571 Analytic number Theory I, Spring Term 2025, Problems 11

Return by Monday 14th April

Throughout, n is a large integer, $N_k = \lfloor n^{1/k} \rfloor$, $f_k(\alpha) = \sum_{x \le N_k} e(\alpha x^k)$, and $R(n;s,t)$ denotes the number of solutions in positive integers $x_1, \dots, x_s, y_1, \dots, y_t$ of

$$x_1^2 + \cdots + x_s^2 + y_1^3 + \cdots + y_t^3 = n. \tag{1}$$

Also we define $\nu = \frac{1}{100}$, $\mathfrak{M}(q,a) = \left\{\alpha : \left|\alpha - \frac{a}{q}\right| \le N_3^{\nu-3}\right\}$ and $\mathfrak{M}$ denotes the union of all $\mathfrak{M}(\mathfrak{q},\mathfrak{a})$ with $1 \le a \le q \le N_3^{\nu}$, $(a,q) = 1$, and $\mathfrak{m} = \left(N_3^{\nu-3}, 1 + N_3^{\nu-3}\right] \backslash \mathfrak{M}$. Let $S_k(q,a) = \sum_{x=1}^{q} e(ax^k/q)$, $v_k(\beta) = \sum_{m=1}^{n} \frac{1}{k} m^{\frac{1}{k}-1} e(m\beta)$,

$$V_k(\alpha, q, a) = q^{-1} S_k(q,a) v_k(\alpha - a/q)$$

and define $V_k(\alpha)$ on $\mathfrak{M}$ by taking $V_k(\alpha)$ to be $V_k(\alpha, q, a)$ on $\mathfrak{M}(\mathfrak{q},\mathfrak{a})$. s, t is always one of the pairs $1,7$; $2,5$; $3,3$

1. Show that

$$R(n;s,t) = \int_0^1 f_2(\alpha)^s f_3(\alpha)^t e(-\alpha n)\, d\alpha.$$

2. Show that (i) $\displaystyle\int_0^1 \left|f_2(\alpha) f_3(\alpha)^6\right| d\alpha \ll n^{\frac{3}{2}+\varepsilon}$,

$$\text{(ii)} \quad \int_0^1 \left|f_2(\alpha)^2 f_3(\alpha)^4\right| d\alpha \ll n^{\frac{4}{3}+\varepsilon}, \quad \text{(iii)} \quad \int_0^1 \left|f_2(\alpha)^3 f_3(\alpha)^2\right| d\alpha \ll n^{\frac{7}{6}+\varepsilon}.$$

Hint: Combine Hölder's inequality and Hua's lemma.

3. Show that there is a positive number δ such that

$$\text{(i)} \quad \int_{\mathfrak{m}} \left|f_2(\alpha) f_3(\alpha)^7\right| d\alpha \ll n^{\frac{11}{6}-\delta}, \quad \text{(ii)} \quad \int_{\mathfrak{m}} \left|f_2(\alpha)^2 f_3(\alpha)^5\right| d\alpha \ll n^{\frac{5}{3}-\delta},$$

$$\text{(iii)} \quad \int_{\mathfrak{m}} \left|f_2(\alpha)^3 f_3(\alpha)^3\right| d\alpha \ll n^{\frac{3}{2}-\delta}.$$

Hint: Dirichlet's theorem on diophantine approximation and Weyl's inequality are relevant.

4. Suppose that $1 \le a \le q \le N_3^{\nu}$, $(q,a) = 1$, $\alpha \in \mathfrak{M}(q,a)$.
(i) Show that $f_k(\alpha) - V_k(\alpha, q, a) \ll n^{\nu}$.
(ii) Show that $f_2(\alpha)^s f_3(\alpha)^t - V_2(\alpha,q,a)^s V_3(\alpha,q,a)^t \ll n^{\frac{s}{2}+\frac{t}{3}-\frac{1}{3}+\nu}$.
Hint: Observe that $f_2^s f_3^t - V_2^s V_3^t = (f_2^s - V_2^s) f_3^t + V_2^s (f_3^t - V_3^t)$.

5. Prove that

$$\text{(i)} \quad \int_{\mathfrak{M}} \left|f_2(\alpha)^s f_3(\alpha)^t - V_2(\alpha)^s V_3(\alpha)^t\right| d\alpha \ll n^{\frac{s}{2}+\frac{t}{3}-1-\delta},$$

$$\text{(ii)} \quad R(n;s,t) = \int_{\mathfrak{M}} V_2(\alpha)^s V_3(\alpha)^t e(-\alpha n)\, d\alpha + O\left(n^{\frac{s}{2}+\frac{t}{3}-1-\delta}\right).$$