MATH 571 ANALYTIC NUMBER THEORY, SPRING 2025, SOLUTIONS 2

1. Let $k \in \mathbb{N}$. Prove that there are infinitely many n such that $\mu(n+1) = \mu(n+2) = \cdots = \mu(n+k)$.

Let $p_1, \ldots, p_k$ be k different primes. Then by the Chinese Remainder Theorem there are infinitely many n with $n \equiv -j \pmod{p_j^2}$ $(1 \le j \le k)$ and so $p_j^2 | n+j$ and $\mu(n+j) = 0$.

2. Suppose that the arithmetical function $\eta(n)$ satisfies $\sum_{m|n} \eta(m) = \phi(n)$. Show that $\eta(n)$ is multiplicative and evaluate $\eta(p^k)$.

By the Möbius inversion formula we have $\eta = \phi * \mu$, and since both ϕ and μ are multiplicative so is η by Theorem 1.4. Thus it suffices to evaluate $\eta(p^k) = \sum_{j=0}^{k} \mu(p^j)\phi(p^{k-j})$. For $k \ge 1$ this is $\mu(1)\phi(p^k) + \mu(p)\phi(p^{k-1})$ and so $\eta(p) = (p-1) - 1 = p-2$ and, $k \ge 2$, $\eta(p^k) = p^{k-1}(p-1) - p^{k-2}(p-1) = p^{k-2}(p-1)^2$.

3. This question investigates whether there exists an arithmetic function θ such that $\theta * \theta = \mu$ and $\theta(1) \ge 0$. (i) Prove that θ exists and is uniquely determined. (ii) Prove that $\theta(p^k) = (-1)^k \binom{\frac{1}{2}}{k}$. This is the coefficient of z^k in the Taylor expansion of $(1-z)^{1/2}$ centred at 0. (iii) By considering the function $\theta_1(n) = \prod_{p^k \| n} \theta(p^k)$, or otherwise, show that $\theta \in \mathcal{M}$.

(i) We can define θ iteratively by $\theta(1) = 1$ and

$$\theta(n) = \frac{1}{2}\left(\mu(n) - \sum_{\substack{m|n \\ 1<m<n}} \theta(m)\theta(n/m)\right).$$

That this determines θ uniquely follows by induction. For suppose that θ^* is another solution. Then $\theta^*(1) = 1 = \theta(1)$ and if $\theta^*(m) = \theta(m)$ for $m < n$, then, by the above formula, $\theta^*(n) = \theta(n)$. (ii) & (iii) Suppose that $|z| < 1$. Then $f(z) = \sum_{k=0}^{\infty} (-1)^k \binom{\frac{1}{2}}{k} z^k$ converges absolutely to $(1-z)^{1/2}$ and so by rearrangement

$$\sum_{k=0}^{\infty} \mu(p^k) z^k = 1 - z = f(z)^2 = \sum_{k=0}^{\infty} z^k \sum_{\substack{j,k \ge 0 \\ j+l=k}} (-1)^j \binom{\frac{1}{2}}{j} (-1)^l \binom{\frac{1}{2}}{l}.$$

Thus, by the identity theorem for power series

$$\mu(p^k) = \sum_{\substack{j,k\geq 0\\ j+l=k}} (-1)^j \binom{\frac{1}{2}}{j} (-1)^l \binom{\frac{1}{2}}{l}.$$

Thus the choice $\theta(p^k) = (-1)^j \binom{\frac{1}{2}}{j}$ and θ multiplicative will satisfy (i) and so by uniqueness is the only solution.

4. A number $n \in \mathbb{N}$ is *squarefree* when it has no repeated prime factors. For $X \in \mathbb{R}$, $X \geq 1$ let $Q(X)$ denote the number of squarefree numbers not exceeding X.

(i) Prove that

$$Q(X) = \frac{6}{\pi^2} X + O\left(\sqrt{X}\right).$$

(ii) Prove that if $n \in \mathbb{N}$, then

$$Q(n) \geq n - \sum_p \left\lfloor \frac{n}{p^2} \right\rfloor.$$

(iii) Prove that

$$\sum_p \frac{1}{p^2} < \frac{1}{4} + \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} < \frac{1}{4} + \sum_{k=1}^{\infty} \frac{1}{4k(k+1)} = \frac{1}{2}.$$

(iv) Prove that $Q(n) > n/2$ for all $n \in \mathbb{N}$.

(v) Prove that every integer $n > 1$ is a sum of two squarefree numbers.

(i) We have $Q(x) = \sum_{n\leq x} \sum_{m^2|n} \mu(m) = \sum_{m\leq\sqrt{x}} \mu(m) \left\lfloor \frac{x}{m^2} \right\rfloor$. (ii) $Q(n) = n - N$ where N is the number of $n \leq N$ for which there exists a prime number p with $p^2|n$. Thus $N \leq \sum_p \left\lfloor \frac{n}{p^2} \right\rfloor$. (iii) Separate out the prime $p = 2$ and use $\lfloor y \rfloor \leq y$. (iv) We have immediately from (ii) and (iii) that $Q(n) > n - \frac{n}{2} = \frac{n}{2}$. Consider the $n-1$ boxes "k" with $1 \leq k \leq n-1$. If $m \leq n-1$ and m is squarefree put m in "m" and if $m' \leq n-1$ and m' is squarefree put $n - m'$ in "$n - m'$". We have just put $2Q(n-1) > n-1$ objects into $n-1$ boxes so one box contains two objects, i.e. $m = n - m'$ with both m and m' squarefree.