MATH 571 ANALYTIC NUMBER THEORY, SPRING 2025, PROBLEMS 3

Due Monday 3rd February

We use $e(\alpha) = \exp(2\pi i\alpha)$.

1. Let f denote a polynomial with integer coefficients and $f(0) = 0$ and for $q \in \mathbb{N}$ define

$$S(q; f) = \sum_{x=1}^{q} e\big(f(x)/q\big),\ W(q; f) = \sum_{\substack{x=1\\(x,q)=1}}^{q} e\big(f(x)/q\big).$$

(i) Suppose that $q_1,\ q_2 \in \mathbb{N}$ with $(q_1, q_2) = 1$. Choose r_1, r_2 so that $r_1q_2 \equiv 1 \pmod{q_1}$ and $r_2q_1 \equiv 1 \pmod{q_2}$. Prove that

$$S(q_1q_2; f) = S(q_1; r_1f)S(q_2; r_2f) \ \&\ W(q_1q_2; f) = W(q_1; r_1f)W(q_2; r_2f)$$

(ii) Note that Ramanujan's sum $c_q(n)$ is W with $f(x) = nx$. Prove that if $q = p^k$ for some prime p and positive integer k, then

$$c_q(n) = \sum_{m|(q,n)} m\mu(q/m).$$

Deduce that this holds for general q.

2. (Hooley (1972), Montgomery & Vaughan (1979)) By lower and upper bound sifting functions we mean functions $\lambda^{\pm} : \mathbb{N} \to \mathbb{R}$ such that

$$\sum_{m|n} \lambda^-_m \le \sum_{m|n} \mu(m) \le \sum_{m|n} \lambda^+_m.$$

(i) Let λ^+_d be such that $\lambda^+_d = 0$ for all $d > z$. Show that for any q,

$$0 \le \frac{\varphi(q)}{q} \sum_{\substack{d\\(d,q)=1}} \frac{\lambda^+_d}{d} \le \sum_{d} \frac{\lambda^+_d}{d}.$$

(Hint: Multiply both sides by $P/\varphi(P) = \sum 1/m$ where m runs over all integers composed of the primes dividing P, and $P = \prod_{p\le z} p$.)

(ii) Let η_d be real with $\eta_d = 0$ for $d > z$. Show that for any q,

$$0 \le \frac{\varphi(q)}{q} \sum_{\substack{d,\,e\\(de,q)=1}} \frac{\eta_d\eta_e}{[d, e]} \le \sum_{d,\,e} \frac{\eta_d\eta_e}{[d, e]}.$$

(iii) Let λ_d^- be a lower bound sifting function such that $\lambda_d^- = 0$ for $d > z$. Show that for any q,

$$\frac{\varphi(q)}{q} \sum_{\substack{d \\ (d,q)=1}} \frac{\lambda_d^-}{d} \geq \sum_d \frac{\lambda_d^-}{d}.$$