Math 571 Analytic Number Theory, Spring 2023, Problems 3

Due Monday 30th January

Let $e(\alpha) = \exp(2\pi i\alpha)$ and $c_q(m)$ denote Ramanujan's sum $c_q(m) = \displaystyle\sum_{\substack{r=1\\(r,q)=1}}^{q} e\left(\frac{m}{q}r\right)$.

1. (i) Let $f : \mathbb{N} \to \mathbb{C}$. Prove that $\displaystyle\sum_{\substack{r=1\\(r,q)=1}}^{q} f(r) = \sum_{d|q} \mu(d) \sum_{s=1}^{q/d} f(ds)$.

(ii) Prove that $\displaystyle\sum_{r=1}^{q} e\left(\frac{m}{q}r\right) = \begin{cases} q & \text{when } q|m, \\ 0 & \text{when } q \nmid n. \end{cases}$

(iii) Prove that $c_q(m) = \displaystyle\sum_{k|(q,m)} \mu(q/k)k$.

(iv) Prove that for m fixed, $c_q(m)$ is a multiplicative function of q.

(v) Prove that $c_q(m) = \dfrac{\phi(q)\mu(q/(q,m))}{\phi(q/(q,m)}$.

2. (Hooley (1972), Montgomery & Vaughan (1979)) By lower and upper bound sifting functions we mean functions $\lambda^{\pm} : \mathbb{N} \to \mathbb{R}$ with the properties

$$\sum_{m|n} \lambda_m^- \le \sum_{m|n} \mu(m) \le \sum_{m|n} \lambda_m^+.$$

(i) Let λ_d^+ be an upper bound sifting function such that $\lambda_d^+ = 0$ for all $d > z$. Show that for any q,

$$0 \le \frac{\varphi(q)}{q} \sum_{\substack{d\\(d,q)=1}} \frac{\lambda_d^+}{d} \le \sum_{d} \frac{\lambda_d^+}{d}.$$

(Hint: Multiply both sides by $P/\varphi(P) = \sum 1/m$ where m runs over all integers composed of the primes dividing P, and $P = \prod_{p\le z} p$.)

(ii) Let η_d be real with $\eta_d = 0$ for $d > z$. Show that for any q,

$$0 \le \frac{\varphi(q)}{q} \sum_{\substack{d,\, e\\(de,q)=1}} \frac{\eta_d\eta_e}{[d,e]} \le \sum_{d,\, e} \frac{\eta_d\eta_e}{[d,e]}.$$

(iii) Let λ_d^- be a lower bound sifting function such that $\lambda_d^- = 0$ for $d > z$. Show that for any q,

$$\frac{\varphi(q)}{q} \sum_{\substack{d\\(d,q)=1}} \frac{\lambda_d^-}{d} \ge \sum_{d} \frac{\lambda_d^-}{d}.$$