambda_d^+}{d}$. Primes $p > z$ make the factor $\phi(q)/q$ smaller, but still positive, without changing the sum.

Alternative proof. We have

$$\frac{\varphi(q)}{q} \sum_{\substack{d \\ (d,q)=1}} \frac{\lambda_d^+}{d} = \sum_{\substack{d \\ (d,q)=1}} \lambda_d^+ \lim_{X\to\infty} \sum_{\substack{m \le X/d \\ (m,q)=1}} 1/X = \lim_{X\to\infty} X^{-1} \sum_{\substack{n \le X \\ (n,q)=1}} \sum_{d|n} \lambda_d^+$$

$$\le \lim_{X\to\infty} X^{-1} \sum_{n \le X} \sum_{d|n} \lambda_d^+ = \lim_{X\to\infty} \sum_d \lambda_d^+ X^{-1} \lfloor X/d \rfloor = \sum_d \lambda_d^+/d.$$

(ii) Note that in the above proof we only used the property $\sum_{d|n} \lambda_d^+ \ge 0$, not the property $\lambda_1 \ge 1$. Now $(\sum_{d|n} \eta_d)^2 \ge 0$ and the LHS is $\sum_{d|n} \sum_{e|n} \eta_d \eta_e = \sum_{m|n} \sum_{d,e,[d,e]=m} \eta_d \eta_e$. Let $\lambda_m^+ = \sum_{d,e,[d,e]=m} \eta_d \eta_e$. Then $\sum_{m|n} \lambda_m^+ \ge 0$ and the support of λ_n^+ is contained in $[1, z^2]$, so we can apply (i) (with z replaced by z^2).

(iii) Either proof of (i) adapts. For example

$$\frac{\phi(q)}{q} \sum_{(d,q)=1} \lambda_d^- d^{-1} = \sum_{(d,q)=1} \lambda_d^- \lim_{X\to\infty} \sum_{m \le X/d, (m,q)=1} X^{-1}$$

$$= \lim_{X\to\infty} X^{-1} \sum_{n \le X, (n,q)=1} \sum_{d|n} \lambda_d^- \ge \sum_{n \le X} \sum_{d|n} \lambda_d^-,$$

etc.