Math 568 Introduction to Analytic Number Theory, Spring 2020, Problems 2

Due Tuesday 28th January

1. (cf Hille (1937)) Suppose that $f(x)$ and $F(x)$ are complex-valued functions defined on $[1,\infty)$. Prove that

$$F(x) = \sum_{n\le x} f(x/n)$$

for all x if and only if

$$f(x) = \sum_{n\le x} \mu(n)F(x/n)$$

for all x.

2. (cf Hartman & Wintner (1947)) Suppose that $\sum |f(n)|d(n) < \infty$, and that $\sum |F(n)|d(n) < \infty$. Prove that

$$F(n) = \sum_{\substack{m \\ n|m}} f(m)$$

for all n if and only if

$$f(n) = \sum_{\substack{m \\ n|m}} \mu(m/n)F(m)$$

for all n

3. Suppose that $z \in \mathbb{C}$ and $|z| < 1$. Prove that

$$\sum_{n=1}^{\infty} \frac{\mu(n)z^n}{1-z^n} = z.$$

4. (a) Let $d_n = \operatorname{lcm}[1, 2, \dots, n]$. Show that $d_n = e^{\psi(n)}$.
(b) Let $P \in \mathbb{Z}[x]$, $\deg P \le n$. Put $I = I(P) = \int_0^1 P(x)\,dx$. Show that $Id_{n+1} \in \mathbb{Z}$, and hence that $d_{n+1} \ge 1/|I|$ if $I \ne 0$.
(c) Show that there is a polynomial P as above so that $Id_{n+1} = 1$.
(d) Verify that $\max_{0\le x\le 1} |x^2(1-x)^2(2x-1)| = 5^{-5/2}$.
(e) For $P(x) = \big(x^2(1-x)^2(2x-1)\big)^{2n}$, verify that $0 < I < 5^{-5n}$.
(f) Show that $\psi(10n+1) \ge (\frac{1}{2}\log 5)\cdot 10n$.