571 Analytic number Theory I, Spring Term 2025, Problems 13

Return by Monday 28th April

This homework is part of the transition from the discrete to the continuous. See §5.2 of HLM.

1. Suppose that $k \in \mathbb{N}$, $\delta_j > 0$ $(1 \le j \le k)$ and Γ is a set of points $\boldsymbol{\gamma} \in \mathbb{R}^k$ with the property that the open sets $\{\boldsymbol{\beta} \in \mathbb{R}^k : \|\beta_j - \gamma_j\| < \delta_j, 0 \le \beta_j < 1\}$ are pairwise disjoint. Let $N_j \in \mathbb{N}$ and for $\mathbf{n} \in \mathcal{N} = \prod_j^k [1, N_j]$ let $a(\mathbf{n})$ be arbitrary complex numbers. Define

$$S(\boldsymbol{\alpha}) = \sum_{\mathbf{n}\in\mathcal{N}} a(\mathbf{n}) e(\boldsymbol{\alpha}.\mathbf{n}).$$

Prove that $\displaystyle\sum_{\boldsymbol{\gamma}\in\Gamma} |S(\boldsymbol{\gamma})|^2 \ll \sum_{\mathbf{n}\in\mathcal{N}} |a(\mathbf{n})|^2 \prod_{j=1}^{k} (N_j + \delta_j^{-1}).$

Hint: Try adapting to k-dimensions one of the proofs of the large sieve inequality.

2. The notation $\boldsymbol{\nu}(x) = \boldsymbol{\nu}_k(x)$ represents the vector $(x, x^2, \dots, x^k)$. Thus the polynomial $\alpha_1 x + \dots + \alpha_k x^k = \boldsymbol{\alpha}.\boldsymbol{\nu}(x)$. Let

$$f(\boldsymbol{\alpha}; N) = \sum_{n=1}^{N} e(\boldsymbol{\alpha}.\boldsymbol{\nu}(n))$$

and suppose that $1 \le m \le N$.
(i) Prove that

$$f(\boldsymbol{\alpha}; N) = \sum_{n=1+m}^{N+m} e(\boldsymbol{\alpha}.\boldsymbol{\nu}(n-m)) = \int_0^1 g(\boldsymbol{\alpha}, \beta; m) \sum_{y=1+m}^{N+m} e(-y\beta) d\beta.$$

where

$$g(\boldsymbol{\alpha}, \beta; m) = \sum_{n=1}^{2N} e(\boldsymbol{\alpha}.\boldsymbol{\nu}(n-m) + n\beta).$$

(ii) Let $\mathcal{M} \subset (\mathbb{N} \cap [1, N])$ and $M = \mathrm{card}(\mathcal{M})$. Prove that

$$f(\boldsymbol{\alpha}) \ll M^{-1} (\log 2N) \sup_{\beta\in[0,1]} \sum_{m\in\mathcal{M}} |g(\boldsymbol{\alpha}, \beta; m)|.$$

3. Define the $k-1$ dimensional vector $\boldsymbol{\gamma}(m)$ by

$$\boldsymbol{\alpha}.\boldsymbol{\nu}(x-m) = \sum_{j=1}^{k} \alpha_j (-m)^j + \sum_{h=1}^{k-1} x^h \gamma_h(m) + \alpha_k x^k.$$

(i) Prove that $\gamma_h(m) = \sum_{j=h}^{k} \alpha_j \binom{j}{h} (-m)^{j-h}$.

(ii) Specialise to the case $k = 3$. Prove that if m, m' are two different elements of $\mathcal{M}$, then

$$\|3\alpha_3(m - m')\| = \|\gamma_2(m) - \gamma_2(m')\|,$$
$$\|2\alpha_2(m - m')\| \le \|\gamma_1(m) - \gamma_1(m')\| + 2N\|\gamma_2(m) - \gamma_2(m')\|,$$

so that if the $\alpha_2 m$ or α_3 are well spaced modulo 1, so are the γ_1 or the γ_2 and we can make use of question 1. This will happen when α_2 or α_3 are on minor arcs.

(i) Apply the binomial theorem and interchange the order of summation. (ii)