Math 571, Spring 2023, Project
Do one of the following by Friday 28th April

1. **Project 1**

Let

$$A(n)=\sum_{\substack{p_1,p_2,p_3\le n\\ p_1+p_3=2p_2}}(\log p_1)(\log p_2)(\log p_3).$$

Since $p_3-p_2=p_2-p_1=d$ the p_j are three successive members of the arithmetic progression p_1+xd. In fact we are counting, with weight $(\log p_1)(\log p_2)(\log p_3)$ all such triples of primes not exceeding n. Note that $d<0$ and $d=0$ are allowed, so each triple with $d\ne 0$ is being counted essentially twice. The terms with $d=0$ only contribute $\pi(n)$. It is this which Green and Tao famously generalised in 2004 to k primes in a.p. The object is to show that for any fixed $B\ge 1$,

$$A(n)=\frac{C_2}{2}n^2+O_B\big(n^2(\log n)^{-B}\big) \qquad (1.1) \qquad \texttt{eq:one}$$

where C_2 is the twin prime constant $C_2=2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)$.

It is easily deduced that $\operatorname{card}\{p_1<p_2<p_3\le n : p_1+p_2=2p_3\}\sim\frac14 C_2n^2(\log n)^{-3}$.

Suggested outline:

1. In the notation of Theorem 7.6, show that $\int_{\mathfrak{m}}S(\alpha)^2S(-2\alpha)d\alpha\ll n^2(\log n)^{(7-A)/2}$.

2. Show that $\int_{\mathfrak{M}}S(\alpha)^2S(-2\alpha)d\alpha=C_2J(n)+O\big(n^2(\log n)^{1-A}\big)$ where

$$J(n)=\int_{-(\log n)^An^{-1}}^{(\log n)^An^{-1}}T(\beta)^2T(-2\beta)d\beta.$$

3. There is a problem in that $T(2\times 1/2)=n$. To get round this, prove that

$$\int_{-1/2}^{1/2}|T(2\beta)|^2d\beta=\frac12\int_{-1}^{1}|T(\beta)|^2d\beta=\int_{-1/2}^{1/2}|T(\beta)|^2d\beta=n \quad \text{and}$$

$$\int_{(\log n)^An^{-1}\le|\beta|\le 1/2}|T(\beta)^2T(-2\beta)|d\beta\ll\frac{n}{(\log n)^A}\int_{-1/2}^{1/2}|T(\beta)T(-2\beta)|d\beta\ll\frac{n^2}{(\log n)^A}.$$

4. Prove that $\int_{-1/2}^{1/2}T(\beta)^2T(-2\beta)d\beta=\operatorname{card}\{n_1,n_3\le n : 2|n_1+n_3\}=\frac12n^2+O(1)$ and

$$\int_{\mathfrak{M}}S(\alpha)^2S(-2\alpha)d\alpha=\frac12C_2n^2+O\big(n^2(\log n)^{1-A}\big).$$

5. Deduce (1.1).

2. **Project 2 (Pillai 1936)**

The object here is to prove the analogue of the Hardy and Littlewood prime k-tuples conjecture for squarefree numbers. Given a set $\mathbf{h}$ of distinct non-negative integers $h_1, \dots, h_k$, for each prime p let $\nu_p(\mathbf{h})$ denote the number of different residue classes modulo p^2 amongst them. If $\nu_p(\mathbf{h}) < p^2$, then we call $\mathbf{h}$ *sf–admissible*. Let $S(x;\mathbf{h})$ be the number of $n \le x$ so that $n+h_1, \dots, n+h_k$ are simultaneously squarefree.

Suggested outline. 1. Let $f(n)$ denote the characteristic function of the squarefree numbers. Prove that $S(x;\mathbf{h}) = \sum_{n\le x} f(n+h_1)\dots f(n+h_k)$ and $f(n) = \sum_{d^2|n} \mu(d)$.

2. Suppose that $0 < \delta < 1/(3k)$ and let $y = x^\delta$ and $f(n;y) = \sum_{\substack{d\le y\\ d^2|n}} \mu(d)$. Prove that for $j=1,\dots,k$ $S(x;\mathbf{h}) = T_j(x;y) + O(x^{1+\varepsilon}y^{-1})$ where $T_j(x;y) = \sum_{n\le x} f(n+h_1;y)\dots f(n+h_j;y)f(n+h_{j+1})\dots f(n+h_k)$.

3. Given a k–tuple of positive integers $\mathbf{d} = d_1,\dots,d_k$ let $d = d_1\dots d_k$ and given another one $\mathbf{r}$ we use $\mathbf{d}|\mathbf{r}$ to mean $d_j|r_j$ $(j=1,\dots,k)$ and $\mathbf{d}^2$ to mean $d_1^2,\dots,d_k^2$. Write $n+\mathbf{h}$ for the k–tuple $n+h_1,\dots,n+h_k$. Let $\rho(\mathbf{d})$ denote the number of solutions of $\mathbf{d}^2|n+\mathbf{h}$ in n modulo d^2. Prove that $\rho(\mathbf{d}) \le d^2$ and $T_k(x;y) = x\sum_{d_1\le y,\dots,d_k\le y} \frac{\mu(d_1)\dots\mu(d_k)}{d^2}\rho(\mathbf{d}) + O(y^{3k})$.

4. Let $\rho^*(\mathbf{d})$ denote the number of solutions of $\mathbf{d}^2|n+\mathbf{h}$ in n modulo $\mathrm{lcm}[d_1,\dots,d_k]^2$. Prove that $\rho(\mathbf{d}) = d^2\mathrm{lcm}[d_1,\dots,d_k]^{-2}\rho^*(\mathbf{d})$ and $\rho^*(\mathbf{d}) \le 1$.

5. Prove $\sum_{\max d_j>y} \frac{\mu(d_1)..\mu(d_k)}{d^2}\rho(\mathbf{d}) \ll \sum_{\max d_j>y} \frac{\mu(d_1)^2..\mu(d_k)^2}{[d_1,..,d_k]^2} \ll \sum_{m>y} \frac{2^{k\omega(m)}}{m^2} \ll y^{\varepsilon-1}$ and $T_k(x,y) = x\sum_{m=1}^{\infty} \frac{g(m)}{m^2} + O\left(xy^{\varepsilon-1}\right)$ where $g(m) = \sum_{\substack{\mathbf{d}\\ [d_1,\dots,d_k]=m}} \mu(d_1)\dots\mu(d^k)\rho^*(\mathbf{d})$.

6. Prove that $\rho(\mathbf{d})$ is multiplicative, i.e. given $\mathbf{d}$, $\mathbf{e}$, define $\mathbf{de} = d_1e_1,\dots,d_ke_k$ and deduce that if $(d,e)=1$, then $\rho(\mathbf{de}) = \rho(\mathbf{d})\rho(\mathbf{e})$.

7. Prove that $g(m)$ is multiplicative and has its support on the squarefree numbers.

8. Deduce that $\sum_{m=1}^{\infty} \frac{g(m)}{m^2} = \prod_p \left(1+g(p)p^{-2}\right)$ and $1+g(p)p^{-2} = 1-\nu_p(\mathbf{h})p^{-2}$.

9. Prove that $S(x;h) = x\prod_p \left(1-\frac{\nu_p(\mathbf{h})}{p^2}\right) + O(x^{1-\delta})$ and that if $\mathbf{h}$ is sf–admissible, then there are infinitely many n such that $n+h_j$ are simultaneously square free for $j=1,\dots,k$.