Math 571 Analytic Number Theory, Spring 2023, Problems 5

Return by Monday 13th February

Let
$$\mathcal{A}(X,Y)=\{n\le X : p|n\Rightarrow p\le Y\},$$
sometimes called "Y-factorable or Y-smooth numbers", and let
$$\psi(X,Y)=\operatorname{card}\mathcal{A}(X,Y)$$
be their counting function.

1. Suppose that u and v are real numbers with $1\le u\le v$. Prove the following variant of Buchstab's identity
$$\psi\big(X,X^{1/v}\big)=\psi\big(X,X^{1/u}\big)-\sum_{X^{1/v}<p\le X^{1/u}}\psi(X/p,p).$$
Hint: Sort the elements of
$$\mathcal{A}\big(X,X^{1/v}\big)\backslash\mathcal{A}\big(X,X^{1/u}\big)$$
according to their largest prime factor.

2. Suppose that $u\le 2$ and $u\ne 0$. Prove that
$$\psi(X,X^{1/u})=\begin{cases}1 & (u<0),\\ X+O(1) & (0<u\le 1),\\ (1-\log u)X+O(X/\log X) & (1<u\le 2).\end{cases}$$
One might guess that there is a function $\rho(\alpha)$ such that if $1\le u\le v$, then
$$\rho(v)=\rho(u)-\int_{X^{1/v}}^{X^{1/u}}\rho\left(\frac{\log(X/\alpha)}{\log\alpha}\right)\frac{d\alpha}{\alpha\log\alpha}=\rho(u)-\int_u^v\frac{\rho(\beta-1)}{\beta}d\beta$$
and that, at least for fixed u,
$$\psi(X,X^{1/u})\sim\rho(u)X.$$

Thus we define $\rho(u):\mathbb{R}\to\mathbb{R}$ to be continuous for all $u\ne 0$, differentiable for all $u\ne 0,1$ and to satisfy
$$\rho(u)=\begin{cases}0 & (u\le 0),\\ 1 & (0<u\le 1),\\ u\rho'(u)=-\rho(u-1) & (u>1).\end{cases}$$

3. (i) Prove that ρ is positive and strictly decreasing on $[1,\infty)$.

(ii) Prove that $\rho(u)\le u^{-1}$.

By working harder one can show that $\rho(u)\ll\exp\big(-u(\log u-1)\big)$.

It is also possible to prove using only elementary prime number theory that there is a real number $B > 1$ such that for all $X \geq 1$ and $u > 0$,

$$\bigl|\psi\bigl(X, X^{1/u}\bigr) - X\rho(u)\bigr| \leq B^u X(\log 2X)^{-1}.$$

In particular one can use questions 1-3 and induction on $n = 0, 1, 2, \ldots$ to show that

$$\bigl|\psi\bigl(X, X^{1/u}\bigr) - X\rho(u)\bigr| \leq B^n X(\log 2X)^{-1} \quad (n < u \leq n+1).$$

There is quite a bit of detail. For those interested in doing research in analytic number trheory it is a useful example of what one can get in to.