**Math 571, Spring 2025, Problems 8**
**Due Monday 24th March**

1. Suppose that $\alpha$ is a real number,$1 < u < \sqrt{X}$, and

$$S(\alpha) = \sum_{n \le X} \mu(n) e(\alpha n).$$

Prove that

$$S(\alpha) = S_1 - S_2 + 2S_3$$

where

$$S_1 = \sum_{m > u} \sum_{u < n \le X/m} a_m \mu(n) e(\alpha m n),\ S_2 = \sum_{m \le u^2} c_m \sum_{n \le X/m} e(\alpha m n),$$

$$S_3 = \sum_{n \le u} \mu(n) e(\alpha n)$$

$$a_m = \sum_{m|n, m \le u} \mu(m),\ c_m = \sum_{k\ell = m, k \le u, \ell \le u} \mu(k)\mu(\ell).$$

2. Suppose that $\alpha$ is a real number, $a \in \mathbb{Z}$, $q \in \mathbb{N}$ with $(a, q) = 1$ and $|\alpha - a/q| \le q^{-2}$. Prove that for every $\varepsilon > 0$

$$S(\alpha) \ll_\varepsilon (Xq^{-1/2} + X^{4/5} + X^{1/2}q^{1/2})X^\varepsilon.$$

You might want to try and refine this to obtain

$$S(\alpha) \ll (Xq^{-1/2} + X^{4/5} + X^{1/2}q^{1/2})(\log X)^C$$

for some positive constant $C$.

From a result of the above kind, Davenport showed in 1937 that for any fixed positive number $A$

$$S(\alpha) \ll X(\log X)^{-A}$$

uniformly for $\alpha \in \mathbb{R}$.