Math 571 Chapter 6 The Bombieri-Vinogradov Theorem

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- and define

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  which counts the prime powers $p^k \le x$ with $p^k \equiv a \pmod p$ with weight $\log p$.
- The higher powers of primes contribute a relatively small amount, and

$$\vartheta(x;q,a) = \psi(x;q,a) + O(x^{\frac{1}{2}})$$

  where $\vartheta(x;q,a) = \sum_{\substack{p \le x \\ p \equiv a \pmod q}} \log p.$

- All the main theorems stated here can be restated with $\psi(x;q;a)$ replaced by $\vartheta(x;q,a)$ or

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- Note that

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- The main reason for preferring Λ is that it arises naturally as the coefficient in the Dirichlet series expansion of the logarithmic derivative of

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

viz.

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

when $\Re s > 1$.

- The best general estimate we have for an individual pair q, a, which is uniform in q, is the

Theorem 1

Siegel [1935]–Walfisz [1936] Theorem Suppose that $A > 0$ is a fixed real number. When $(a, q) = 1$ and $q \le (\log x)^A$ we have

$$\psi(x; q, a) = \frac{x}{\phi(q)} + O_A\Big(x \exp\big(-c_1\sqrt{\log x}\big)\Big)$$

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where c_1 is an absolute positive constant.

- It is possible to extend the range for q, but weaken the error term, and be forced to include extra terms from zeros close to 1 which could almost cancel the main term, but the above is the most convenient balanced result we have.

- Let χ denote a Dirichlet character modulo q and put

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- Then, by orthogonality

$$\psi(x;q,a) = \frac{1}{\phi(q)} \sum_{\chi \pmod q} \overline{\chi}(a)\psi(x;\chi), \tag{1}$$

and clearly

$$\psi(x;\chi) = \sum_{a=1}^{q} \chi(a)\psi(x;q,a).$$

- The proof of the above is established by applying the following.

Theorem 2

Siegel–Walfisz Theorem variant Suppose that $A > 0$ is a fixed real number. When $q \le (\log x)^A$ and χ is a Dirichlet character modulo q we have

$$\psi(x;\chi) - \delta(\chi)x \ll_A x\exp\left(-c_1\sqrt{\log x}\right)$$

where c_1 is an absolute positive constant and $\delta(\chi)$ is 1 or 0 according as χ is principal or non-principal.

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- Good references for these two results are Davenport [2000] or Estermann [1952] or Montgomery and Vaughan [2006].

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- This has an analytic continuation to $\mathbb{C}$, and is entire except when χ is principal, in which case it is analytic except at 1 where it has a simple pole with residue

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$$\frac{\phi(q)}{q}.$$
- Indeed,
$$L(s,\chi_0) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right).$$

- GRH states that $L(s,\chi) \neq 0$ when $\Re s > \frac{1}{2}$.

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- If GRH holds for $L(s,\chi)$, then we know (Titchmarsh [1930]) that

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- We can compare this with

Theorem 3

Bombieri [1965] For any fixed positive number A,

$$\sum_{q\le Q} \max_{(a,q)=1} \sup_{y\le x} \left|\psi(y;q,a) - \frac{y}{\phi(q)}\right|$$
$$\ll_A x(\log x)^{-A} + x^{1/2}Q(\log xQ)^4.$$

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- Also, apart from the log power there is no known way in general of improving the crucial term $x^{1/2}Q(\log xQ)^4$ even if one assumes GRH.
- Something can be done if one fixes a for all q, replaces y by x or does not take absolute values, but such results are of limited applicability.

- The crude estimate $(x/q+1)\log x$ for each term gives
$$x(\log xQ)^2 + Q\log x$$
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- All the proofs of the above start off the same way.
- One observes that, by (1),
$$\left|\psi(y;q,a) - \frac{y}{\phi(q)}\right| \le \frac{1}{\phi(q)} \sum_{\chi \pmod q} |\psi(y;\chi) - \delta(\chi)y|$$

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$$\sum_{q\le Q} \frac{1}{\phi(q)} \sum_{\chi \pmod q} \sup_{y\le x} |\psi(y;\chi) - \delta(\chi)y| \qquad (2)$$

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- This already throws away some cancellation in the summation over χ. Almost certainly any improvements will have to make some use of it.

- When χ is induced by the primitive character χ^*, so that the conductor q^* divides q we have

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- The error term here is $\ll (\log q)\log y$ and so (2) is
$$= \sum_{q\le Q} \frac{1}{\phi(q)} \sum_{q^*|q} \sum_{\chi^* \pmod{q^*}}^{*} \sup_{y\le x} |\psi(y;\chi^*) - \delta(\chi^*)y| + O\left(Q(\log Q)(\log x)\right)$$
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- The error term here is more than acceptable, and on interchanging the order of summation and replacing q by $q^* r$, the main term becomes

$$\sum_{q^*\le Q} \sum_{r\le Q/q^*} \frac{1}{\phi(q^* r)} \sum_{\chi \pmod{q^*}}^{*} \sup_{y\le x} |\psi(y;\chi) - \delta(\chi)y|. \tag{3}$$

- Now

$$\frac{1}{\phi(q^*r)} \le \frac{1}{\phi(q^*)\phi(r)}$$

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- Then the sum is $\sum_{r\le Q} \mu(r)^2 r^{-2} \sum_{m\le Q/r} \frac{1}{m}$.
- Hence, on replacing q^* by q (9) is
$$\ll \sum_{q\le Q} \frac{\log 2Q}{\phi(q)} \sum\nolimits^*_{\chi \pmod q} \sup_{y\le x} |\psi(y;\chi) - \delta(\chi)y|.$$
Let $R = (\log x)^{6+A}$.

- Then, by the variant Siegel–Walfisz theorem we have

$$\sum_{q\le R} \frac{\log 2Q}{\phi(q)} \sum_{\chi \pmod q}^{*} \sup_{y\le x} |\psi(y;\chi) - \delta(\chi)y| \ll_A (\log x) Rx \exp\big(-c_2\sqrt{\log x}\big)$$

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- If $y \le \sqrt{x}$, then we get the conclusion at once.
- If $\sqrt{x} \le y \le x$, then the conditions of the Siegel-Walfsiz theorem are satisfied, possibly with a slightly large value of A.
- Hence

$$\sum_{q\le R}\frac{\log 2Q}{\phi(q)}\sum_{\chi \pmod q}^{*}\sup_{y\le x}|\psi(y;\chi)-\delta(\chi)y| \ll_A x(\log x)^{-A},$$

which is acceptable.

- Everything so far is classical and could have been done in 1935.

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- By definition $\delta(\chi) = 0$ for primitive characters with conductor $q > 1$. Thus it remains (!) to deal with the sum

$$(\log 2Q) \sum_{R<q\le Q} \frac{1}{\phi(q)} \sum_{\chi \pmod q}{}^{*} \sup_{y\le x} |\psi(y;\chi)|\,.$$

- The essential extra ingredient is the following

Theorem 4 (Basic Mean Value Theorem)

Let

$$T(x,Q) = \sum_{q\le Q} \frac{q}{\phi(q)} \sum_{\chi \pmod q}^{*} \sup_{y\le x} |\psi(y;\chi)|$$

where $\sum^*$ indicates that the sum is over primitive characters modulo q, and suppose that $Q\ge 1$, $x\ge 2$. Then

$$T(x,Q) \ll \left(x + x^{5/6}Q + x^{1/2}Q^2\right)(\log xQ)^3.$$

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$$T(x,Q)\ll\left(x+x^{5/6}Q+x^{1/2}Q^2\right)(\log xQ)^3.$$

- We remark in passing that by working harder it is possible to replace the middle term by $x^{4/5}Q$.
- The desired conclusion now follows from the above by partial summation.

- To see this, let

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- Then the sum in question is $(\log 2Q) \sum_{R<q\le Q} f(q) =$

$$(\log 2Q) \sum_{R<q\le Q} f(q)q \left(\frac{1}{Q} + \int_q^Q \frac{dt}{t^2} \right) =$$

$$\frac{\log 2Q}{Q} \sum_{R<q\le Q} qf(q) + (\log 2Q) \int_R^Q \sum_{R<q\le t} qf(q) \frac{dt}{t^2}$$

$$\le (\log 2Q) Q^{-1} T(x,Q) + \int_R^Q t^{-2} T(x,t) dt.$$

- We have

$$(\log 2Q) \sum_{R<q\le Q} f(q)$$

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- By the Basic Mean Value Theorem this is

$$\ll Q^{-1}\left(x + x^{5/6}Q + x^{1/2}Q^2\right)(\log x)^4$$
$$+ \int_R^Q t^{-2}\left(x + x^{5/6}t + x^{1/2}t^2\right)(\log x)^4 dt$$
$$\ll \left(xR^{-1} + x^{5/6}\log(2Q/R) + x^{1/2}Q\right)(\log x)^4.$$

- We have

$$(\log 2Q) \sum_{R<q\le Q} f(q) \le (\log 2Q)Q^{-1}T(x,Q) + \int_R^Q t^{-2}T(x,t)dt.$$

- By the Basic Mean Value Theorem this is

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$$\ll \left(xR^{-1} + x^{5/6}\log(2Q/R) + x^{1/2}Q\right)(\log x)^4.$$

- We recall our choice $R = (\log x)^{6+A}$ to conclude that

$$(\log 2Q) \sum_{R<r\le Q} f(r) \ll x(\log x)^{-A} + x^{1/2}Q(\log x)^4$$

as required.

von Mangoldt

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- Thus we want to convert our sums involving $\Lambda(n)$ into double sums.
- One, possibly naive, way of doing this is *via* the formula

$$\Lambda(n) = \sum_{lm=n} \mu(l) \log m$$

so that, for example,

$$\sum_{n \le x} \Lambda(n) f(n) = \sum_{l \le x} \sum_{m \le x/l} \mu(l)(\log m) f(lm)$$

and we would think of $\mu(l)$ and $\log m$ as being values of the variables in the bilinear form and f(lm) as being the coefficient of the bilinear form.

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and we would think of $\mu(l)$ and $\log m$ as being values of the variables in the bilinear form and f(lm) as being the coefficient of the bilinear form.
- The first person to successfully attack such a problem was I. M. Vinogradov [1937].

- Vinogradov needed to bound $\sum_{p\le x} e(p\alpha)$.

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- It was while examining Vinogradov's methods that RCV[1977] found a way of dealing with

$$\sum_{n\le x} \Lambda(n)e(n\alpha)$$

which was intrinsically more direct, and focussed towards the available information on bilinear forms.

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- Here we should think of the c_{mn} as oscillating and potentially giving some cancellation.

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- Here we should think of the c_{mn} as oscillating and potentially giving some cancellation.
- Typical examples are additive or multiplicative characters.
- It is useful to divide bilinear forms into two categories.

- **Type I**. In these one of the variables is smooth, ideally always 1, such as

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- **Type II**. In these we are not lucky enough to find that one of the variables is congenial. One needs to use quite general bounds, such as those provided by the large sieve.

- To illustrate this let us look at the bound provided by Lemma 5.5. For sake of argument, lets suppose that $MN \asymp x$, and

$$\sum_m |a_m|^2 \ll M, \quad \sum_n |b_n|^2 \ll N.$$

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- and this is a good bound (cf BMVT) provided that M and N are both large (or equivalently M is large but not too close to x).
- In effect we are saying that the rectangular coefficient matrix (c_{mn}) should not be too "thin".

- We can "partition" Λ so as to obtain "good" bilinear type I and II forms.

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Lemma 5

Suppose $u>0$, $v>0$, $y\ge 2$ *and* $f:\mathbb{N}\to\mathbb{C}$. *Then* $\sum_{n\le y}\Lambda(n)f(n)=S_1-S_2-S_3+S_4$ *where*

$$S_1=\sum_{m\le u}\mu(m)\sum_{n\le y/m}(\log n)f(mn),$$

$$S_2=\sum_{m\le uv}c_m\sum_{n\le y/m}f(mn)\text{ where } c_m=\sum_{\substack{k\le u,l\le v\\ kl=m}}\Lambda(k)\mu(l),$$

$$S_3=\sum_{m>u}\sum_{\substack{n>v\\ mn\le y}}\Big(\sum_{\substack{k|m\\ k>u}}\Lambda(k)\Big)\mu(n)f(mn),\quad S_4=\sum_{n\le v}\Lambda(n)f(n).$$

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- Presumably the number of terms in S_4 will be relatively small so it can be bounded trivially.

- **Proof** Consider the identity

$$-\frac{\zeta'}{\zeta}(s) = \big(-\zeta'(s)\big)G(s) - F(s)G(s)\zeta(s) - \big(-\zeta'(s) - F(s)\zeta(s)\big)\left(G(s) - \frac{1}{\zeta(s)}\right) + F(s)$$

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- Multiply by $f(n)$ and sum over n. By inspection of each of the Dirichlet series $D_j(s)$ we can see that each S_j satisfies $S_j = \sum_n \Lambda_j(n)f(n)$. $\square$

- We now return to the proof of the theorem, that is, we bound

$$T(x,Q)=\sum_{q\le Q}\frac{q}{\phi(q)}\sum_{\chi \pmod q}^{*}\sup_{y\le x}|\psi(y;\chi)|$$

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- If $Q^2>x$, then using Lemma 5.6 directly with $M=1$, $a_1=1$, $N=\lfloor x\rfloor$, $b_n=\Lambda(n)$ gives the bound

$$\ll\left(Q^2(x+Q^2)\sum_{n\le x}\Lambda(n)^2\log x\right)^{\frac12}\ll x^{\frac12}Q^2\log Qx.$$

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- In view of Lemma 5 with $f(n)=\chi(n)$ when $n\le y$ and $f(n)=0$ otherwise it then suffices to bound

$$T_j=\sum_{q\le Q}\frac{q}{\phi(q)}\sum_{\chi}{}^{*}\sup_{u^2\le y\le x}|S_j(\chi)|$$

for $j=1,2,3,4$.

- The case $j = 4$ is easy since

$$S_4(\chi) = \sum_{n \le u} \chi(n)\Lambda(n) = \psi(u; \chi)$$

and $u \le u^2$, and so we can appeal to (4).

- The expression T_1 is also fairly easy, since log is smooth and

$$S_1(\chi) = \sum_{m\le u} \mu(m)\chi(m) \sum_{n\le y/m} \chi(n) \int_1^n \frac{dt}{t}$$

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- This together with the trivial bound $x(\log x)^2$ for the term $q = 1$ gives

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- Note that here the upper limit x/M is never smaller than y/m and will only come into play after we have used Lemma 5.6 to remove the condition $mn \le y$.

- It follows now that $T_3 \le \sum_{M \in \mathcal{M}} T_3(M)$ where

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- We have

$$T_3(M) \ll (\log x)^2 \left(x + \frac{x}{M^{1/2}} Q + x^{\frac12} M^{\frac12} Q + x^{\frac12} Q^2 \right).$$

- Now we sum over $m \in \mathcal{M}$ to obtain

$$T_3 \ll (\log x)^3 \left(x + x u^{-1/2} Q + x^{1/2} Q^2 \right).$$

- Recall that $u = Q^2$, $u = x^{1/3}$, $u = xQ^{-2}$ according as $Q \le x^{1/6}$, $x^{1/6} < Q \le x^{1/3}$ and $Q > x^{1/3}$.
- When $Q \le x^{1/6}$, we have $xu^{-1/2}Q = x$.
- When $x^{1/6} < Q \le x^{1/3}$, we have $xu^{-1/2}Q = x^{5/6}Q$.
- When $x^{1/3} < Q \le x^{1/2}$, we have $xu^{-1/2}Q = x^{1/2}Q^2$.

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- Note that $|c_m| = \left| \sum_{\substack{k\le u, l\le u \\ kl=m}} \mu(k)\Lambda(l) \right| \le \sum_{l|m} \Lambda(l) = \log m$ and completes the proof of Bombieri's theorem.

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