MATH 571, ANALYTIC NUMBER THEORY, SPRING 2023, PROBLEMS 7

Due Monday 27th February

1. (Gallagher) Suppose that α, $\delta \in \mathbb{R}$ with $\delta > 0$, and suppose that f has a continuous derivative on $[\alpha - \delta, \alpha + \delta]$.

(i) Prove that $\displaystyle\int_0^\delta (\delta - \beta)\big(f'(\alpha + \beta) - f'(\alpha - \beta)\big)d\beta = -2\delta f(\alpha) + \int_{\alpha-\delta}^{\alpha+\delta} f(\gamma)d\gamma.$

(ii) Prove that $\displaystyle 2\delta|f(\alpha)| \le \int_{\alpha-\delta}^{\alpha+\delta} |f(\alpha)|d\alpha + \delta\int_{\alpha-\delta}^{\alpha+\delta} |f'(\alpha)|d\alpha.$

(iii) Suppose that $x_r \in \mathbb{R}$ $(1 \le r \le R)$, $\delta > 0$, $x_r + \delta \le x_{r+1}$, $x_R + \delta \le x_1 + 1$. Let $\tau = x_1 - \delta/2$. Prove that $\displaystyle\sum_{r=1}^R |f(x_r)| \le \delta^{-1}\int_\tau^{1+\tau} |f(\alpha)|d\alpha + \frac{1}{2}\int_\tau^{1+\tau} |f'(\alpha)|d\alpha.$

(iv) Suppose that $K \in \mathbb{N}$ and $b_k \in \mathbb{C}$ $(-K \le k \le K)$. Let $G(\alpha) = \displaystyle\sum_{k=-K}^K b_k e(\alpha k)$.

Prove that

$$\int_0^1 |G(\alpha)|^2 d\alpha = \sum_{k=-K}^K |b_k|^2, \quad \int_0^1 |G'(\alpha)|^2 d\alpha \le 4\pi^2 K^2 \sum_{k=-K}^K |b_k|^2.$$

(v) Prove that $\displaystyle\sum_{r=1}^R |G(x_r)|^2 \le (\delta^{-1} + 2\pi K)\sum_{k=-K}^K |b_k|^2.$

(vi) Let $M \in \mathbb{Z}$, $N \in \mathbb{N}$, $a_n \in \mathbb{C}$ $(M+1 \le n \le M+N)$ and $S(\alpha) = \displaystyle\sum_{n=M+1}^{M+N} a_n e(n\alpha)$.

Prove that $\displaystyle\sum_{r=1}^R |S(x_r)|^2 \le (\delta^{-1} + \pi N)\sum_{n=M+1}^{M+N} |a_n|^2.$

Hint: Choose $N_0 = \lfloor (N+1)/2 \rfloor$ and replace n by $k = n - N_0 - M$.

2. Let K, $R \in \mathbb{N}$, $M \in \mathbb{Z}$, $N = KR + 1$, $x_r = r/R$ $(1 \le r \le R)$, $a_n = 1$ when $n \equiv M + 1 \pmod{R}$ and $a_n = 0$ otherwise. Show that

$$\sum_{r=1}^R \left|\sum_{n=M+1}^{M+N} a_n e(nx_r)\right|^2 = (N - 1 + 1/\delta)\sum_{n=M+1}^{M+N} |a_n|^2 \text{ where } \delta = \min_{r \ne s} \|x_r - x_s\|.$$

E. Bombieri & H. Davenport, Some inequalities involving trigonometrical polynomials, Annali Sc. Norm. Pisa, 23(1969), 223–241.