MATH 571, SPRING 2023, PROBLEMS 10

Due Monday 27th March

This is a continuation of the previous two homeworks. Suppose below that α is a real number, $a \in \mathbb{Z}$, $q \in \mathbb{N}$ with $(a,q) = 1$ and $|\alpha - a/q| \le q^{-2}$, $X \ge 2$ and $1 < u < \sqrt{X}$. We assume also that

$$S = S(\alpha) = \sum_{n \le X} \Lambda(n) e(\alpha n) = S_1 + S_2 - S_3 + S_4,$$

$$S_1 = \sum_{m > u} \sum_{u < n \le X/m} a_m \mu(n) e(\alpha m n), \quad S_2 = \sum_{m \le u} \mu(m) \sum_{n \le X/m} (\log n) e(\alpha m n),$$

$$S_3 = \sum_{m \le u^2} c_m \sum_{n \le X/m} e(\alpha m n), \quad S_4 = \sum_{n \le u} \Lambda(n) e(\alpha n),$$

$$a_m = \sum_{\substack{k \mid m \\ k > u}} \Lambda(k), \quad c_m = \sum_{k \le u} \sum_{\substack{l \le u \\ kl = m}} \Lambda(k)\mu(l).$$

1. Prove that (i) $0 \le a_m \le \log m$, (ii) $|c_m| \le \log m$.
2. Let $\mathcal{M} = \{2^j u : 0 \le j,\ 2^j \le X u^{-2}\}$ and write $S_1 = \sum_{M \in \mathcal{M}} T(M)$ where

$$T(M) = \sum_{M < m \le 2M} \sum_{u < n \le X/m} a_m \mu(n) e(\alpha m n).$$

Prove that (homework 9 is useful here)

$$S_1 \ll \sum_{M \in \mathcal{M}} \left(M (\log X)^2\right)^{\frac{1}{2}} (X/M)^{\frac{1}{2}} \left(Xq^{-1} + M + X/M + q\right)^{\frac{1}{2}} (\log X)^{1/2}$$
$$\ll (Xq^{-1/2} + Xu^{-1/2} + X^{1/2} q^{1/2})(\log X)^{5/2}.$$

3. Prove that $S_2 = \displaystyle\int_1^X \sum_{m \le \min(u, X/v)} \mu(m) \sum_{v < n \le X/m} e(\alpha m n) \frac{dv}{v}$ and hence that

$$S_2 \ll (Xq^{-1} + u + q)(\log X)^2.$$

The results of homework 8 are useful here and in the next question.

4. Prove that $S_3 \ll (Xq^{-1} + u^2 + q)(\log X)^2$.
5. Prove that $S \ll (Xq^{-1/2} + X^{4/5} + X^{1/2} q^{1/2})(\log X)^{5/2}$.