Appendix D

Topics In Harmonic Analysis

1. Pointwise convergence of Fourier series

Let $f \in L^1(\mathbb{T})$, and suppose that

$$\widehat{f}(k) = \int_{\mathbb{T}} f(x)e(-kx)\,dx \tag{1}$$

are the Fourier coefficients of f. Here $e(\theta) = e^{2\pi i\theta}$ is the complex exponential with period 1. It is a familiar fact in the theory of Fourier series that if f has bounded variation on $\mathbb{T}$ then

$$\lim_{K\to\infty} \sum_{k=-K}^{K} \widehat{f}(k)e(k\alpha) = \frac{f(\alpha^+) + f(\alpha^-)}{2}\,. \tag{2}$$

Less familiar is the strong quantitative version of this that we now derive.

Let $D_K(x) = \sum_{k=-K}^{K} e(kx)$. This is the *Dirichlet kernel*. We multiply both sides of (1) by $e(k\alpha)$ and sum, to see that

$$\sum_{k=-K}^{K} f(k)e(k\alpha) = \int_{\mathbb{T}} f(x)D_K(\alpha - x)\,dx = \int_{\mathbb{T}} D_K(x)f(\alpha - x)\,dx.$$

Since D_K is an even function, the above is

$$= \int_{\mathbb{T}} D_K(x)f(\alpha + x)\,dx. \tag{3}$$

Clearly $D_K(0) = 2K + 1$. If $x \notin \mathbb{Z}$ then $D_K(x)$ is the sum of a segment of a geometric progression, which permits us to write D_K in closed form,

$$\begin{aligned} D_K(x) &= \frac{e\big((K+1)x\big) - e(-Kx)}{e(x) - 1} = \frac{e\big((K+\frac{1}{2})x\big) - e\big(-(K+\frac{1}{2})x\big)}{e(x/2) - e(-x/2)} \\ &= \frac{\sin(2K+1)\pi x}{\sin \pi x}. \end{aligned} \tag{4}$$

Our analysis of the pointwise convergence of Fourier series is based on the behaviour of the the Fourier series of one particular function, namely the 'saw-tooth function' $s(x)$ given by

$$s(x) = \begin{cases} \{x\} - \frac{1}{2} & (x \notin \mathbb{Z}), \\ 0 & (x \in \mathbb{Z}). \end{cases}$$

Lemma 1. *Let*

$$E_K(x) = s(x) + \sum_{k=1}^{K} \frac{\sin 2\pi kx}{\pi k}\,.$$

Then $|E_K(x)| \le \min\big(1/2, 1/((2K+1)\pi|\sin \pi x|)\big)$.

It is easy to compute the Fourier coefficients of $s(x)$; we find that $\widehat{s}(0) = 0$, and that $\widehat{s}(k) = -1/(2\pi i k)$ for $k \ne 0$. Thus the above lemma constitutes a quantitative form of (2), for the function $s(x)$.

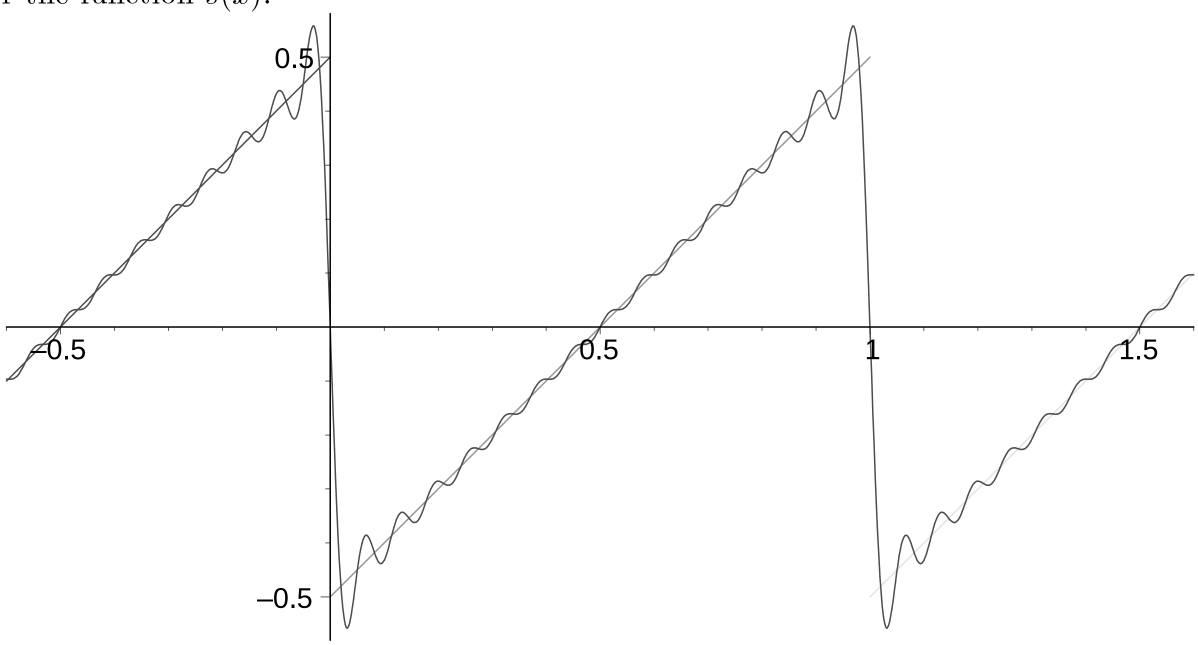

FIGURE 1. Graph of $s(x)$ and its Fourier approximation $-\sum_{k=1}^{15} \sin 2\pi kx/(\pi k)$.

Proof. All terms comprising $E_K(x)$ are odd, and hence E_K is odd. Thus we may suppose that $0 \le x \le 1/2$. The case $x = 0$ is clear. We observe that if $x \notin \mathbb{Z}$ then

$$E_K'(x) = 1 + 2\sum_{k=1}^{K} \cos 2\pi kx = D_K(x).$$

Hence if $0 < x \le 1/2$ then by (4) we see that

$$E_K(x) = -\frac{1}{2}\int_x^{1-x} D_K(z)\,dz$$

$$= \frac{-1}{2}\int_x^{1-x} \frac{\sin(2K+1)\pi z}{\sin \pi z}\,dz = \frac{i}{2}\int_x^{1-x} \frac{e\big((K+\frac{1}{2})z\big)}{\sin \pi z}\,dz.$$

The integrand is analytic in the rectangle $x \le \Re z \le 1-x$, $0 \le \Im z \le Y$, so by letting $Y \to \infty$ and applying Cauchy's theorem we see that the above is

$$= \frac{i}{2}\int_x^{x+i\infty} \frac{e\big((K+\frac12)z\big)}{\sin \pi z}\,dz - \frac{i}{2}\int_{1-x}^{1-x+i\infty} \frac{e\big((K+\frac12)z\big)}{\sin \pi z}\,dz.$$

On writing $z = x+iy$ in the first integral, and $z = 1-x+iy$ in the second, we see that the above is

$$= \frac{-1}{2}\int_0^\infty \Big(\frac{e\big((K+\frac12)x\big)}{\sin \pi(x+iy)} - \frac{e\big(-(K+\frac12)x\big)}{\sin \pi(1-x+iy)}\Big)e^{-(2K+1)\pi y}\,dy. \tag{5}$$

But $\sin \pi(x+iy) = (\sin \pi x)\cosh \pi y - i(\cos \pi x)\sinh \pi y$, so that $|\sin \pi(x+iy)| \ge \sin \pi x$ for all real y. Hence the expression above has absolute value not exceeding

$$\frac{1}{\sin \pi x}\int_0^\infty e^{-(2K+1)\pi y}\,dy = \frac{1}{(2K+1)\pi \sin \pi x}.$$

This gives the second part of the bound. The first bound, $|E_K(x)| \le 1/2$, is weaker if $1/(2K+1) \le x \le 1/2$, since $\sin \pi x \ge 2x$ in this range. Thus it suffices to show that $|E_K(x)| \le 1/2$ when $0 < x < 1/(2K+1)$. Since $0 < \sin u < u$ for $0 \le u \le \pi$, it follows from the definition of $E_K(x)$ that

$$x - \frac12 \le E_K(x) \le (2K+1)x - \frac12$$

for $0 \le x \le 1/(2K+1)$. This gives the desired bound.

We now establish an analogue of Lemma 1 for arbitrary functions of bounded variation.

Theorem 2. *If f has bounded variation on $\mathbb{T}$, with $\widehat{f}(k)$ given by (1), then for any α,*

$$\Big|\frac{f(\alpha^+)+f(\alpha^-)}{2} - \sum_{k=-K}^{K}\widehat{f}(k)e(k\alpha)\Big| \le \int_{0^+}^{1^-} \min\Big(\frac12, \frac{1}{(2K+1)\pi \sin \pi x}\Big)\,|df(\alpha+x)|.$$

Since the right hand side here tends to 0 as $K \to \infty$, this inequality implies the qualitative relation (2).

Proof. As $E_K'(x) = D_K(x)$ when $x \notin \mathbb{Z}$, the integral (3) is

$$\int_{0^+}^{1^-} E_K'(x) f(\alpha+x)\,dx = \int_{0^+}^{1^-} f(\alpha+x)\,dE_K(x),$$

by Theorem A.3. But $E_K(0^+) = -1/2$, $E_K(1^-) = 1/2$. Hence by integrating by parts (as in Theorem A.2) we see that the above is

$$\frac12 f(\alpha^+) + \frac12 f(\alpha^-) - \int_{0^+}^{1^-} E_K(x)\,df(\alpha+x).$$

To complete the proof it suffices to apply the triangle inequality (as in Theorem A.4) and the bound of Lemma 1.

2. The Poisson summation formula

The formula in question asserts that under suitable conditions,

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \widehat{f}(k) \tag{6}$$

where f is a function of a real variable, and $\widehat{f}$ is its Fourier transform,

$$\widehat{f}(t) = \int_{\mathbb{R}} f(x)e(-tx)\,dx. \tag{7}$$

To ensure that $\widehat{f}$ is well-defined, we impose the condition $f \in L^1(\mathbb{R})$, i.e. that the integral $\int_{\mathbb{R}} |f(x)|\,dx$ is finite. Put

$$F(\alpha) = \sum_{n\in\mathbb{Z}} f(n+\alpha). \tag{8}$$

This sum is absolutely convergent for almost all α, since

$$\int_0^1 \sum_{n\in\mathbb{Z}} |f(n+\alpha)|\,d\alpha = \sum_{n\in\mathbb{Z}} \int_n^{n+1} |f(\alpha)|\,d\alpha = \int_{\mathbb{R}} |f(\alpha)|\,d\alpha < \infty.$$

Moreover, $F(\alpha)$ has period 1, $\int_{\mathbb{T}} |F(\alpha)|\,d\alpha < \infty$, and F has Fourier coefficients

$$\begin{aligned}\widehat{F}(k) &= \int_0^1 F(\alpha)e(-k\alpha)\,d\alpha = \sum_{n\in\mathbb{Z}} \int_0^1 f(n+\alpha)e(-k\alpha)\,d\alpha = \int_{\mathbb{R}} f(x)e(-kx)\,dx \\ &= \widehat{f}(k).\end{aligned} \tag{9}$$

Here the interchange of the integral and the sum is justified by absolute convergence. Thus the Fourier expansion of F is

$$\sum_{k\in\mathbb{Z}} \widehat{f}(k)e(k\alpha).$$

The Poisson summation formula (6) is simply the assertion that this Fourier expansion converges to $F(\alpha)$ when $\alpha = 0$. Our hypotheses thus far do not ensure this, but in this direction we establish the following two precise results.

Theorem 3. *Suppose that $f \in L^1(\mathbb{R})$, and that f is of bounded variation on $\mathbb{R}$. Then*

$$\sum_{n\in\mathbb{Z}} \frac{f(n^+) + f(n^-)}{2} = \lim_{K\to\infty} \sum_{k=-K}^{K} \widehat{f}(k).$$

If in addition f is continuous then we have a result which is close to (6), although it is still necessary to restrict ourselves to symmetric partial sums on the right hand side.

Proof. We first note that if $n \le \alpha \le n+1$ then

$$f(\alpha) = \int_n^{n+1} f(x)\,dx + \int_n^{\alpha} (x-n)\,df(x) + \int_{\alpha}^{n+1} (x-n-1)\,df(x),$$

as can readily be seen by integration by parts. Hence

$$|f(\alpha)| \le \int_n^{n+1} |f(x)|\,dx + \operatorname{var}_{[n,n+1]} f, \tag{10}$$

and it follows from our hypotheses that the sum

$$\sum_{n\in\mathbb{Z}} f(n+\alpha)$$

is absolutely convergent for all α, and uniformly convergent in compact regions. Hence $F(\alpha)$ can be taken to be the value of this sum for all α, not merely for almost all α. By the triangle inequality, $\operatorname{var}_{\mathbb{T}} F \le \operatorname{var}_{\mathbb{R}} f$, so that F is of bounded variation on $\mathbb{T}$, and hence the relation (2) applies to F. Thus we see that the Fourier series of F converges to $\big(F(\alpha^+) + F(\alpha^-)\big)/2$ for all α. Using the fact that f is of bounded variation once more, we see that $F(\alpha^+) = \sum_{n\in\mathbb{Z}} f((n+\alpha)^+)$, and similarly for $F(\alpha^-)$. Hence we have the stated result.

Theorem 4. *Suppose that f is continuous, and that the series $\sum_{n\in\mathbb{Z}} f(n+\alpha)$ is uniformly convergent for $0 \le \alpha \le 1$. Then*

$$\sum_{n\in\mathbb{Z}} f(n) = \lim_{K\to\infty} \sum_{k=-K}^{K} \Big(1 - \frac{|k|}{K}\Big)\widehat{f}(k).$$

Proof. Clearly $F(\alpha)$ given in (8) is continuous. Since we have not assumed that $f \in L^1(\mathbb{R})$, the Fourier transform $\widehat{f}(t)$ may not exist. However, if k is an integer then $\widehat{f}(k)$ exists as a convergent improper integral. To see this we first note that $\sum_{n=M}^{N} f(n+\alpha)$ is small if M and N are large integers and $0 \le \alpha \le 1$. Then

$$\int_0^1 \sum_{M}^{N} f(n+\alpha)e(-k\alpha)\,d\alpha = \int_M^{N+1} f(x)e(-kx)\,dx$$

is small. The hypothesis that $\sum_n f(n+\alpha)$ converges uniformly implies that $f(x) \to 0$ as $|x| \to \infty$. Hence $\int_u^v f(x)e(-kx)\,dx \to 0$ as u, v tend to infinity through real values. The calculation of $\widehat{F}(k)$ in (9) is still valid, but is now justified by uniform convergence. Next we appeal to a theorem of Fejér, which asserts that the Fourier series of a continuous function $F(\alpha)$ with period 1 is uniformly $(C,1)$-summable to F. That is,

$$\sum_{k=-K}^{K} \Big(1 - \frac{|k|}{K}\Big)\widehat{F}(k)e(k\alpha) \longrightarrow F(\alpha)$$

uniformly as $K \to \infty$. The stated identity follows on taking $\alpha = 0$.

D.2. Exercises

1. Show that if f satisfies the hypotheses of Theorem 2, and α and β are real numbers, then the function $f(x+\alpha)e(\beta x)$ does also. Specify conditions under which

$$\sum_n f(n+\alpha)e(\beta n) = \sum_k \widehat{f}(k-\beta)e((k-\beta)\alpha).$$

2. Suppose that f has bounded variation on $[-A, A]$, for every $A > 0$. Show that

$$\lim_{N\to\infty} \sum_{n=-N}^{N} f(n) = \lim_{T\to\infty} \sum_{k=-\infty}^{\infty} \int_{-T}^{T} f(x)e(-kx)\,dx$$

provided that either limit exists.

3. Suppose that $f \in L^1(\mathbb{R}^n)$, and for $\boldsymbol{x} \in \mathbb{T}^n$ put

$$F(\boldsymbol{x}) = \sum_{\boldsymbol{\lambda}\in\mathbb{Z}^n} f(\boldsymbol{\lambda}+\boldsymbol{x})\,.$$

(a) Show that the sum $F(\boldsymbol{x})$ is absolutely convergent for almost all $\boldsymbol{x}$.
(b) Show that $F \in L^1(\mathbb{T}^n)$ and that $\|F\|_{L^1(\mathbb{T}^n)} \le \|f\|_{L^1(\mathbb{R}^n)}$.
(c) Show that $\widehat{F}(\boldsymbol{k}) = \widehat{f}(\boldsymbol{t})$.

4. (a) Suppose that there is a $\delta > 0$ such that $c(\boldsymbol{k}) \ll (1+|\boldsymbol{k}|)^{-n-\delta}$. Show that

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^n} c(\boldsymbol{k})e(\boldsymbol{k}\cdot\boldsymbol{x})$$

is a continuous function of $\boldsymbol{x} \in \mathbb{T}^n$.
(b) Suppose that there is a $\delta > 0$ such that $f(\boldsymbol{x}) \ll (1+|\boldsymbol{x}|)^{-n-\delta}$ for $\boldsymbol{x} \in \mathbb{R}^n$. Suppose also that $f(\boldsymbol{x})$ is continuous. Show that

$$F(\boldsymbol{x}) = \sum_{\boldsymbol{\lambda}\in\mathbb{Z}^n} f(\boldsymbol{\lambda}+\boldsymbol{x})$$

is a continuous function for $\boldsymbol{x} \in \mathbb{T}^n$.
(c) Suppose that in addition to the hypotheses in (b), the function f also has the property that $\widehat{f}(\boldsymbol{t}) \ll (1+|\boldsymbol{t}|)^{-n-\delta}$. Show that

$$\sum_{\boldsymbol{\lambda}\in\mathbb{Z}^n} f(\boldsymbol{\lambda}+\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^n} \widehat{f}(\boldsymbol{k})e(\boldsymbol{k}\cdot\boldsymbol{x})$$

for all $\boldsymbol{x} \in \mathbb{T}^n$.

5. A *lattice* in $\mathbb{R}^n$ is a set of points of the form $A\mathbb{Z}^n$ where A is a non-singular $n \times n$ matrix. Thus $\mathbb{Z}^n$ is an example of a lattice, called the *lattice of integral points*.
(a) Suppose that $\Lambda_1 = A\mathbb{Z}^n$ and $\Lambda_2 = B\mathbb{Z}^n$ are two lattices. Show that $\Lambda_2 \subseteq \Lambda_1$ if and only if there is an $n \times n$ matrix K with integral entries such that $B = AK$.
(b) An $n \times n$ matrix U is said to be *unimodular* if (i) its entries are integers, and (ii) $\det U = \pm 1$. Show that if $\Lambda_1 = A\mathbb{Z}^n$ and $\Lambda_2 = B\mathbb{Z}^n$ are two lattices, then $\Lambda_1 = \Lambda_2$ if and only if there is a unimodular matrix U such that $B = AU$.
(c) Let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ denote the columns of A. These vectors are said to form a basis for Λ_1, because every member of Λ_1 has a unique representation in the form $c_1\boldsymbol{a}_1 + \cdots c_n\boldsymbol{a}_n$ where the c_i are integers. If $\Lambda = A\mathbb{Z}^n$, we say that the *determinant* of Λ is $d(\Lambda) = |\det A|$. Show that the determinant of a lattice is independent of the basis by which it is presented.
(d) Suppose that $\Lambda = A\mathbb{Z}^n$ is a lattice in $\mathbb{R}^n$. Let Λ^* be the set of all those points $\boldsymbol{\mu} \in \mathbb{R}^n$ such that $\boldsymbol{\mu} \cdot \boldsymbol{\lambda} \in \mathbb{Z}$ for all $\boldsymbol{\lambda} \in \Lambda$. Show that Λ^* is a lattice, and indeed that $\Lambda^* = \left(A^{-1}\right)^{\mathrm{T}}\mathbb{Z}^n$.
(e) Suppose that f is a continuous function on $\mathbb{R}^n$ such that

$$f(\boldsymbol{x}) \ll (1 + |\boldsymbol{x}|)^{-n-\delta},$$
$$\widehat{f}(\boldsymbol{t}) \ll (1 + |\boldsymbol{t}|)^{-n-\delta}$$

for some $\delta > 0$. Let $\Lambda = A\mathbb{Z}^n$ be a lattice. Show that

$$\sum_{\boldsymbol{\lambda} \in \Lambda} f(\boldsymbol{\lambda} + \boldsymbol{x}) = \frac{1}{d(\Lambda)} \sum_{\boldsymbol{\mu} \in \Lambda^*} \widehat{f}(\boldsymbol{\mu}) e(\boldsymbol{\mu} \cdot \boldsymbol{x})$$

for all $\boldsymbol{x}$.

D. Notes

§1. The relation (2) is the famous Dirichlet–Jordan test, which is usually derived with much less effort. Theorem 2 generalizes and refines an argument of Pólya (1918), who estimated the rate of convergence of the Fourier series (9.18). For more on the convergence of Fourier series, see Katznelson (1968, Chapter 2), Körner (1988, Part I), or Zygmund (1968, Chapter II).

§2. For more on the Poisson summation formula, see Katznelson (1968, VI.1.15), Körner (1988, §27), or Zygmund (1968, Chapter II§13). For a discussion of the Poisson summation formula in higher dimensions, see Stein & Weiss (1971, Chapter VII§2). Siegel (1935) showed that Minkowski's convex body theorem could be derived by applying the Poisson summation formula. Cohn & Elkies (2003), Cohn (2002) and Cohn & Kumar (2004) have applied the Poisson summation formula in $\mathbb{R}^n$ to limit the density of sphere packings.

D. Literature

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