Appendix E

Further Topics In Harmonic Analysis

E.1 Quantitative Trigonometric Approximation

Let $I = [\alpha, \beta]$ be an interval of $\mathbb{R}$ with χ_I its characteristic function, and suppose that $\delta > 0$ is given. Our object is to construct functions $S_+(x)$ and $S_-(x)$ such that

$$\widehat{S}_\pm(t) = 0 \text{ when } |t| \ge \delta,$$
$$S_-(x) \le \chi_I(x) \le S_+(x) \text{ for all } x,$$

and such that the integrals

$$\int_{\mathbb{R}} S_+(x) - \chi_I(x)\,dx, \qquad \int_{\mathbb{R}} \chi_I(x) - S_-(x)\,dx$$

are small. We do not attempt to determine exactly the extreme values of these integrals, but the functions constructed are elegant and close to optimal. With S_+ and S_- in hand, we use the Poisson summation formula to derive corresponding trigonometric polynomials $T_\pm$ that approximate closely the characteristic function of an arc of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. These $T_\pm$ are useful in a number of connections. We employ them in discussing uniform distribution (Chapter 16), in deriving the large sieve (Chapter 19), and in determining the mean square behaviour of Dirichlet polynomials (Chapter 21).

We begin by defining *Beurling's function*,

$$B(z) = \Big(\frac{\sin \pi z}{\pi}\Big)^2 \Big(\frac{2}{z} + \sum_{n=0}^{\infty} \frac{1}{(z-n)^2} - \sum_{n=1}^{\infty} \frac{1}{(z+n)^2}\Big), \tag{E.1}$$

whose basic properties are as follows.

Theorem E.1 *The function $B(z)$ above is an entire function such that*:
- (a) $B(n) = 1$ *for all integers* $n \ge 0$, $B(n) = -1$ *for all integers* $n < 0$;
- (b) $B'(n) = 0$ *for all integers* $n \ne 0$, $B'(0) = 2$;
- (c) $B(x) \ge \operatorname{sgn}(x)$ *for all real* x;
- (d) $B(x) - \operatorname{sgn}(x) \ll \min(1, x^{-2})$;

(e) $B'(x) \ll \min(1, x^{-2})$;
(f) $B(z) - \operatorname{sgn}(x) \ll |z|^{-2} e^{2\pi|y|}$ *where* $z = x + iy$;
(g) $\int_{-\infty}^{\infty} B(x) - \operatorname{sgn}(x)\, dx = 1$.

An entire function $f(z)$ belongs to the class E^σ of *functions of exponential type* σ if for every constant $\varepsilon > 0$ the inequality $|f(z)| < \exp((\sigma + \varepsilon)|z|)$ holds for all z with $|z|$ large. Thus we see that $B(x) \in E^{2\pi}$. Other examples of functions of exponential type are provided by observing that if $f \in L^1([-c, c])$, then its Fourier transform

$$\widehat{f}(z) = \int_{-c}^{c} f(u) e^{-2\pi i z u}\, du$$

is an entire function of the class $E^{2\pi c}$. In the case of $B(z)$, we note that $B \notin L^1(\mathbb{R})$, and also that there is no $f \in L^1(\mathbb{R})$ of which $B(z)$ is the Fourier transform (since $B(x) \not\to 0$ as $x \to \infty$). Nevertheless, the estimate (f) above may be thought of as asserting that $\operatorname{supp} \widehat{B} \subseteq [-1, 1]$.

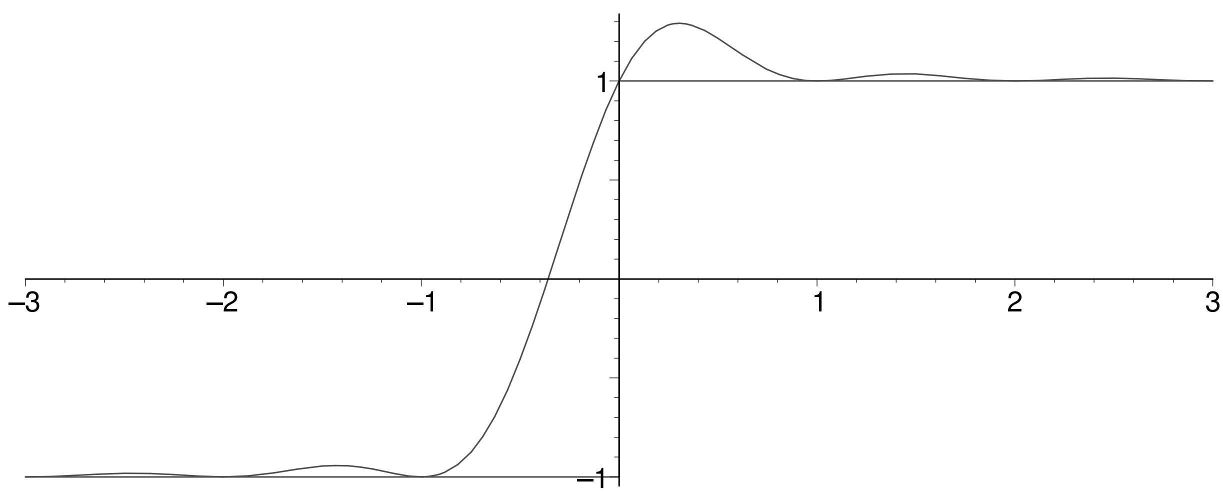

Figure E.1 Graph of Beurling's function $B(x)$ for $-3 \le x \le 3$.

Proof We begin by establishing further formulae for $B(z)$. We recall the partial fraction formula

$$\Big(\frac{\pi}{\sin \pi z}\Big)^2 = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}.$$

(This may be proved by noting that the difference between the two sides is a bounded entire function that tends to 0 as $z \to i\infty$.) On combining this with (E.1) we find that

$$B(z) = 1 + 2\Big(\frac{\sin \pi z}{\pi}\Big)^2 \Big(\frac{1}{z} - \sum_{n=1}^{\infty} \frac{1}{(z+n)^2}\Big). \tag{E.2}$$

Suppose that $z \notin (-\infty, 0]$. The integral test suggessts that the sum above is approximately

$$\int_0^\infty (u+z)^{-2}\, du = \frac{1}{z}\,.$$

Hence the second factor on the right hand side is the difference between this approximation and the sum. To express this quantity more explicitly, we observe that if f has continuous first derivative on an interval $[\alpha, \beta]$, then

$$\int_\alpha^\beta f(u)\,du = f(\beta)(\beta-\alpha) - \int_\alpha^\beta f'(u)(u-\alpha)\,du$$

by integration by parts. By taking $\alpha = n-1$, $\beta = n$, $f(u) = (u+z)^{-2}$, it follows that

$$\int_{n-1}^{n} (u+z)^{-2}\,du = (z+n)^{-2} + 2\int_{n-1}^{n} (z+u)^{-3}\{u\}\,du$$

provided that $z \notin [-n, -n+1]$. If $z \notin (-\infty, 0]$, then we may sum over $n = 1, 2, \ldots$, and thus we deduce from (E.2) that

$$B(z) = 1 + 4\Big(\frac{\sin \pi z}{\pi}\Big)^2 \int_0^\infty \frac{\{u\}}{(u+z)^3}\,du. \tag{E.3}$$

Similarly from (E.1) and (E.2) we find that

$$B(z) = -1 + 2\Big(\frac{\sin \pi z}{\pi}\Big)^2 \Big(\frac{1}{z} + \sum_{n=0}^{\infty} \frac{1}{(z-n)^2}\Big), \tag{E.4}$$

and that if $z \notin [0, \infty)$, then

$$B(z) = -1 + 4\Big(\frac{\sin \pi z}{\pi}\Big)^2 \int_0^\infty \frac{1-\{u\}}{(u-z)^3}\,du. \tag{E.5}$$

The assertions (a) and (b) are immediate from the definition (E.1) of $B(z)$. For $x \ge 0$ the inequality (c) and the estimate (d) follow from (E.3), since the value of the integral lies between 0 and $\frac{1}{2}x^{-2}$. For $x < 0$ these assertions follow similarly from (E.5). To obtain the estimate (e) it suffices to differentiate the formulae (E.3), (E.5), and then estimate the quantities that arise. As for (f), we note that $(\sin \pi z)^2 \ll e^{2\pi|y|}$, and that if $\Re z \ge 0$, then

$$\int_0^\infty \frac{\{u\}}{(u+z)^3}\,du \ll \int_0^\infty \frac{du}{(u+|z|)^3} \ll |z|^{-2}.$$

Thus we obtain (f) from (E.3) when $\Re z \ge 0$, and similarly from (E.5) when $\Re z < 0$. As for (g), let

$$V(z) = \Big(\frac{\sin \pi z}{\pi}\Big)^2 \Big(\frac{2}{z} + \sum_{n=-\infty}^{\infty} \frac{\operatorname{sgn}(n)}{(z-n)^2}\Big), \tag{E.6}$$

so that $B(z) = V(z) + (\sin \pi z)^2/(\pi z)^2$. Since $V(x)$ and $\operatorname{sgn}(x)$ are odd functions, we know that

$$\int_{-X}^{X} V(x) - \operatorname{sgn}(x)\,dx = 0$$

for any X. Hence

$$\begin{aligned}\int_{-\infty}^{\infty} B(x) - \operatorname{sgn}(x)\,dx &= \lim_{X\to\infty}\int_{-X}^{X} B(x) - \operatorname{sgn}(x)\,dx \\ &= \lim_{X\to\infty}\int_{-X}^{X} V(x) - \operatorname{sgn}(x) + (\sin \pi x)^2/(\pi x)^2\,dx \\ &= \lim_{X\to\infty}\int_{-X}^{X} \Big(\frac{\sin \pi x}{\pi x}\Big)^2\,dx \\ &= \int_{-\infty}^{\infty} \Big(\frac{\sin \pi x}{\pi x}\Big)^2\,dx = 1.\end{aligned}$$

The evaluation of this last definite integral can be accomplished by means of the calculus of residues. □

Although the proof is now complete, it is instructive to note that (c) can be derived from (E.2) and (E.4) by appealing to the integral test. For example, if $x > 0$, then

$$\sum_{n=1}^{\infty}\frac{1}{(x+n)^2} < \int_0^{\infty}\frac{du}{(x+u)^2} = \frac{1}{x}.$$

We now use the function $B(z)$ to construct approximations to the characteristic function χ_I of an interval $[\alpha, \beta]$.

Theorem E.2 *Let $I = [\alpha, \beta]$ be a finite interval, and suppose that $\delta > 0$ is given. Then there exist entire functions $S_+(z)$ and $S_-(z)$ such that:*
(a) $S_\pm(x) \ll_{\alpha,\beta,\delta} \min(1, x^{-2})$ *for real* x;
(b) $S_-(x) \le \chi_I(x) \le S_+(x)$ *for real* x;
(c) $\int_{-\infty}^{\infty} S_\pm(x)\,dx = \beta - \alpha \pm 1/\delta$;
(d) $\widehat{S}_\pm(t) = 0$ *when* $|t| \ge \delta$;
(e) $S_\pm(x)$ *is of bounded variation on* $\mathbb{R}$.

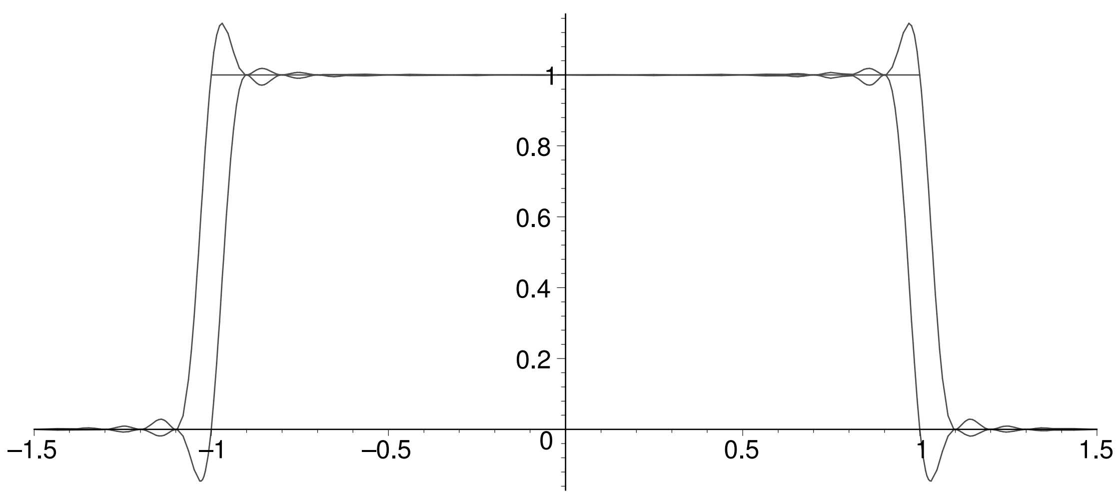

Figure E.2 Graph of $\chi_I(x)$ and Selberg's functions $S_\pm(x)$ for $I = [-1, 1]$ and $\delta = 10$.

Proof We take

$$S_+(z) = \frac{1}{2}B(\delta(z-\alpha)) + \frac{1}{2}B(\delta(\beta - z)),$$
$$S_-(z) = -\frac{1}{2}B(\delta(\alpha - z)) - \frac{1}{2}B(\delta(z-\beta));$$

these are the *Selberg functions*. Then the assertion (a) follows immediately from Theorem E.1(d). To obtain the inequalities (b) we note that

$$S_+(x) \ge \frac{1}{2}\operatorname{sgn}(\delta(x-\alpha)) + \frac{1}{2}\operatorname{sgn}(\delta(\beta - x))$$

by Theorem E.1(c). Here the right hand side is $\chi_I(x)$ unless $x = \alpha$ or $x = \beta$. If $\alpha < \beta$, then we may conclude that $S_+(\alpha) \ge 1$, $S_+(\beta) \ge 1$, because S_+ is continuous. If $\alpha = \beta$, then $S_+(\alpha) = 1$ because $B(0) = 1$. Similarly we see that $S_-(x) \le \chi_I(x)$ for all x. As for (c), we note that

$$\begin{aligned}\int_{-\infty}^{\infty} S_+(x)\,dx &= \int_{-\infty}^{\infty} \chi_I(x)\,dx + \int_{-\infty}^{\infty} S_+(x) - \chi_I(x)\,dx \\ &= \beta - \alpha + \frac{1}{2}\int_{-\infty}^{\infty} B(\delta(x-\alpha)) - \operatorname{sgn}(\delta(x-\alpha))\,dx \\ &\qquad + \frac{1}{2}\int_{-\infty}^{\infty} B(\delta(\beta - x)) - \operatorname{sgn}(\delta(\beta - x))\,dx \\ &= \beta - \alpha + 1/\delta,\end{aligned}$$

by Theorem E.1(g), and similarly for S_-. Since the functions $S_\pm$ are in $L^1(\mathbb{R})$, we can define their Fourier transforms,

$$\widehat{S}_\pm(t) = \int_{-\infty}^{\infty} S_\pm(x)e(-tx)\,dx.$$

Here $S_\pm(z)e^{-2\pi i t z}$ is an entire function, and if $t \ge \delta$, then by Theorem E.1(f) we see that this function is $\ll_{\alpha,\beta,\delta} |z|^{-2}$ in the lower half-plane $\Im z \le 0$. We consider the integral above to be a contour integral in the complex plane, and on replacing this path by a semicircle in the lower half-plane we conclude that $\widehat{S}_\pm(t) = 0$ if $t \ge \delta$. Similarly $S_\pm(t) = 0$ if $t \le -\delta$, so we have (d). Finally, from Theorem E.1(e) we see that $B(x)$ is of bounded variation on $\mathbb{R}$, and hence the same is true of $S_\pm$. □

We now derive analogous results for approximations in $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ by trigonometric polynomials.

Theorem E.3 *For any arc $I = [\alpha, \beta]$ in $\mathbb{T}$ with length $\beta - \alpha < 1$, and for any positive integer K, there are trigonometric polynomials*

$$T_\pm(x) = \sum_{k=-N}^{N} \widehat{T}_\pm(k)e(kx) \tag{E.7}$$

of degree at most N such that:

(a) $T_-(x) \le \chi_I(x) \le T_+(x)$ *for all* x;

(b) $\int_0^1 T_\pm(x)\,dx = \beta - \alpha \pm 1/(N+1)$.

Proof. Take $\delta = N+1$, and let $S_\pm$ be the functions described in Theorem E.2. Put

$$T_\pm(x) = \sum_n S_\pm(x+n).$$

From Theorem E.2(a) we see that this series is uniformly convergent for x in a compact set, so that $T_\pm(x)$ is continuous. The inequalities (a) follow from Theorem E.2(b). From Theorem E.2(a),(e) we see that the Poisson summation formula, in the form given in Theorem D.3, applies to $S_\pm$. Thus

$$T_\pm(x) = \lim_{K\to\infty} \sum_{k=-K}^{K} \widehat{S}_\pm(k) e(kx).$$

But $\widehat{S}_\pm(k) = 0$ for $|k| \ge \delta = N+1$, and $\widehat{T}_\pm(k) = \widehat{S}_\pm(k)$ for all k, so we find that $T_\pm$ is a trigonometric polynomial, as in (E.7). Finally, the integral in (b) is

$$\widehat{T}_\pm(0) = \widehat{S}_\pm(0) = \int_{-\infty}^{\infty} S_\pm(x)\,dx,$$

and the stated result follows from Theorem E.2(c). □

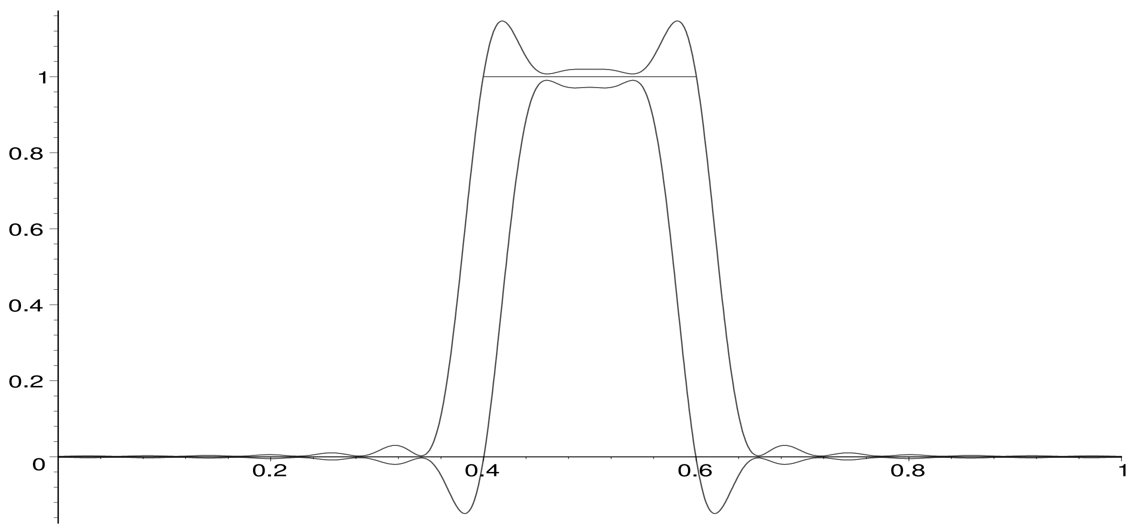

Figure E.3 Graph of $\chi_I(x)$ and $T_\pm(x)$ for $I = [2/5, 3/5]$ with $K = 16$.

In the above situation, the interval I is short, δ is large, the graphs of $S_\pm(x)$ are repeated with period 1, and the $\widehat{S}_\pm(k)$ become Fourier coefficients. With an alternative application of the Poisson Summation Formula we reverse this, so that I is long, δ is small, the graphs of $\widehat{S}_\pm$ are repeated with period 1, and the $S_\pm$ are Fourier coefficients.

Theorem E.4 *Let M and N be integers, $N \ge 1$. Suppose that $0 < \delta \le 1/2$. There exist functions $W_\pm(x)$ with period 1 and absolutely convergent Fourier expansions $W_\pm(x) = \sum_n w_\pm(n)e(nx)$ such that*
(a) $w_-(n) \le \chi_{[M+1,M+N]}(n) \le w_+(n)$ *for all integers n;*
(b) $W_\pm(x) = 0$ *if* $\|x\| \ge \delta$;
(c) $\sum_n w_\pm(n) = W_\pm(0) = N - 1 \pm 1/\delta$.

Proof Let $S_\pm(u)$ be the Selberg functions for the interval $I = [M+1, M+N]$, and set $w_\pm(u) = S_\pm(u)$. Thus we have (a). We apply the Poisson Summation Formula to $f(u) = S_\pm(u)e(ux)$. Hence by Theorem D.3 we see that

$$\sum_{n=-\infty}^{\infty} w_\pm(n)e(nx) = \sum_{k=-\infty}^{\infty} \widehat{S}_\pm(k-x),$$

and then properties (b) and (c) are immediate. □

E.1.1 Exercises

1. Suppose that $I = [\alpha, \beta]$ is an interval on the real line, put $K = (\beta - \alpha)/\delta$, and suppose that K is a positive integer. Suppose that $f \in L^1(\mathbb{R})$, that $f(x) \ge \chi_I(x)$ for all x, that f has bounded variation on $\mathbb{R}$, and that $\widehat{f}(t) = 0$ when $|t| \ge \delta$.
(a) Show that
$$\sum_{n=-\infty}^{\infty} f(n/\delta + x) = \delta \widehat{f}(0)$$
for all x.
(b) Show that x can be chosen so that $n/\delta + x \in I$ for $K+1$ values of n.
(c) Deduce that
$$\int_{-\infty}^{\infty} f(u)\,du \ge \beta - \alpha + 1/\delta\,.$$
That is, the function S_+ described in Theorem E.2 is optimal when $(\beta-\alpha)/\delta$ is an integer.

2. Prove the following identities:

(a)
$$\Big(\frac{\sin \pi x}{\pi x}\Big)^2 = \int_{-1}^{1} (1 - |t|)e(tx)\,dt;$$

(b)
$$\frac{(\sin \pi x)^2}{x} = \pi \int_0^1 \sin 2\pi tx\,dt;$$

(c)
$$\sum_{n=-N}^{N} \operatorname{sgn}(n)e(-nt) = -i \cot \pi t + i\frac{\cos \pi(2N+1)t}{\sin \pi t};$$

(d)
$$\operatorname{sgn}(x) = \frac{2}{\pi}\int_0^\infty \frac{1}{t}\sin 2\pi tx\,dt.$$

3. Let $V(z)$ be *Vaaler's function* as defined in (E.6), and put

$$V_N(z) = \Big(\frac{\sin \pi z}{\pi}\Big)^2 \Big(\frac{2}{z} + \sum_{-N}^{N} \frac{\operatorname{sgn}(n)}{(z-n)^2}\Big).$$

(a) Using the identities in Exercise 2, or otherwise, show that

$$\begin{aligned} V_N(x) = 2 &\int_0^1 \big((1-t)\cot \pi t + 1/\pi\big) \sin 2\pi t x \, dt \\ &- 2\int_0^1 \frac{\cos \pi(2N+1)t}{\sin \pi t}(1-t)\sin 2\pi t x\, dt. \end{aligned}$$

(b) By using the Riemann–Lebesgue lemma, show that

$$V(x) = 2\int_0^1 \big((1-t)\cot \pi t + \frac{1}{\pi}\big) \sin 2\pi t x\, dt.$$

(c) Let

$$\varphi(t) = \begin{cases} 1 & \text{if } t = 0, \\ \pi(1-|t|)t\cot \pi t + |t| & \text{if } 0 < |t| \le 1, \\ 0 & \text{if } |t| > 1. \end{cases}$$

Show that

$$V'(x) = 2\int_{-1}^{1} \varphi(t) e(xt)\, dt.$$

(d) Show that $\varphi(t)$ is non-negative, continuously differentiable on $\mathbb{R}$ and that is is strictly decreasing on $[0, 1]$.
(e) Show that $V(z)$ is an odd entire function, and that

$$V(z) = 1 - 6\Big(\frac{\sin \pi z}{\pi}\Big)^2 \int_0^\infty \frac{\{u\}(1-\{u\})}{(z+u)^4}\, du$$

provided that $z \notin (-\infty, 0]$.
(f) Show that $V(n) = \operatorname{sgn}(n)$ for all integers n, that $V'(n) = 0$ for all integers $n \ne 0$, that $V'(0) = 2$, and that $0 \le V(x) \le 1$ for $x > 0$.
(g) Show that if $x > 0$, then

$$\begin{aligned} V(x) - 1 &\ll \min(1, x^{-3}), \\ V'(x) &\ll \min(1, x^{-3}). \end{aligned}$$

(h) Show that all zeros of $V'(x)$ lie on the real axis.
(i) Show that

$$V(x) - \operatorname{sgn}(x) = \int_{-\infty}^{\infty} \frac{\varphi(t) - 1}{\pi i t} e(tx)\, dt.$$

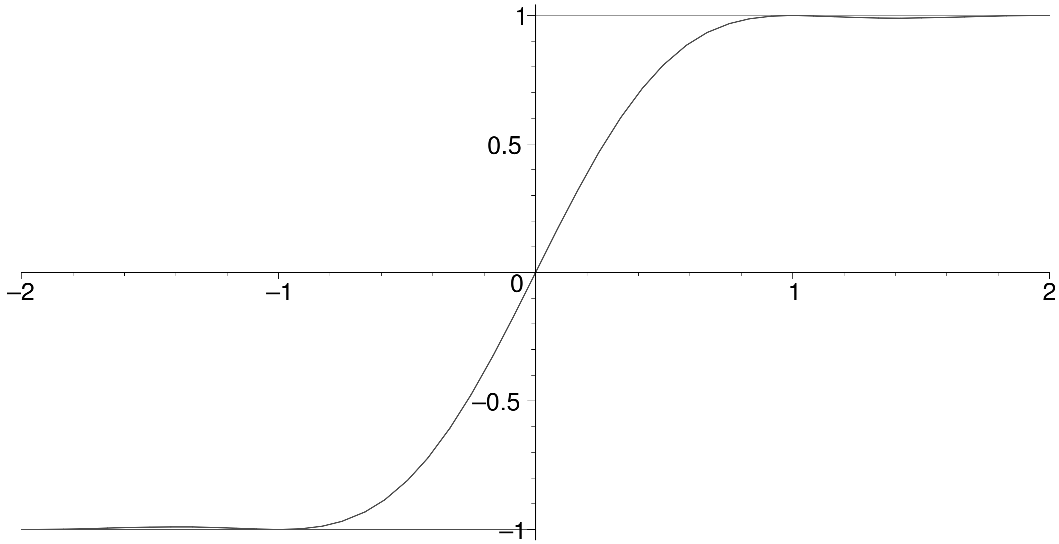

Figure E.4 Graph of Vaaler's function $V(x)$ for $-2 \le x \le 2$.

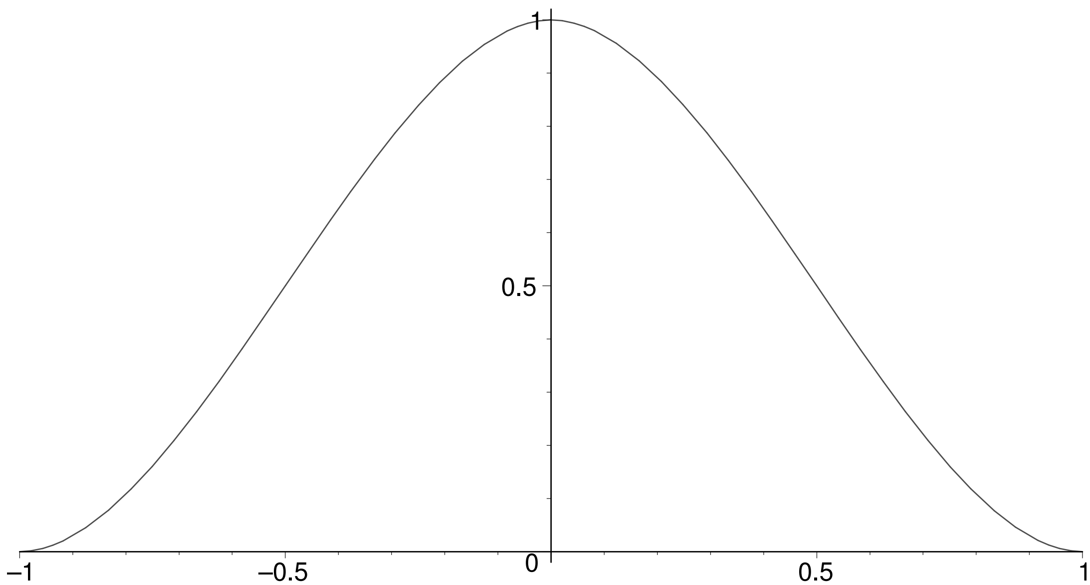

Figure E.5 Graph of $\varphi(t)$ for $-1 \le t \le 1$.

4. Let

$$P(x) = \frac{K+1}{2} \sum_{n=-\infty}^{\infty} V'((K+1)(n+x)),$$

$$Q(x) = \frac{1}{2} \sum_{n=-\infty}^{\infty} V((K+1)(n+x)) - \operatorname{sgn}(n+x),$$

$$R(x) = Q(x) - \{x\} + 1/2.$$

(a) Show that $P(x)$ is a trigonometric polynomial of degree K, with coefficients $\widehat{P}(k) = \varphi(k/(K+1))$.

(b) Show that $Q(x)$ has Fourier coefficients

$$\widehat{Q}(k) = \frac{\varphi(\frac{k}{K+1}) - 1}{2\pi i k}$$

for $k \neq 0$, and that $\widehat{Q}(0) = 0$.
(c) Show that $R(x)$ is a trigonometric polynomial of degree K with coefficients

$$\widehat{R}(k) = \frac{\varphi(\frac{k}{K+1})}{2\pi i k}$$

for $k \neq 0$, $\widehat{R}(0) = 0$, and that $R'(x) = P(x) - 1$.
(d) Show that for all x,

$$R(x) - \frac{\Delta_{K+1}(x)}{2(K+1)} \le 1/2 - \{x\} \le R(x) + \frac{\Delta_{K+1}(x)}{2(K+1)}.$$

5. Let $P(x)$ and $Q(x)$ be as above. Suppose that f is of bounded variation on $\mathbb{T}$.
(a) Show that if f is continuous at x, then

$$f(x) = \int_0^1 f(x+u)P(u)\,du + \int_0^1 Q(u)\,df(x+u).$$

(b) Suppose that f is a real-valued function of bounded variation on $\mathbb{T}$. Show that

$$-\int_0^1 \frac{\Delta_{K+1}(x-u)}{2(K+1)}\,|df(u)| \le f(x) - \int_0^1 f(x+u)P(u)\,du \le \int_0^1 \frac{\Delta_{K+1}(x-u)}{2(K+1)}\,|df(u)|.$$

for all x.
(c) Show that $\int_0^1 f(x+u)P(u)\,du$ is a trigonometric polynomial of degree at most K with coefficients $\varphi(k/(K+1))\widehat{f}(k)$.
(d) Show that $\int_0^1 \Delta_{K+1}(x-u)\,|df(u)|$ is a trigonometric polynomial of degree at most K with coefficients

$$\frac{1 - \frac{|k|}{K+1}}{2(K+1)} \int_0^1 e(-ku)\,|df(u)|.$$

(e) Let

$$T_\pm(x) = \int_0^1 f(x+u)P(u)\,du \pm \int_0^1 \frac{\Delta_{K+1}(x-u)}{2(K+1)}\,|df(u)|.$$

Show that $T_\pm$ is a trigonometric polynomial of degree at most K such that $T_-(x) \le f(x) \le T_+(x)$ for all x, and that

$$\int_0^1 T_\pm(u)\,du = \int_0^1 f(u)\,du \pm \frac{\mathrm{Var}_{\mathbb{T}}(f)}{2(K+1)}.$$

(f) Show that if $f = \chi_{[\alpha,\beta]}$, then the $T_\pm$ above are the same as in Theorem E.3, and hence that the trigonometric polynomials in that theorem have coefficients

$$\widehat{T}_\pm(k) = \Big(\varphi(\frac{k}{K+1})\frac{\sin \pi k(\beta-\alpha)}{\pi k} \pm \frac{1-\frac{|k|}{K+1}}{K+1}\cos \pi k(\beta-\alpha)\Big)e(-k(\beta+\alpha)/2)$$

for $0 < |k| \le K$, $\widehat{T}_\pm(0) = \beta - \alpha \pm 1/(K+1)$.

6. (a) Suppose that $T(x)$ is a trigonometric polynomial of degree at most K, and that $N > K$. Show that for any real α,

$$\frac{1}{N}\sum_{n=1}^{N} T(\alpha + n/N) = \int_0^1 T(x)\,dx.$$

(b) Suppose that $I = [\alpha, \beta]$ is an arc of $\mathbb{T}$, and that $(\beta-\alpha)N$ is an integer $< N$. Show that if a function $T \in L^1(\mathbb{T})$ has the property that $T(x) \ge \chi_I(x)$ for all $x \in \mathbb{T}$, then

$$\sum_{n=1}^{N} T(\alpha + n/N) \ge (\beta - \alpha)N + 1.$$

(c) Suppose that $T(x)$ is a trigonometric polynomial of degree at most K, that $(\beta-\alpha)(K+1)$ is an integer $< K+1$, and that $T(x) \ge \chi_{[\alpha,\beta]}(x)$ for all $x \in \mathbb{T}$. Show that $\int_0^1 T(x)\,dx \ge \beta - \alpha + 1/(K+1)$.
(d) Suppose that $T(x)$ is a trigonometric polynomial of degree at most K, that $(\beta-\alpha)(K+1)$ is an integer $< K+1$, and that $T(x) \le \chi_{[\alpha,\beta]}(x)$ for all $x \in \mathbb{T}$. Show that $\int_0^1 T(x)\,dx \le \beta - \alpha - 1/(K+1)$.

E.2 Maximal inequalities

We develop a technique that allows us to truncate a given sum. The penalty we pay for this is generally a factor of one logarithm. This method applies to additive characters (the finite Fourier transform), the Fourier expansion of periodic functions, and to Dirichlet polynomials. We begin with additive characters.

Let f be an arithmetic function with period q, and let $\widehat{f}$ denote its finite Fourier transform,

$$\widehat{f}(k) = \sum_{n=1}^{q} f(n)e(-nk/q)\,.$$

We have the finite Fourier expansion

$$f(n) = \sum_{k=1}^{q} \widehat{f}(k)e(kn/q),$$

as in (4.3). Let M and N be integers, with $1 \le N \le q$. Thus

$$\sum_{n=M+1}^{M+N} f(n)e(an/q) = \sum_{n=M+1}^{M+N} e(an/q) \sum_{k=1}^{q} \widehat{f}(k)e(nk/q)$$
$$= \sum_{k=1}^{q} \widehat{f}(k) \sum_{n=M+1}^{M+N} e(n(a+k)/q)\,.$$

By (16.5), the inner sum is $\ll \min(N, 1/\|(a+k)/q\|)$, so the above is

$$\ll \sum_{k=1}^{q} |\widehat{f}(k)| \min\big(N, 1/\|(a+k)/q\|\big)\,.$$

Here the right hand side is independent of M, and can be made independent of N by replacing N by q. That is, we have shown that

$$\max_{\substack{1\le M\le q\\ 1\le N\le q}} \Big|\sum_{n=M+1}^{M+N} f(n)e(an/q)\Big| \ll \sum_{k=1}^{q} |\widehat{f}(k)| \min\big(q, 1/\|(a+k)/q\|\big)\,. \tag{E.8}$$

If we take $a = 0$ and $f(n) = \chi(n)$ where χ is a primitive character modulo q, then the estimate above becomes the key step in our proof of the Pólya–Vinogradov inequality (cf Theorem 9.18).

The estimate (E.8) may be used in several ways. If we take $|\widehat{f}(k)|$ out at its maximum, then we are left with a harmonic sum, so

$$\sum_{n=M+1}^{M+N} f(n)e(an/q) \ll q(\log 2q) \max_k |\widehat{f}(k)| \tag{E.9}$$

By taking $M = 0$, $N = q$, and a suitably, the left hand side can be made as large as $q \max |\widehat{f}(k)|$, so the above is within a factor $\log 2q$ of being best-possible. Alternatively, we can sum both sides of (E.8) over a to see that

$$\sum_{a=1}^{q} \max_{\substack{1\le M\le q\\ 1\le N\le q}} \Big|\sum_{n=M+1}^{M+N} f(n)e(an/q)\Big| \ll q(\log 2q) \sum_{k=1}^{q} |\widehat{f}(k)|\,. \tag{E.10}$$

Moreover, by Cauchy's inequality we see that

$$\sum_{k=1}^{q} |\widehat{f}(k)| \min\big(q, 1/\|(a+k)/q\|\big) \le \Big(\sum_{k=1}^{q} |\widehat{f}(k)|^2 \min\big(q, 1/\|(a+k)/q\|\big)\Big)^{1/2}$$
$$\times \Big(\sum_{k=1}^{q} \min\big(q, 1/\|(a+k)/q\|\big)\Big)^{1/2}$$
$$\ll (q\log 2q)^{1/2} \Big(\sum_{k=1}^{q} |\widehat{f}(k)|^2 \min\big(q, 1/\|(a+k)/q\|\big)\Big)^{1/2}.$$

Thus by squaring both sides of (E.8), and then summing over a, we find that

$$\sum_{a=1}^{q} \max_{\substack{1\le M\le q\\ 1\le N\le q}} \Big| \sum_{n=M+1}^{M+N} f(n)e(an/q)\Big|^2 \ll (q\log 2q)^2 \sum_{k=1}^{q} |\widehat{f}(k)|^2 . \tag{E.11}$$

If we take $M = 0$ and $N = q$, then the left hand side becomes $q^2 \sum |\widehat{f}(k)|^2$, so the upper bound is never larger than the truth by more than a factor of $(\log 2q)^2$.

For $f \in L^1(\mathbb{T})$ we recall the definition of the Fourier coefficient $\widehat{f}(k)$, the Dirichlet kernel $D_K(x)$, and its closed-form formula, as in (D.1)–(D.4). We observe that $D_K(x) \ll \min(K, 1/\|x\|)$. Thus

$$\begin{aligned}\sum_{k=-K}^{K} \widehat{f}(k)e(k\alpha) &= \int_{\mathbb{T}} D_k(x)f(\alpha+x)\,dx\\ &\ll \int_{\mathbb{T}} |f(x+\alpha)|\min(N, 1/\|x\|)\,dx \end{aligned}\tag{E.12}$$

if $K \le N$ and $N \ge 1$. In the same way that we deduced (E.9), (E.10), and (E.11) from (E.8), we deduce that

$$\max_{0\le K\le N} \Big| \sum_{k=-K}^{K} \widehat{f}(k)e(k\alpha)\Big| \ll \|f\|_{L^\infty(\mathbb{T})} \log 2N, \tag{E.13}$$

$$\int_{\mathbb{T}} \max_{0\le K\le N} \Big| \sum_{k=-K}^{K} \widehat{f}(k)e(k\alpha)\Big|\, d\alpha \ll \|f\|_{L^1(\mathbb{T})} \log 2N, \tag{E.14}$$

and

$$\int_{\mathbb{T}} \max_{0\le K\le N} \Big| \sum_{k=-K}^{K} \widehat{f}(k)e(k\alpha)\Big|^2 d\alpha \ll \|f\|_{L^2(\mathbb{T})}^2 (\log 2N)^2, \tag{E.15}$$

Finally, we consider truncations of a Dirichlet polynomial

$$D(s) = \sum_{n=1}^{N} a_n n^{-s} . \tag{E.15}$$

Our discussion here has some points of contact with our treatment of the quantitative truncation of Perron's formula (cf Theorem 5.2). We begin by observing that

$$\begin{aligned}\int_{-U}^{U} e^{i\beta u}\frac{\sin\alpha u}{u}\,du &= \int_{-U}^{U} \frac{\cos\beta u\,\sin\alpha u}{u}\,du\\ &= \frac{1}{2}\int_{-U}^{U} \frac{\sin(\alpha+\beta)u + \sin(\alpha-\beta)u}{u}\,du\\ &= \operatorname{sgn}(\alpha+\beta)\int_0^{|\alpha+\beta|U} \frac{\sin u}{u}\,du + \operatorname{sgn}(\alpha-\beta)\int_0^{|\alpha-\beta|U}\frac{\sin u}{u}\,du\,. \end{aligned}\tag{E.16}$$

We recall that $\int_0^\infty \frac{\sin u}{u}\,du = \pi/2$, and that the *sine integral* $\mathrm{si}(x)$ is defined to be

$$\mathrm{si}(x) = -\int_x^\infty \frac{\sin u}{u}\,du\,.$$

Thus the expression (E.15) is

$$\mathrm{sgn}(\alpha+\beta)\Big(\frac{\pi}{2} + \mathrm{si}(|\alpha+\beta|U)\Big) + \mathrm{sgn}(\alpha-\beta)\Big(\frac{\pi}{2} + \mathrm{si}(|\alpha-\beta|U)\Big)\,.$$

Let χ_I denote the characteristic function of the interval $[-\alpha,\alpha]$, and recall that it is evident that $\mathrm{si}(x) \ll \min(1, 1/x)$ for $x \ge 0$, as was recorded already in (5.6). Thus the above is

$$= \pi\chi_I(\beta) + O\Big(\min\Big(1, \frac{1}{U|\alpha-\beta|}\Big)\Big) + \Big(\min\Big(1, \frac{1}{U|\alpha+\beta|}\Big)\Big)\,.$$

For integers K, $0 \le K < N$, we take $\alpha = \log(K+1/2)$, $\beta = -\log n$, multiply by a_n, and sum over n. Thus we find that

$$\sum_{n=1}^K a_n = \int_{-U}^U D(iu)\frac{\sin\alpha u}{u}\,du + O\Big(\sum_{n=1}^N |a_n| \min\Big(1, \frac{1}{U|\log n/(K+1/2)|}\Big)\Big)\,.$$

Now $(\sin\alpha u)/u \ll \min(|\alpha|, 1/u)$, and $|\log n/(K+1/2)| \gg 1/N$. Hence

$$\max_{y\le N}\Big|\sum_{n\le y} a_n\Big| \ll \int_{-U}^U |D(iu)| \min(\log N, 1/|u|)\,du + \frac{N}{U}\sum_{n=1}^N |a_n|\,. \tag{E.17}$$

By replacing a_n by $a_n n^{-it}$ it follows that

$$\max_{y\le N}\Big|\sum_{n\le y} a_n n^{-it}\Big| \ll \int_{-U}^U |D(it+iu)| \min(\log N, 1/|u|)\,du + \frac{N}{U}\sum_{n=1}^N |a_n|\,. \tag{E.18}$$

By Cauchy's inequality we see that

$$\begin{aligned}\Big(\int_{-U}^U |D(it+iu)| \min(\log N, 1/|u|)\,du\Big)^2 \\ &\le \int_{-U}^U |D(it+iu)|^2 \min(\log N, 1/|u|)\,du \int_{-U}^U \min(\log N, 1/|u|)\,du \\ &\ll \int_{-U}^U |D(it+iu)|^2 \min(\log N, 1/|u|)\,du \log(U\log N)\,.\end{aligned}$$

Thus if we square both sides of (E.18) and integrate with respect to t, we find that

$$\begin{aligned}\int_0^T \max_{y\le N}\Big|\sum_{n\le y} a_n n^{-it}\Big|^2 dt &\ll \int_{-U}^U \int_0^T |D(it+iu)|^2\,dt\,\min(\log N, 1/|u|)\,du \log(U\log N) \\ &\quad + \frac{N^3T}{U^2}\sum_{n=1}^N |a_n|^2\,.\end{aligned} \tag{E.19}$$