MATH 568 ANALYTIC NUMBER THEORY I, SPRING 2020, PROBLEMS 9

1. Let χ denotes a non-principal character modulo q, $M, N \in \mathbb{Z}$, $x \in \mathbb{R}$, $x, N > M \geq 0$, and $S(x;\chi) = \displaystyle\sum_{M<n\leq x} \chi(n)$. For brevity define $\tau = 2 + |t|$. Also, recall that for all M, x and non–principal χ modulo q, $|S(x;\chi)| \leq q$.

(i) Prove that $\displaystyle\sum_{M+1}^{N} \chi(n)n^{-s} = S(N;\chi)N^{-s} + \int_{M+1}^{N} S(x;\chi)sx^{-s-1}dx$.

(ii) Prove that if $\sigma > 0$, then $L(s,\chi) = \displaystyle\sum_{N=1}^{M} n^{-s}\chi(n) + \int_{M+1}^{\infty} S(x;\chi)sx^{-s-1}dx$.

(iii) Let $T = \sum_{n=1}^{M} \chi(n)n^{-s}$. Prove that if $0 < \sigma < 1$, then $|T| < \frac{M^{1-\sigma}}{1-\sigma}$, if $\sigma > 1$, then $|T| < \frac{\sigma}{\sigma-1}$, and if $|\sigma - 1| \leq \frac{1}{\log M}$, then $|T| \leq 1 + e\log M$.

(iv) Prove that $\left|\displaystyle\int_{M+1}^{\infty} S(x;\chi)sx^{-s-1}dx\right| \leq |s|q(M+1)^{-\sigma}\sigma^{-1}$.

(v) Prove that if $\sigma \leq 1 - \frac{1}{\log q\tau}$, then $|L(s;\chi)| \leq (q\tau)^{1-\sigma}\left(\frac{1}{1-\sigma} + \frac{1}{\sigma}\right)$.

(vi) Prove that if $\sigma \geq 1 + \frac{1}{\log q\tau}$, then $|L(s;\chi)| \leq \frac{1}{\sigma-1} + 1$.

(vii) Prove that if $|\sigma - 1| \leq \frac{1}{\log q\tau}$, then $|L(s;\chi)| \leq 1 + e\log q\tau + e\sigma^{-1}$.

(viii) Suppose that $0 < \delta < 1$. Prove that uniformly for $\sigma \geq \delta$ we have $|L(s;\chi)| \ll \left(1 + (q\tau)^{1-\sigma}\right)\min\left(1 + \frac{1}{|1-\sigma|}, \log q\tau\right)$.

By the way, note the symmetry between the q–aspect and t–aspect of these bounds. Also that we showed in class that they hold for $\zeta(s) - \frac{1}{s-1}$ (with $q = 1$).