Math 568, Analytic Number Theory I, Spring 2020, Problems 10

This homework is a continuation of the previous one. Here we investigate the zero–free region for L–functions formed from quadratic characters. We suppose throughout that χ is real but non–principal and that $\rho_0 = \beta_0 + i\gamma_0$ is a zero of $L(s, \chi)$. Now we have an additional problem in that $\chi^2 = \chi_0$, so that $L(s, \chi^2)$ has a pole at $s = 1$. This homework is an example of how a simple idea which works in an original situation can be adapted and amended to deal with more awkward ones.

1. Suppose that $|\gamma_0| \geq 6(1 - \beta_0)$. Since we know that $L(1, \chi) \neq 0$ we may suppose that $\gamma_0 \neq 0$.
(i) Prove that there is a positive constant C such that if $1 < \sigma < 2$, then
$$-\Re \frac{L'}{L}(\sigma + 2i\gamma_0, \chi^2) \leq \frac{\sigma - 1}{(\sigma - 1)^2 + 4\gamma_0^2} + C \log(q(4 + |\gamma_0|)).$$
(ii) Prove that there is a positive constant C such that if $\sigma > 1$, then
$$0 \leq \frac{3}{\sigma - 1} - \frac{4}{\sigma - \beta_0} + \frac{\sigma - 1}{(\sigma - 1)^2 + 4|\gamma_0|^2} + C \log(q(4 + |\gamma_0|))$$
and deduce that $\beta_0 \neq 1$ and that $0 \leq \dfrac{3}{\sigma - 1} - \dfrac{4}{\sigma - \beta_0} + \dfrac{\sigma - 1}{(\sigma - 1)^2 + 144(1 - \beta_0)^2} + C \log(q(4+|\gamma_0|))$.
(iii) Prove that there is a positive constant c such that $\beta_0 \leq 1 - c/\log(q(4 + |\gamma_0|))$.

2. Suppose that $0 < |\gamma_0| \leq 6(1 - \beta_0)$. Note that then $\beta_0 \neq 1$. (i) Prove that $L(\beta_0 - i\gamma_0, \chi) = 0$.
(ii) Prove that if $\sigma > 1$, then $-\dfrac{L'}{L}(\sigma, \chi_0) - \dfrac{L'}{L}(\sigma, \chi) \geq 0$.
(iii) Prove that if $\sigma > 1$, then
$$\frac{1}{\sigma - \beta_0 - i\gamma_0} + \frac{1}{\sigma - \beta_0 + i\gamma_0} = \frac{2(\sigma - \beta_0)}{(\sigma - \beta_0)^2 + \gamma_0^2} \geq \frac{2(\sigma - \beta_0)}{(\sigma - \beta_0)^2 + 36(1 - \beta_0)^2}.$$
(iv) Prove that there is a positive constant C such that if $\sigma > 1$, then
$$0 \leq \frac{1}{\sigma - 1} - \frac{2(\sigma - \beta_0)}{(\sigma - \beta_0)^2 + 36(1 - \beta_0)^2} + C \log(q(4 + |\gamma_0|)).$$
(v) Prove that there is a positive constant c such that $\beta_0 \leq 1 - c/\log(q(4 + |\gamma_0|))$.

3. Suppose the $L(s, \chi)$ has two real zeros β_0 and β_1 with $\beta_0 \leq \beta_1 \leq 1$. Note that from the proof of Dirichlet's theorem we have $\beta_1 < 1$. (i) Prove that there is a positive constant C such that if $1 < \sigma < 2$, then $-\Re \dfrac{L'}{L}(\sigma, \chi) \leq -\dfrac{1}{\sigma - \beta_0} - \dfrac{1}{\sigma - \beta_1} + C \log 4q \leq -\dfrac{2}{\sigma - \beta_0} + C \log 4q$.
(ii) Prove that there is a constant $C > 0$ such that if $\sigma > 1$, then (2(ii) is useful)
$$0 \leq \frac{1}{\sigma - 1} - \frac{2}{\sigma - \beta_0} + C \log(4q).$$
(iii) Prove that there is a positive constant c such that $\beta_0 \leq 1 - c/\log(4q)$.

To summarise. The above shows that there is a region $\{s : \sigma \geq 1 - c/(\log(q(4 + |t|)))\}$ in which $L(s, \chi)$ has at most one zero, and if such a zero exists, then it is real and χ is real but non–principal. Such a zero is known as a Siegel zero. No such zero has ever been found.