MATH 568 INTRODUCTION TO ANALYTIC NUMBER THEORY, SPRING 2020, PROBLEMS 3

Due Tuesday 4th February

1. (i) Prove that if $x \geq 1$, then

$$\int_1^x \frac{\psi(u)}{u^2} du = \log x + O(1).$$

(ii) Prove that $\limsup_{x\to\infty} \frac{\psi(x)}{x} \geq 1$ and $\liminf_{x\to\infty} \frac{\psi(x)}{x} \leq 1$.
(iii) Prove that if there is a constant c such that $\psi(x) \sim cx$ as $x \to \infty$, then $c = 1$.
(iv) Prove that if there is a constant c such that $\pi(x) \sim c\frac{x}{\log x}$ as $x \to \infty$, then $c = 1$.

2. Suppose that $k \geq 2$ and let m be the product of the first k primes, say $m = p_1 p_2 \ldots p_k$ where $p_1 = 2$, $p_2 = 3$ and so on. Then $k = \pi(p_k)$ and $\log m = \vartheta(p_k)$.
(i) Prove that $\log p_k = \log\log m + O(1)$.
(ii) Prove that

$$k = \frac{\log m}{\log p_k} + \int_2^{p_k} \frac{\vartheta(t)dt}{t\log^2 t} = \frac{\log m}{\log\log m}\left(1 + O\left(\frac{1}{\log\log m}\right)\right)$$

and deduce that

$$\omega(m) = \frac{\log m}{\log\log m}\left(1 + O\left(\frac{1}{\log\log m}\right)\right).$$

(iii) Let $n \geq 3$. Prove that

$$\omega(n) \leq \frac{\log n}{\log\log n}\left(1 + O\left(\frac{1}{\log\log n}\right)\right)$$

and that this is essentially best possible.
Hint: Let $k = \omega(n)$ and observe that then $n \geq m$ and the functions $\frac{\log x}{(\log\log x)^r}$ $(r = 1, 2)$ are increasing for $x \geq e^{e^r}$.