MATH 571 ANALYTIC NUMBER THEORY, SPRING 2025, PROBLEMS 1

Due Wednesday 22nd January

Throughout this course we will use $e(\alpha)$ to denote $e^{2\pi i\alpha}$.

1. (i) Prove that

$$\frac{1}{q}\sum_{a=1}^{q} e(an/q) = \begin{cases} 1 & \text{when } q|n, \\ 0 & \text{when } q\nmid n. \end{cases}$$

(ii) Let $\sigma(n) = \sum_{m|n} m$ denote the sum of the divisors of n. Prove that $\sigma(n)$ is a multiplicative function.

(iii) Prove that $\sigma(n) = n\sum_{m|n}\frac{1}{m}$.

(iv) (Ramanujan) Prove that

$$\sigma(n) = \frac{\pi^2 n}{6}\sum_{q=1}^{\infty} q^{-2}c_q(n)$$

where $c_q(n)$ denotes Ramanujan's sum $\sum_{\substack{a=1\\(a,q)=1}}^{q} e(an/q)$.

2. Let $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$ with $n \geq 0$, $m \geq 0$. The binomial coefficient $\binom{n}{m}$ is defined inductively by

$$\binom{0}{0} = 1, \quad \binom{n}{-1} = \binom{0}{m} = 0\,(m > 0), \quad \binom{n+1}{m} = \binom{n}{m-1} + \binom{n}{m}.$$

(i) Prove that $\binom{n}{m} \in \mathbb{N}$.

(ii) Prove that if p is a prime and $1 \leq m \leq p-1$, then $p|\binom{p}{m}$.

3. Prove that the number $\rho(n)$ of solutions of the equation

$$x_1 + \ldots + x_k = n$$

in non-negative integers $x_1, \ldots, x_k$ is

$$(-1)^n\binom{-k}{n} = \binom{k+n-1}{n}$$

4. Prove that no polynomial $f(x)$ of degree at least 1 with integral coefficients can be prime for every positive integer x.