Math 571, Spring 2025, Problems 7
Due Monday 17th March

1. Suppose α, x, $y \in \mathbb{R}$ with $x \ge 1$, $y \ge 1$ and $A \in \mathbb{R}$ is such that for $m \le y$ the complex numbers a_m satisfy $|a_m| \le A$. Define $\|\alpha\| = \min_{n\in\mathbb{Z}} |\alpha - n|$.
(i). Prove the triangle inequality $\|\alpha+\beta\| \le \|\alpha\| + \|\beta\|$.
(ii) Prove that $\displaystyle\sum_{m\le y} a_m \sum_{n\le x/m} e(\alpha m n) \ll A \sum_{m\le y} \min\left(\frac{x}{m}, \frac{1}{\|\alpha m\|}\right)$.

Suppose $q \in \mathbb{N}$, $a \in \mathbb{Z}$ are such that $(q,a) = 1$ and $|\alpha - a/q| \le q^{-2}$. When $1 \le m \le y$ put $m = hq + j$ where $-q/2 < j \le q/2$ so that $0 \le h \le \frac{y}{q} + \frac{1}{2}$, and $j > 0$ if $h = 0$.
(iii) Suppose $h = 0$. Prove that $\|\alpha m\| \ge \|aj/q\| - \frac{1}{2q} \ge \frac{1}{2}\|aj/q\|$ and

$$\sum_{m\le\min(y,q/2)} \min\left(\frac{x}{m}, \frac{1}{\|\alpha m\|}\right) \ll \sum_{1\le j\le\min(y,q/2)} \|aj/q\|^{-1} \ll \sum_{l\le\min(y,q)} \frac{q}{l} \ll q\log 2y.$$

(iv) Suppose $h > 0$. Put $\beta = q^2(\alpha - a/q)$ and let k be a nearest integer to βh. Prove that if $\left\|\frac{ja+k}{q}\right\| \ge \frac{2}{q}$, then $\|\alpha m\| \ge \frac{1}{2}\left\|\frac{ja+k}{q}\right\|$. Deduce that

$$\sum_{hq-q/2<m\le hq+q/2} \min\left(\frac{x}{m}, \frac{1}{\|\alpha m\|}\right) \ll \frac{x}{hq} + \sum_{l=1}^{q-1} \frac{q}{l} \ll \frac{x}{hq} + q\log 2q.$$

(v) Prove that $\displaystyle\sum_{m\le y} a_m \sum_{n\le x/m} e(\alpha m n) \ll \left(\frac{x}{q} + y + q\right) A\log 2x$.

2. Suppose that $a_1, a_2, \ldots, b_1, b_2, \ldots$ are complex numbers and $\alpha \in \mathbb{R}$, and given $M \le X$ write $N = \lfloor X/M \rfloor$ define $T = \displaystyle\sum_{M<m\le 2M} \sum_{n\le X/m} a_m b_n e(\alpha m n)$.

(i) Prove $\displaystyle |T|^2 \le \left(\sum_{M<m\le 2M} |a_m|^2\right) \sum_{n_1\le N} \sum_{n_2\le N} b_{n_1}\overline{b_{n_2}} \sum_{M<m\le\min(2M, \frac{X}{n_1}, \frac{X}{n_2})} e\big(\alpha m(n_1-n_2)\big)$

$\displaystyle \ll \left(\sum_{M<m\le 2M} |a_m|^2 \sum_{n=1}^{N} |b_n|^2\right)\left(M + \sum_{h=1}^{N} \min\left(\frac{X}{h}, \frac{1}{\|\alpha h\|}\right)\right)$.

(ii) Again suppose $(q,a) = 1$ and $|\alpha - a/q| \le q^{-2}$. Prove that

$$|T|^2 \ll \left(\sum_{M<m\le 2M} |a_m|^2 \sum_{n\le X/M} |b_n|^2\right)\left(Xq^{-1} + M + X/M + q\right)\log(2X).$$

The bounds that were obtained in question 1 (iii) & (iv) are useful here.

3. Suppose below that α is a real number, $a \in \mathbb{Z}$, $q \in \mathbb{N}$ with $(a,q)=1$ and $|\alpha - a/q| \le q^{-2}$, $X \ge 2$ and $1 < u < \sqrt{X}$.
(i) Prove that $S = S(\alpha) = \sum_{n \le X} \Lambda(n) e(\alpha n) = S_1 + S_2 - S_3 + S_4$, where

$$S_1 = \sum_{m>u} \sum_{u<n\le X/m} a_m \mu(n) e(\alpha m n), \quad S_2 = \sum_{m \le u} \mu(m) \sum_{n \le X/m} (\log n) e(\alpha m n),$$

$$S_3 = \sum_{m \le u^2} c_m \sum_{n \le X/m} e(\alpha m n), \quad S_4 = \sum_{n \le u} \Lambda(n) e(\alpha n),$$

$$a_m = \sum_{\substack{k|m \\ k>u}} \Lambda(k), \quad c_m = \sum_{k \le u} \sum_{\substack{l \le u \\ kl=m}} \Lambda(k)\mu(l).$$

(ii) Prove that $0 \le a_m \le \log m$ and $|c_m| \le \log m$.
(iii) Let $\mathcal{M} = \{2^j u : 0 \le j,\, 2^j \le X u^{-2}\}$ and write $S_1 = \sum_{M \in \mathcal{M}} T(M)$ where

$$T(M) = \sum_{M<m\le 2M} \sum_{u<n\le X/m} a_m \mu(n) e(\alpha m n).$$

Prove that (question 2 is useful here)

$$\begin{aligned} S_1 &\ll \sum_{M \in \mathcal{M}} \left(M(\log X)^2\right)^{\frac{1}{2}} (X/M)^{\frac{1}{2}} \left(Xq^{-1} + M + X/M + q\right)^{\frac{1}{2}} (\log X)^{1/2} \\ &\ll (Xq^{-1/2} + Xu^{-1/2} + X^{1/2}q^{1/2})(\log X)^{5/2}. \end{aligned}$$

(iv) Prove that $S_2 = \displaystyle\int_1^X \sum_{m \le \min(u, X/v)} \mu(m) \sum_{v<n\le X/m} e(\alpha m n) \frac{dv}{v}$ and hence that

$$S_2 \ll (Xq^{-1} + u + q)(\log X)^2.$$

The results of question 1 are useful here and in the next question.
(v) Prove that $S_3 \ll (Xq^{-1} + u^2 + q)(\log X)^2$.
(vi) Prove that $S \ll (Xq^{-1/2} + X^{4/5} + X^{1/2}q^{1/2})(\log X)^{5/2}$.