MATH 568, INTRODUCTION TO ANALYTIC NUMBER THEORY, SPRING 2020, PROBLEMS 7

Due Tuesday 3rd March

The whole of this homework is due to Ingham (1929). Homework 4.3 and Landau's theorem will be useful. Note that $|\sigma_{i\gamma}(n)|^2 = \sigma_{i\gamma}(n)\sigma_{-i\gamma}(n)$

1. Suppose that γ is a non-zero real number, let

$$f(s) = \sum_{n=1}^{\infty} |\sigma_{i\gamma}(n)|^2 n^{-s}$$

and let σ_c (apologies for using σ in two different ways but the notations are standard) denote the abscissa of convergence of this series.

(i) Deduce that f has an analytic continuation to $\sigma > 0$ and that it is analytic for $\sigma > \frac{1}{2}$ except possibly at $s = 1$ and $s = 1 \pm i\gamma$. Prove also that it has a removable singularity at $s = \frac{1}{2}$ and that $f(1/2) = 0$.

Henceforward suppose that $\zeta(1 + i\gamma) = 0$.

(ii) Prove that f has removable singularities at $s = 1$ and $s = 1 \pm i\gamma$ as well as $s = 1/2$, and so represents a function which is also analytic at those points.

(iii) Prove that $\sigma_c < \frac{1}{2}$.

(iv) Prove that $f(1/2) \geq 1$.

(v) Conclude that $\zeta(1 + i\gamma) \neq 0$ whenever $\gamma \neq 0$.

It is very pretty that a zero at $\frac{1}{2}$ should imply no zeros on the 1–line.

2. Let χ be a non-principal Dirichlet character and suppose that γ is a real number. By considering

$$f(s) = \sum_{n=1}^{\infty} \left| \sum_{d|n} \chi(d) d^{-i\gamma} \right|^2 n^{-s}$$

prove that $L(1 + i\gamma, \chi) \neq 0$.

This gives a uniform proof for all characters that $L(1, \chi) \neq 0$, and again a zero at $\frac{1}{2}$ implies no zeros on the 1–line.