MATH 571, SPRING 2025, PROBLEMS 9

Due Monday 31st March

Let

$$A(n) = \sum_{\substack{p_1,p_2,p_3 \le n \\ p_1+p_3=2p_2}} (\log p_1)(\log p_2)(\log p_3).$$

Since we can write $p_3 - p_2 = p_2 = p_1 = d$ the p_j are three successive members of the arithmetic progression $p_1 + xd$. In fact we are counting, with weight $(\log p_1)(\log p_2)(\log p_3)$ all the triples of primes not exceeding n which are in arithmetic progression. Note that we are allowing $d < 0$ and $d = 0$, so each triple with $d \ne 0$ is being counted essentially twice. The terms with $d = 0$ only contribute $\pi(n)$. It is this which Green and Tao famously generalised in 2004 to k primes in a.p. The object of this homework is to show that for any fixed $B \ge 1$ we have

$$A(n) = \frac{1}{2}C_2 n^2 + O_B\big(n^2(\log n)^{-B}\big) \tag{1}$$

where C_2 is the twin prime constant $C_2 = 2\prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right)$. Note that most authors call $C_2' = \prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right)$ the twin prime constant, but then write $\pi_2(x) \sim 2C_2' x(\log x)^{-2}$(!).

It is easily deduced from (1) that $a(n) = \mathrm{card}\{p_1 < p_2 < p_3 \le n : p_1 + p_2 = 2p_3\} \sim \frac{1}{4}C_2 n^2(\log n)^{-3}$.

1. In the notation of Theorem 7.6, show that $\int_{\mathfrak{m}} S(\alpha)^2 S(-2\alpha)d\alpha \ll n^2(\log n)^{(7-A)/2}$.

2. Show that $\int_{\mathfrak{M}} S(\alpha)^2 S(-2\alpha)d\alpha = C_2 J(n) + O\big(n^2(\log n)^{1-A}\big)$ where

$$J(n) = \int_{-(\log n)^A n^{-1}}^{(\log n)^A n^{-1}} T(\beta)^2 T(-2\beta)d\beta.$$

3. We have a problem in that $T(2 \times 1/2) = n$. To get round this, prove that

$$\int_{-1/2}^{1/2} |T(2\beta)|^2 d\beta = \frac{1}{2}\int_{-1}^{1} |T(\beta)|^2 d\beta = \int_{-1/2}^{1/2} |T(\beta)|^2 d\beta = n$$

and $\int_{(\log n)^A n^{-1} \le |\beta| \le 1/2} |T(\beta)^2 T(-2\beta)|d\beta \ll n(\log n)^{-A}\int_{-1/2}^{1/2} |T(\beta)T(-2\beta)|d\beta \ll n^2(\log n)^{-A}$.

4. Prove that $\int_{-1/2}^{1/2} T(\beta)^2 T(-2\beta)d\beta = \mathrm{card}\{n_1, n_3 \le n : 2|n_1 + n_3\} = \frac{1}{2}n^2 + O(1)$ and

$$\int_{\mathfrak{M}} S(\alpha)^2 S(-2\alpha)d\alpha = \frac{1}{2}C_2 n^2 + O\big(n^2(\log n)^{1-A}\big).$$

5. Deduce (1).