MATH 571 ANALYTIC NUMBER THEORY, SPRING 2025, PROBLEMS 2

Due Monday 27th January

1. Let $k \in \mathbb{N}$. Prove that there are infinitely many n such that $\mu(n+1) = \mu(n+2) = \cdots = \mu(n+k)$.

2. Suppose that the arithmetical function $\eta(n)$ satisfies $\sum_{m|n} \eta(m) = \phi(n)$. Show that $\eta(n)$ is multiplicative and evaluate $\eta(p^k)$.

3. This question investigates whether there exists an arithmetic function θ such that $\theta * \theta = \mu$ and $\theta(1) \geq 0$.

(i) Prove that θ exists and is uniquely determined.

(ii) Prove that $\theta(p^k) = (-1)^k \binom{\frac{1}{2}}{k}$. This is the coefficient of z^k in the Taylor expansion of $(1-z)^{1/2}$ centred at 0. It is easily checked that

$$\theta(p^k) = -\frac{(2k)!}{2^{2k}(k!)^2} = -\frac{1}{2^{2k}}\binom{2k}{k}.$$

(iii) By considering the function $\theta_1(n) = \prod_{p^k \| n} \theta(p^k)$, or otherwise, show that $\theta \in \mathcal{M}$.

4. A number $n \in \mathbb{N}$ is *squarefree* when it has no repeated prime factors. For $X \in \mathbb{R}$, $X \geq 1$ let $Q(X)$ denote the number of squarefree numbers not exceeding X.

(i) Prove that

$$Q(X) = \frac{6}{\pi^2}X + O\left(\sqrt{X}\right).$$

(ii) Prove that if $n \in \mathbb{N}$, then

$$Q(n) \geq n - \sum_p \left\lfloor \frac{n}{p^2} \right\rfloor.$$

(iii) Prove that

$$\sum_p \frac{1}{p^2} < \frac{1}{4} + \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} < \frac{1}{4} + \sum_{k=1}^{\infty} \frac{1}{4k(k+1)} = \frac{1}{2}.$$

(iv) Prove that $Q(n) > n/2$ for all $n \in \mathbb{N}$.

(v) Prove that every integer $n > 1$ is a sum of two squarefree numbers.