MATH 571 ANALYTIC NUMBER THEORY, SPRING 2023, PROBLEMS 2

Return by Monday 23rd January

1. Let $\sigma(n)=\sum_{m|n} m$. Prove that, for every $n\in\mathbb{N}$ we have

$$\sum_{m|n}\mu(m)\sigma(n/m)=n.$$

2. For each $n\in\mathbb{N}$, let $f(n)=\sum_{m|n} d(m)^3$ and $g(n)=\sum_{m|n} d(m)$.
(i) Prove that $f\in\mathcal{M}$ and $g\in\mathcal{M}$.
(ii) Prove that

$$\sum_{j=0}^{k}(j+1)=\frac{1}{2}(k+1)(k+2).$$

(iii) Prove that

$$\sum_{j=0}^{k}(j+1)^3=\frac{1}{4}(k+1)^2(k+2)^2.$$

(iv) Prove that, for every $n\in\mathbb{N}$, $f(n)=g(n)^2$.

3. Let $\chi_1(n)=1$ when $n\equiv 1 \pmod 4$, $=-1$ when $n\equiv 3 \pmod 4$ and $=0$ otherwise. Prove that $\chi_1(n)$ is totally multiplicative.

4. Let χ be totally multiplicative, i.e. $\chi(mn)=\chi(m)\chi(n)$ for all m,n, and suppose further that for every n, $\chi(n)=\pm1$ or 0. Let

$$\rho(n)=\sum_{m|n}\chi(m).$$

(i) Prove that ρ is multiplicative.
(ii) Prove that for every n, $\rho(n)\ge 0$.
(iii) Prove that for every n, $\rho(n^2)\ge 1$.