MATH 571, SPRING 2023, PROBLEMS 9

Due Monday 20th March

Suppose throughout that $a_1, a_2, \dots, b_1, b_2, \dots$ are complex numbers and $\alpha \in \mathbb{R}$, and given $M \le X$ write $N = \lfloor X/M \rfloor$ define

$$T = \sum_{M<m\le 2M} \sum_{n \le X/m} a_m b_n e(\alpha m n).$$

1. Prove that

$$\begin{aligned}
|T|^2 &\le \left(\sum_{M<m\le 2M} |a_m|^2 \right) \sum_{n_1 \le N} \sum_{n_2 \le N} b_{n_1} \overline{b_{n_2}} \sum_{M<m\le \min(2M, X/n_1, X/n_2)} e(\alpha m (n_1 - n_2)) \\
&\le \left(\sum_{M<m\le 2M} |a_m|^2 \sum_{n=1}^{N} |b_n|^2 \right) \max_{n_1 \le N} \sum_{n_2=1}^{N} \left| \sum_{M<m\le \min(2M, X/n_1, X/n_2)} e\big(\alpha m (n_1 - n_2)\big) \right| \\
&\ll \left(\sum_{M<m\le 2M} |a_m|^2 \sum_{n=1}^{N} |b_n|^2 \right) \max_{n_1 \le N} \sum_{n_2=1}^{N} \min\left(M, \frac{X}{n_1}, \frac{X}{n_2}, \frac{1}{\|\alpha(n_1 - n_2)\|} \right) \\
&\ll \left(\sum_{M<m\le 2M} |a_m|^2 \sum_{n=1}^{N} |b_n|^2 \right) \left(M + \sum_{h=1}^{N} \min\left(\frac{X}{h}, \frac{1}{\|\alpha h\|} \right) \right)
\end{aligned}$$

2. Suppose further that there are $q \in \mathbb{N}$, $a \in \mathbb{Z}$ with $(q,a) = 1$ such that $|\alpha - a/q| \le q^{-2}$. Prove that

$$|T|^2 \ll \left(\sum_{M<m\le 2M} |a_m|^2 \sum_{n \le X/M} |b_n|^2 \right) \big(X q^{-1} + M + X/M + q \big) \log(2X).$$

The bound that was obtained for $\sum_{m \le y} \min(x/m, 1/\|\alpha m\|)$ in last week's homework is useful here.