MATH 571 ANALYTIC NUMBER THEORY, SPRING 2023, PROBLEMS 4

Return by Monday 6th February

1. (i) Prove that every positive integer can be written uniquely as the product of a squarefree number and a perfect square.

(ii) Let $\Omega(n)$ denote the total number of prime factors of n and let

$$S(X) = \sum_{n \le X} (-1)^{\Omega(n)}.$$

Prove that if $X \ge 2$, then

$$S(X) = \sum_{uv^2 \le X} \mu(u)^2 (-1)^{\Omega(u)} = \sum_{uv^2 \le X} \mu(u) = \sum_{v \le \sqrt{X}} \sum_{u \le X/v^2} \mu(u).$$

(iii) By splitting the sum over v at, say $X^{1/3}$ and applying the bound

$$\sum_{n \le Y} \mu(n) \ll Y \exp(-c\sqrt{\log Y})$$

from prime number theory (Math 568), or otherwise, deduce that

$$\sum_{n \le X} (-1)^{\Omega(n)} \ll X \exp(-c' \sqrt{\log X}).$$

(iv) [Selberg's example]. Prove that if $\theta = \pm 1$ and $d \le \sqrt{X}$, then

$$\sum_{n \le X, d|n} \left(1 + \theta(-1)^{\Omega(n)}\right) = \frac{X}{d} + O\left(\frac{X}{d} \exp(-c'' \sqrt{\log X})\right).$$

If $\theta = 1$, then the sum only counts n with an even number of prime factors. On the other hand if $\theta = -1$ it only counts n with an odd number of prime factors. Thus the input to the sieve cannot distinguish between on odd number or an even number of prime factors. Selberg used this to show that the upper bound factor 2 in the Brun-Titchmarsh theorem cannot be improved by sieve theory alone.

2. Let $\mathcal{P}$ be a set of primes, let $P(z) = \prod_{p \in \mathcal{P}, p \le z} p$, and define

$$S(\mathcal{A}, \mathcal{P}, z) = \sum_{n, (n, P(z))=1} a(n).$$

Also let p_- denote $p - 1$. Prove that if $z < w$, then [Buchstab]

$$S(\mathcal{A}, \mathcal{P}, z) = \sum_{z < p \le w} S(\mathcal{A}_p, \mathcal{P}, p_-) + S(\mathcal{A}, \mathcal{P}, w).$$

Hint. Sort the terms in $S(\mathcal{A}, \mathcal{P}, z) - S(\mathcal{A}, \mathcal{P}, w)$ according to the smallest prime factor $p \in \mathcal{P}$ of n.