MATH 571, SPRING 2023, PROBLEMS 8

Due Monday 13th March

1. Suppose α, x, $y \in \mathbb{R}$ with $x \geq 1$, $y \geq 1$ and that there is an $A \in \mathbb{R}$ such that for $m \leq y$ the complex numbers a_m satisfy $|a_m| \leq A$. $\|\alpha\| = \min_{n\in\mathbb{Z}} |\alpha - n|$.

(i). Prove the triangle inequality $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$.

(ii) Prove that

$$\sum_{m\leq y} a_m \sum_{n\leq x/m} e(\alpha mn) \ll A \sum_{m\leq y} \min\left(\frac{x}{m}, \frac{1}{\|\alpha m\|}\right).$$

Suppose further that there are $q \in \mathbb{N}$, $a \in \mathbb{Z}$ with $(q,a) = 1$ such that $|\alpha - a/q| \leq q^{-2}$. When $1 \leq m \leq y$ put $m = hq + j$ where $-q/2 < j \leq q/2$ so that $0 \leq h \leq \frac{y}{q} + \frac{1}{2}$ and $j > 0$ when $h = 0$.

(iii) Suppose $h = 0$. Prove that $\|\alpha m\| \geq \|aj/q\| - \frac{1}{2q} \geq \frac{1}{2}\|aj/q\|$. Deduce that

$$\sum_{m\leq \min(y,q/2)} \min\left(\frac{x}{m}, \frac{1}{\|\alpha m\|}\right) \ll \sum_{1\leq j\leq \min(y,q/2)} \|aj/q\|^{-1} \ll \sum_{l\leq \min(y,q)} \frac{q}{l} \ll q\log 2y.$$

(iv) Suppose $h > 0$. Put $\beta = q^2(\alpha - a/q)$ and let k be a nearest integer to βh. Prove that if

$$\left\|\frac{ja+k}{q}\right\| \geq \frac{2}{q},$$

then

$$\|\alpha m\| \geq \frac{1}{2}\left\|\frac{ja+k}{q}\right\|.$$

Deduce that

$$\sum_{hq-q/2<m\leq hq+q/2} \min\left(\frac{x}{m}, \frac{1}{\|\alpha m\|}\right) \ll \frac{x}{hq} + \sum_{l=1}^{q-1} \frac{q}{l} \ll \frac{x}{hq} + q\log 2q.$$

(v) Prove that

$$\sum_{m\leq y} a_m \sum_{n\leq x/m} e(\alpha mn) \ll \left(\frac{x}{q} + y + q\right) A\log 2x.$$