Math 571 Maier's Theorem

Robert C. Vaughan

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- Daniel Shiu [2000] shows that there are ‘strings’ of congruent primes,
- i.e. that for any k there exist infinitely many n such that

$$p_{n+1} \equiv p_{n+2} \equiv \ldots \equiv p_{n+k} \equiv a \pmod{q}.$$

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- Only 5 is usually covered in Math 568 and we do not have the time to cover all of rest in detail.
- I propose instead to give an account of Maier's theorem and an overview of the necessary background which includes all of the above except 4.
- In addition we get an immediate proof of Linnik's theorem on the least prime in a.p.

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- Maier [1985] showed that this breaks down for short intervals.
- He proved that if $\lambda > 1$, then

$$\liminf_{x\to\infty} \frac{\pi\big(x + (\log x)^\lambda\big) - \pi(x)}{(\log x)^{\lambda-1}} < 1 < \limsup_{x\to\infty} \frac{\pi\big(x + (\log x)^\lambda\big) - \pi(x)}{(\log x)^{\lambda-1}}$$

- Maier showed that

Theorem 1 (Maier)

Let $\lambda > 1$ and ω denote the Buchstab function,

$$\Omega^-(\lambda) = \inf\{\omega(\tau) : \tau > \lambda\},\ \Omega^+(\lambda) = \sup\{\omega(\tau) : \tau > \lambda\}.$$

Then

$$\limsup_{x\to\infty} \frac{\pi\big(x + (\log x)^\lambda\big) - \pi(x)}{(\log x)^{\lambda-1}} \ge e^{C_0}\Omega^+(\lambda)$$

and

$$\liminf_{x\to\infty} \frac{\pi\big(x + (\log x)^\lambda\big) - \pi(x)}{(\log x)^{\lambda-1}} \le e^{C_0}\Omega^-(\lambda).$$

Moreover $\Omega^-(\lambda) < e^{-C_0} < \Omega^+(\lambda)$.

- The Buchstab function $\omega(u)$ is defined for $u \ge 1$ by

$$\omega(u) = \frac{1}{u} \ (1 < u \le 2), \quad (u\omega(u))' = \omega(u-1) \ (u > 2)$$

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$$\omega'(u) = \frac{1 - (u-1)\log(u-1)}{u^2(u-1)}$$

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- We also have for $u > 2$

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- It can also be shown that $\lim_{u\to\infty} \omega(u) = e^{-C_0}$ where C_0 is Euler's constant. For more details see Chapter 7 of MNT.

- The following lemma is particularly useful in the proof of Maier's theorem.

Lemma 2 (de Bruijn)

The function $\omega(u) - e^{-C_0}$ *changes sign in every interval* $[t-1, t]$ *with* $t \geq 2$.

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- Proof. Define $\xi(u)$ for $u > -1$ by

$$\xi(u) = \int_0^\infty \exp\left(-ux - x + \int_0^x \frac{e^{-y} - 1}{y} dy\right) dx.$$

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- Then $\eta'(t) = 0 \quad (t > 2)$ and $\lim_{t\to\infty} \eta(t) = e^{-C_0}$.

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- Then $\nu'(t) = 0$ $(t > 0)$. We also have $\nu(t) \to 1$ as $t \to \infty$. Therefore

$$\int_{t-1}^{t} \xi(u)du + t\xi(t-1) = 1(t > 0)$$

and so

$$\int_{t-1}^{t} \big(\omega(u) - e^{-C_0}\big)\eta(u)du + t\big(\omega(t) - e^{-C_0}\big)\eta(t-1) = 0.$$

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$$\int_{t-1}^{t} (\omega(u) - e^{-C_0})\eta(u)du + t\big(\omega(t) - e^{-C_0}\big)\eta(t-1) = 0.$$

- Since $\eta(t) > 0$ and $\omega(u)$ is not a constant when $2 \leq u \leq 3$, then the lemma follows.

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- **Exceptional Zero Statement.** *By Corollary 11.10 of MNT there is a positive constant c_1 such that*

$$F(s,T)=\prod_{q\le T}\,\prod_{\chi \bmod q}^{*} L(s,\chi)$$

has at most one zero s with $\Re s > 1-\frac{1}{c_1\log T}$, of necessity real and if this "exceptional zero" β_1 exists, then the corresponding character χ_1 is quadratic and, by Corollary 11.12, there is a positive constant c_2 such that $\delta_1 = 1-\beta_1$ satisfies

$$\frac{1}{c_2 q_1^{1/2}(\log q_1)^2}\le \delta_1 < \frac{1}{c_1\log T}$$

where q_1 is the conductor of χ_1.

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$$N^*(\theta, T)$$
denote the number of zeros $\rho = \beta + i\gamma$ of $F(s, T)$, with $\beta \geq \theta$ and $|\gamma| \leq T$, **other** than any exceptional zero.

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- On the other hand if the exceptional moduli form an infinite sequence $\{q_j\}$ and $\{\beta_j\}$ are the corresponding exceptional zeros, then by the same token one would have to have

$$\limsup_{j\to\infty}(1-\beta_j)\log q_j = 0.$$

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- On the other hand if the exceptional moduli form an infinite sequence $\{q_j\}$ and $\{\beta_j\}$ are the corresponding exceptional zeros, then by the same token one would have to have
$$\limsup_{j\to\infty}(1-\beta_j)\log q_j = 0.$$
- So c_1 could be taken to be as small as one pleases.

- We will probably have to skip the proof of the following.

Theorem 3

There are constants $c > 0$, $c_0 > 0$ so that for $\frac{1}{2} \le \theta \le 1$ and $T \ge 2$

$$N^*(\theta, T) \le c_0 T^{c(1-\theta)},$$

and if there is an exceptional real zero β_1, χ_1, q_1, then

$$N^*(\theta, T) \le c_0 \delta_1 (\log T) T^{c(1-\theta)}.$$

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- This gives an effective Deuring-Heilbronn theorem.

Corollary 4

There are $c_0 > 0, c > 0$ so that if β_1 is exceptional, then any other zero $\rho = \beta + i\gamma$ with $|\gamma| \le T$ of $F(s, T)$ satisfies

$$\beta \le 1 - \frac{\log \frac{1}{c_0(1-\beta_1)\log T}}{c \log T}.$$

- An important application of the above is

Theorem 5 (Gallagher)

There are constants $c \ge 1$, $\kappa_0 \ge 3$ so that if $\kappa \ge \kappa_0$ is a constant and Q and x satisfy $1 < Q^{6c} \le x$, then

$$\sum_{q \le Q} \sideset{}{^*}\sum_{\chi \bmod q} |\vartheta(x;\chi) - E_0(\chi)x| \ll x \exp\left(-\frac{\log x}{\kappa \log Q}\right) + \frac{x \log^2 x}{Q \log^2 Q}$$

unless $F(s,q) = \prod_{q \le Q} \sideset{}{^*}\prod_{\chi \bmod q} L(s,\chi)$ *has an exceptional zero* β_1 *with* $1 - \beta_1 < \frac{1}{\kappa \log Q}$ *when the general term on the left is replaced by* $\left|\vartheta(x;\chi_1) + \frac{x^{\beta_1}}{\beta_1}\right|$ *when* $\chi = \chi_1$ *is the exceptional character, and the right hand side by*

$$(1-\beta_1)(\log x)\left(x \exp\left(-\frac{\log x}{\log Q}\right) + \frac{x \log x}{Q \log Q}\right).$$

- The following two theorems are almost immediate

Theorem 6 (Gallagher)

Suppose that $1 < q^{6c} \le x$, $(a,q)=1$, c, κ *is as in Theorem 5 and that there is no exceptional zero* β_1 *with* $1-\beta_1 < \dfrac{1}{\kappa \log q}$. *Then*

$$\vartheta(x;q,a) = \frac{x}{\phi(q)}\left(1 + O\left(\exp\left(-\frac{\log x}{\kappa \log q}\right) + \frac{\log^2 x}{q \log^2 q}\right)\right).$$

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- Given $q \in \mathbb{N}$ and $a \in \mathbb{Z}$ with $(q,a)=1$ we define $p(q,a)$ to be the least prime number p such that $p \equiv a \pmod{q}$.

Theorem 7 (Linnik)

There is a positive constant A *such that whenever* $q \in \mathbb{N}$, $a \in \mathbb{Z}$ *and* $(q,a)=1$ *we have* $p(q,a) \le q^A$.

- **Theorem 5.** *There are* $c \ge 1$, $\kappa_0 \ge 3$ *so if* $\kappa \ge \kappa_0$, $Q^{6c} \le x$, *then* $\sum_{q\le Q} \sideset{}{^*}\sum_{\chi \bmod q} |\vartheta(x;\chi) - E_0(\chi)x|$

$$\ll x \exp\left(-\frac{\log x}{\kappa \log Q}\right) + \frac{x \log^2 x}{Q \log^2 Q}$$

unless there is a $\beta_1 > 1 - \frac{1}{\kappa \log Q}$ *when LHS with* $\chi = \chi_1$ *is* $\left|\vartheta(x;\chi_1) + \frac{x^{\beta_1}}{\beta_1}\right|$ *when* $\chi = \chi_1$ *and the RHS is*

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- **Theorem 5.** *There are $c \ge 1$, $\kappa_0 \ge 3$ so if $\kappa \ge \kappa_0$, $Q^{6c} \le x$, then* $\displaystyle\sum_{q\le Q} \sum_{\chi \bmod q}^{*} |\vartheta(x;\chi) - E_0(\chi)x|$

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$$(1-\beta_1)(\log x)\left(x \exp\left(-\frac{\log x}{\log Q}\right) + \frac{x \log x}{Q \log Q}\right).$$

- **Proof.** By MVvol1 or Math 568, when $x \ge 2$, $T \le x^{1/2}$

$$\vartheta(x;\chi) = E_0(\chi)x - \sum_{\rho\in\mathcal{R}(\chi)} \frac{x^\rho}{\rho} + O\left(\frac{x}{T}(\log x)^2\right) \qquad (0)$$

where $\mathcal{R}(\chi) = \{\rho : L(\rho,\chi) = 0, \beta \ge \frac{1}{2}, |\gamma| \le T\}$.

- Let the constants c, c_0, c_1, c_2 be as in the Exceptional Zero Statement, Theorem 3 and Corollary 4, and let

$$\kappa_0 = 3\max(c, c_1, c_0 e, 1, c_0 e^{3c}). \tag{1}$$

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- The proof divides into two cases.

- First we suppose that
$F = F(s, T) = \prod_{q\le T} \ \prod^*_{\chi \bmod q} L(s,\chi)$
has no zeros $\rho = \beta + i\gamma$ with $|\gamma| \le T$ and

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- By the explicit formula $\displaystyle\sum_{q\le Q} \sum^*_{\chi \bmod q} |\vartheta(x;\chi) - E_0(\chi)x|$

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- By the explicit formula $\sum_{q\le Q} \sum^*_{\chi \bmod q} |\vartheta(x;\chi) - E_0(\chi)x|$
$$\ll QxT^{-1}(\log x)^2 + \sum_{q\le Q} \sum^*_{\chi \bmod q} \sum_{\rho\in\mathcal{R}(\chi)} x^\beta.$$
- We have $x^\beta = x^{1/2} + \int_{1/2}^{\beta} x^u (\log x) du$ and so the above is
$$\le x^{1/2} N^*(1/2, T) + \int_{1/2}^{1-1/(\kappa' \log T)} x^u N^*(u,T)(\log x) du.$$

- Thus $\displaystyle\sum_{q\le Q}\ \sideset{}{^*}\sum_{\chi \bmod q} |\vartheta(x;\chi) - E_0(\chi)x|$

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- By Theorem 3 this is

$$\ll x^{1/2} T^{c/2} + \int_{1/2}^{1-1/(\kappa' \log T)} x^u T^{c(1-u)}(\log x) du.$$

By (2) and the hypothesis on x.

$$xT^{-c} = xQ^{-3c} \ge x^{1/2} \text{ and } x^{1/2} Q^{c/2} \le x^{3/4}.$$

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- Hence the sum of interest is

$$\ll xQ^{-2}(\log x)^2 + x^{1-1/(\kappa'\log T)}T^{c/(\kappa'\log T)}$$
$$= xQ^{-2}(\log x)^2 + x\exp\big(-(\log x)/(\kappa'\log Q) + c/\kappa'\big)$$
$$\ll xQ^{-2}(\log x)^2 + x\exp\big(-(\log x)/(\kappa\log Q) + c/\kappa'\big).$$

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- We can proceed as above, but now the multiple sum is

$$\ll (1-\beta_1)(\log T)x^{1/2}T^{c/2} + \int_{1/2}^{1-\delta}(1-\beta_1)(\log T)x^u T^{c(1-u)}\,du$$

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- So, by the inequality for δ_1 in EZS, the sum is

$$\ll (1-\beta_1)QxT^{-1}(\log x)^2 + (1-\beta_1)(\log x)x^{1-\delta}T^{c\delta}$$

$$\ll (1-\beta_1)(\log x)\frac{x\log x}{Q\log Q} + (1-\beta_1)(\log x)x^{1-\delta}T^{c\delta}.$$

- We have

$$\begin{aligned} x^{1-\delta}T^{c\delta} &= x\exp\left(\frac{\log(xT^{-c})}{c\log T}\log\left(c_0(1-\beta_1)\log T\right)\right) \\ &\le x\exp\left(-\frac{\log x - 3c\log Q}{3c\log Q}\log(\kappa'/c_0)\right) \\ &\ll x\exp\left(-\frac{\log x - 3c\log Q}{\log Q}\right) \\ &\ll x\exp\left(-\frac{\log x}{\log Q}\right) \end{aligned}$$

and that completes the proof.

- As a consequence of Gallagher's theorem.

Lemma 8

Let $P(z) = \prod_{p \le z} p$. *Then there are positive constants* c *and* κ_0 *which have the property that when* $A > \max(2, 6c)$ *and* $\kappa \in [\kappa_0, 2\kappa_0]$ *there are arbitrarily large* $z > z_0(A, \kappa)$ *such that whenever* $(a, P(z)) = 1$ *we have*

$$\pi\big(2P(z)^A, P(z), a)\big) - \pi\big(P(z)^A, P(z), a)\big) - \frac{P(z)^A}{\phi\big(P(z)\big)\big(A \log P(z)\big)} \ll \frac{P(z)^A \exp(-A/\kappa)}{\phi\big(P(z)\big)\big(A \log P(z)\big)}.$$

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- Note that this is not for **every** large z, only some, perhaps very thin, subset of all large z.

- Proof. Let $q = P(p_n)$. If there is no exceptional zero β_1 of

$$F_n(s) = \prod_{m|P(p_n)} \prod_{\chi \bmod m}^{*} L(s,\chi)$$

with $\beta_1 > 1 - \frac{1}{\kappa_0 \log P(p_n)}$, then we have, provided p_n is large enough in terms of A and κ_0.

$$\vartheta\big(2P(p_n)^A, P(p_n), a)\big) - \vartheta\big(P(p_n)^A, P(p_n), a)\big) - \frac{P(p_n)^A}{\phi\big(P(p_n)\big)} \ll \frac{P(p_n)^A \exp(-A/\kappa_0)}{\phi\big(P(p_n)\big)\big)}.$$

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- Moreover, for $P(p_n)^A < p \le 2P(p_n)^A$, we have

$$\log p = A \log P(p_n) + O(1)$$

and the desired conclusion follows with $z = p_n$.

- Now suppose that there is an exceptional zero β_1 of
$$F_n(s) = \prod_{m|P(p_n)} \prod_{\chi \bmod m}^{*} L(x, \chi)$$
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q_1 is large in terms of n.
- Now choose l minimally so that $q_1 | P(p_l)$ and consider $P(p_{l-1})$. The l will also be large in terms of n, and β_1 will satisfy
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so will be exceptional for $F_l(s)$.
- Suppose that $F_{l-1}(s)$ has an exceptional zero β_2, so that
$$\beta_2 > 1 - \frac{1}{\kappa_0 \log P(p_{l-1})}.$$

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- But there cannot be a second exceptional zero of $F_l(s)$, so

$$\beta_2 \le 1 - \frac{1}{\kappa_0 \log P(p_l)} = 1 - \frac{1}{\kappa_0\big(\log P(p_{l-1}) + \log p_l\big)} < 1 - \frac{1}{2\kappa_0 \log P(p_{l-1})}.$$

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- Thus there are no exceptional zeros of the kind

$$\beta_1 > 1 - \frac{1}{2\kappa_0 \log P(p_{l-1})}$$

associated with $P(p_{l-1})$ and we can proceed as in the first part of the proof.

Recall Maier's Theorem

- **Theorem 1 (Maier).** *ω denotes the Buchstab function,*

$$\Omega^-(\lambda) = \inf\{\omega(\tau) : \tau > \lambda\},\ \Omega^+(\lambda) = \sup\{\omega(\tau) : \tau > \lambda\}.$$

Then $\limsup_{x\to\infty} \dfrac{\pi\big(x + (\log x)^\lambda\big) - \pi(x)}{(\log x)^{\lambda-1}} \ge e^{C_0}\Omega^+(\lambda)$ *and* $\liminf_{x\to\infty} \dfrac{\pi\big(x + (\log x)^\lambda\big) - \pi(x)}{(\log x)^{\lambda-1}} \le e^{C_0}\Omega^-(\lambda)$. *Moreover* $\Omega^-(\lambda) < e^{-C_0} < \Omega^+(\lambda)$.

Proof of Maier's Theorem

- Let $\tau > \lambda$ and consider the array $\mathfrak{M} = (a_{uv})$

  $$\big(1 \le u \le P(z)^{A-1},\, 1 \le v \le (A \log P(z))^{\tau},\, \big(v, P(z)\big) = 1\big)$$

  where $a_{uv} = 1$ when $uP(z)^{A-1} + v$ is prime and 0 otherwise.

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  where $a_{uv} = 1$ when $uP(z)^{A-1} + v$ is prime and 0 otherwise.
- By Lemma 8 (Gallagher), the number of non-zero entries in the v-th column is

$$\pi\big(2P(z)^A, P(z), v\big) - \pi\big(P(z)^A, P(z), v\big) = \frac{P(z)^A}{\phi(P(z))A \log P(z)}\big(1 + O\big(\exp(-A/\kappa)\big)\big) \quad (3)$$

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- where Φ is as in:

Theorem 9 (Buchstab)

Let $\Phi(x, y)$ denote the number of positive integers $n \le x$ composed entirely of prime numbers $p \ge y$, and let $\omega(u)$ be Buchstab's function. Then

$$\Phi(x, y) = \frac{\omega(u)x}{\log y} - \frac{y}{\log y} + O\Big(\frac{x}{(\log x)^2}\Big)$$

uniformly for $1 \le u \le U$ and all $y \ge 2$. Here $u = (\log x)/\log y$, i.e. $y = x^{1/u}$.

- Thus the total number of non-zero entries in the array is

$$\frac{P(z)^A \Phi\big((A\log P(z))^\tau, z\big)}{\phi\big(P(z)\big) A\log P(z)}\big(1+O\big(\exp(-A/\kappa)\big)\big) =$$

$$\frac{P(z)^{A-1}\big(A\log(P(z))\big)^\tau \omega\left(\frac{\tau\log\big(A\log P(z)\big)}{\log z}\right)}{A\log P(z)\prod_{p\le z}(1-1/p)}\big(1+O('')\big).$$

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- Moreover, by the prime number theorem and Mertens,

$$\log P(z) = \vartheta(z) = z + O(z/\log z),$$

$$\log(A\log P(z)) = (\log z)\left(1+O\left((\log A)(\log z)^{-1}\right)\right)$$

$$\prod_{p\le z}(1-1/p) = \frac{e^{-C_0}}{\log z}\left(1+O\left(\frac{1}{\log z}\right)\right), \text{ so}$$

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- $\omega(\tau \log(A \log P(z))/\log z) = \omega(\tau)(1 + O(\log A/\log z)))$

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- Hence there are rows with $\geq M$ non-zero entries where

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- By dividing the primes counted in these rows into N subintervals of length $(A\log(P(z)))^{\tau}N^{-1}$ where $N=\left\lceil(A\log(P(z)))^{\tau-\lambda}\right\rceil$ we find that there are intervals $\big(X,X+(A\log(P(z)))^{\tau}N^{-1}\big]$ containing $\geq\dfrac{(A\log(P(z)))^{\lambda}e^{C_0}\omega(\tau)}{A\log P(z)}\big(1+O\big(\exp(-A/\kappa)\big)\big)$ primes where $P(z)^{A}\leq X\leq 2P(z)^{A}+\big(A\log P(z)\big)^{\tau}$.

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- Intervals $\big(X, X + (A\log(P(z)))^\tau N^{-1}\big]$, with $N = \left\lceil (A\log(P(z)))^{\tau-\lambda} \right\rceil$, containing $\geq \dfrac{(A\log(P(z)))^\lambda e^{C_0}\omega(\tau)}{A\log P(z)}\big(1 + O\big(\exp(-A/\kappa)\big)\big)$ primes where $P(z)^A \leq X \leq 2P(z)^A + \big(A\log P(z)\big)^\tau$.
- The length of such intervals is as most

$$\big(A\log P(z)\big)^\lambda \leq (\log X)^\lambda.$$

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- The length of such intervals is as most

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- Thus it follows that there are arbitrarily large X such that

$$\begin{aligned}\pi(X + (\log X)^\lambda) &- \pi(X)\\ &\geq e^{C_0}\omega(\tau)(\log X)^{\lambda-1}\big(1 + O\big(\exp(-A/\kappa)\big)\big).\end{aligned}$$

- In the opposite direction, there are rows with at most

$$\frac{(A\log(P(z)))^\tau e^{C_0}\omega(\tau)}{A\log P(z)}\big(1+O\big(\exp(-A/\kappa)\big)\big)$$

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- The choice $N=\left\lfloor \frac{(A\log(P(z)))^\tau}{\big(\log\big(2P(z)^A+\big(A\log P(z)\big)^\tau\big)\big)^\lambda}\right\rfloor$ produces intervals $\left(X, X+\frac{(A\log(P(z)))^\tau}{N}\right]$ of length at least $\big(\log\big(2P(z)^A+(A\log P(z))^\tau\big)\big)^\lambda \ge (\log X)^\lambda$ containing $\le e^{C_0}\omega(\tau)(\log X)^{\lambda-1}(1+O(\exp(-A/\kappa)))$ primes.

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- Thus it follows that there are arbitrarily large X such that

$$\pi(X+(\log X)^\lambda)-\pi(X)\le e^{C_0}\omega(\tau)(\log X)^{\lambda-1}(1+O(\exp(-A/\kappa))).$$

The theorem now follows.

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