Chapter 7

Applications of the Prime Number Theorem

We now use the Prime Number Theorem, and other estimates obtained by similar methods, to estimate the number of integers whose multiplicative structure is of a specified type.

1. Numbers composed of small primes

Let $\psi(x, y)$ denote the number of integers n, $1 \le n \le x$, all of whose prime factors are $\le y$. Obviously, if $y \ge x$ then

$$\psi(x, y) = [x] = x + O(1). \tag{1}$$

Also, if $n \le x$ then n can have at most one prime factor $p > \sqrt{x}$, and hence if $x^{1/2} \le y \le x$ then

$$\begin{aligned}\psi(x, y) &= [x] - \sum_{y<p\le x} \sum_{\substack{n\le x \\ p\mid n}} 1 \\ &= [x] - \sum_{y<p\le x} [x/p] \\ &= x - x \sum_{y<p\le x} \frac{1}{p} + O(\pi(x)).\end{aligned}$$

By estimates of Chebyshev and Mertens (Corollary 2.6 and Theorem 2.7(d)), this is

$$= x\Big(1 - \log \frac{\log x}{\log y}\Big) + O\Big(\frac{x}{\log x}\Big).$$

Thus if we take $u = (\log x)/(\log y)$, so that $y = x^{1/u}$, then we see that

$$\psi(x, x^{1/u}) = (1 - \log u)x + O\Big(\frac{x}{\log x}\Big) \tag{2}$$

uniformly for $1 \le u \le 2$. We shall show more generally that there is a function $\rho(u) > 0$ such that

$$\psi(x, x^{1/u}) \sim \rho(u)x \tag{3}$$

as $x \to \infty$ with u bounded. The function $\rho(u)$ that arises here is known as the *Dickman function*; it may be defined to be the unique continuous function on $[0, \infty)$ satisfying the differential-delay equation

$$u\rho'(u) = -\rho(u-1) \tag{4}$$

for $u > 1$ together with the initial condition that

$$\rho(u) = 1 \tag{5}$$

for $0 \le u \le 1$. Before proceeding further we note some simple properties of this function. By dividing both sides of (4) by u and then integrating, we find that

$$\rho(v) = \rho(u) - \int_u^v \rho(t-1)\,\frac{dt}{t} \tag{6}$$

for $1 \le u \le v$. Also, from (4) we see that $\big(u\rho(u)\big)' = \rho(u) - \rho(u-1)$, so that by integrating it follows that

$$u\rho(u) = \int_{u-1}^{u} \rho(v)\,dv + C$$

for $u \ge 1$, where C is a constant of integration. On taking $u = 1$ we deduce that $C = 0$, and hence that

$$u\rho(u) = \int_{u-1}^{u} \rho(v)\,dv \tag{7}$$

for $u \ge 1$.

FIGURE 1. The Dickman function $\rho(u)$ for $0 \le u \le 4$.

As might be surmised from Figure 1, $\rho(u)$ is positive and decreasing. To prove this, let u_0 be the infimum of the set of all solutions of the equation $\rho(u) = 0$. By the continuity of ρ it follows that $\rho(u_0) = 0$. But $\rho(u) > 0$ for $0 \le u < u_0$, and hence if we take $u = u_0$ in (7) then the left hand side is 0 while the right hand side is positive, a contradiction. Thus $\rho(u) > 0$ for all $u \ge 0$, and by (4) it follows that $\rho'(u) < 0$ for all $u > 1$. Figure 1 also suggests that $\rho(u)$ tends to 0 rapidly as $u \to \infty$. We now establish a crude estimate in this direction.

Lemma 1. *The function $\rho(u)$ is positive and decreasing for $u \ge 0$, and satisfies the inequalities*

$$\frac{1}{2\Gamma(2u+1)} \le \rho(u) \le \frac{1}{\Gamma(u+1)}.$$

Proof. For positive integers U we prove by induction that the upper bound holds for $0 \le u \le U$. To provide the basis of the induction we need to show that $\Gamma(s) \le 1$ for $1 \le s \le 2$. This is immediate from the relations

$$(8) \qquad \Gamma(1) = \Gamma(2) = 1, \qquad \Gamma''(s) = \int_0^\infty e^{-x} x^{s-1} (\log x)^2 \, dx > 0 \qquad (0 < s < \infty).$$

Since $\rho(u)$ is decreasing, we see by (7) that $u\rho(u) \le \rho(u-1)$. Thus if the desired upper bound holds for $u \le U$ and if $U \le u \le U+1$, then

$$\rho(u) \le \frac{\rho(u-1)}{u} \le \frac{1}{u\Gamma(u)} = \frac{1}{\Gamma(u+1)}$$

by (C.4).

After making the change of variables $u = v/2$, the desired lower bound asserts that $\rho(v/2) \ge 1/(2\Gamma(v+1))$. We let V run through positive integral values, and prove by induction on V that the lower bound holds for $0 \le v \le V$. To establish the lower bound for $0 \le v \le 2$ it suffices to show that $\Gamma(s) \ge 1/2$ for all $s > 0$. From (8) we see that $\Gamma(s) \ge 1$ for $0 < s \le 1$ and for $s \ge 2$; thus it remains to note that if $1 \le s \le 2$ then

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} \, dx \ge \int_0^1 e^{-x} x \, dx + \int_1^\infty e^{-x} \, dx = 1 - \frac{1}{e} > \frac{1}{2}.$$

(The actual fact of the matter is that $\min_{s>0} \Gamma(s) = \Gamma(1.4616\ldots) = 0.8856\ldots$.) Since $\rho(u)$ is decreasing, we see by (7) that $u\rho(u) \ge \rho(u-1/2)/2$. Thus if the lower bound holds for $0 \le v \le V$ and if $V \le v \le V+1$ then

$$\rho(v/2) \ge \frac{\rho((v-1)/2)}{v} \ge \frac{1}{2v\Gamma(v)} = \frac{1}{2\Gamma(v+1)}$$

by (C.4). This completes the inductive step, so the proof is complete.

We now use elementary reasoning to show that (3) holds uniformly for u in bounded intervals.

Theorem 2. (Dickman) *Let $\psi(x, y)$ be the number of positive integers not exceeding x composed entirely of prime numbers not exceeding y, and let $\rho(u)$ be defined as above. Then for any $U \ge 0$ we have*

$$(9) \qquad \psi(x, x^{1/u}) = \rho(u)x + O\Big(\frac{x}{\log x}\Big)$$

uniformly for $0 \le u \le U$ *and all* $x \ge 2$.

Proof. We restrict U to integral values, and induct on U. The basis of the induction is provided by (1) and (5). Also, (2) gives (9) for $1 \le u \le 2$ since from (6) we see that

$$\rho(u) = 1 - \log u \tag{10}$$

for $1 \le u \le 2$. Suppose now that U is an integer, $U \ge 2$, and that (9) holds uniformly for $0 \le u \le U$. We show that (9) holds uniformly for $U \le u \le U+1$. To this end we classify n according to size of the largest prime factor $P(n)$ of n. Thus we see that

$$\psi(x,y) = 1 + \sum_{p \le y} \operatorname{card}\{n \le x : P(n) = p\}.$$

Here the first term on the right reflects the fact that if $x \ge 1$ then $\psi(x,y)$ counts the number $n = 1$ for which $P(1)$ is undefined. In the sum on the right, the summand is $\psi(x/p,p)$, and hence we see that

$$\psi(x,y) = 1 + \sum_{p \le y} \psi(x/p,p). \tag{11}$$

On differencing, it follows that if $y \le z$ then

$$\psi(x,y) = \psi(x,z) - \sum_{y < p \le z} \psi(x/p,p). \tag{12}$$

Suppose that $z = x^{1/U}$ and that $y = x^{1/u}$ with $U \le u \le U+1$. Define u_p by the relation $p = (x/p)^{1/u_p}$. That is,

$$u_p = \frac{\log x}{\log p} - 1,$$

which is $\le u - 1 \le U$ if $p \ge y$. Hence by the inductive hypothesis the right hand side of (12) is

$$\rho(U)x + O\Big(\frac{x}{\log x}\Big) - x \sum_{y < p \le z} \frac{\rho((\log x)/(\log p) - 1)}{p} + O\Big(x \sum_{y < p \le z} \frac{1}{p \log x/p}\Big). \tag{13}$$

Let $s(w) = \sum_{p \le w} 1/p$, and write Mertens' estimate (Theorem 2.7(d)) in the form $s(w) = \log\log w + c + r(w)$. Then the sum in the main term above is

$$\begin{aligned}\int_y^z \rho((\log x)/(\log w) - 1)\, ds(w) = &\int_y^z \rho((\log x)/(\log w) - 1)\, d\log\log w \\ &+ \int_y^z \rho((\log x)/(\log w) - 1)\, dr(w).\end{aligned} \tag{14}$$

We put $t = (\log x)/(\log w)$. Since

$$d \log\log w = \frac{dw}{w \log w} = -\frac{dt}{t},$$

the first integral on the right hand side of (14) is

$$\int_U^u \rho(t-1)\,\frac{dt}{t}. \tag{15}$$

By integrating by parts and the estimate $r(w) \ll 1/\log w$ we see that the second integral on the right hand side of (14) is

$$\rho((\log x)/(\log w)-1)r(w)\Big|_y^z - \int_y^z r(w)\,d\rho((\log x)/(\log w)-1)$$
$$\ll \frac{1}{\log x}\Big(1+\int_y^z 1\,|d\rho((\log x)/(\log w)-1)|\Big) \ll \frac{1}{\log x}$$

since ρ is monotonic and bounded. By Mertens' estimate (Theorem 2.7(d)) we also see that the error term in (13) is

$$\ll \frac{x}{\log x}\sum_{y<p\le z}\frac{1}{p} \ll \frac{x}{\log x}$$

since $\log\log z = \log\log y + O(1)$. On combining our estimates in (12) we find that

$$\psi(x, x^{1/u}) = x\Big(\rho(U) - \int_U^u \rho(t-1)\,\frac{dt}{t}\Big) + O\Big(\frac{x}{\log x}\Big).$$

Thus by (6) we have the desired estimate for $U \le u \le U+1$, and the proof is complete.

As for $\psi(x,y)$ when $y < x^{\varepsilon}$, we show next that

$$\psi(x, (\log x)^a) = x^{1-1/a+o(1)} \tag{16}$$

for any fixed $a \ge 1$. The upper bound portion of this is obtained by means of bounds for an associated Dirichlet series, while the lower bound is derived by combinatorial reasoning.

An upper bound for $\psi(x,y)$ can be constructed by observing that if $\sigma > 0$ then

$$\psi(x,y) \le \sum_{\substack{n\le x\\ p|n\Rightarrow p\le y}} \Big(\frac{x}{n}\Big)^{\sigma} \le x^{\sigma}\sum_{p|n\Rightarrow p\le y}\frac{1}{n^{\sigma}} = x^{\sigma}\prod_{p\le y}\Big(1-\frac{1}{p^{\sigma}}\Big)^{-1}. \tag{17}$$

Rankin used this chain of inequalities to derive an upper bound for $\psi(x,y)$. This approach is fruitful in a variety of settings, and has become known as 'Rankin's method'.

To use the above, we must establish an upper bound for the product on the right hand side. The size of this product is a little difficult to describe, because its behaviour depends on the size of σ. If σ is near 0 then most of the factors are approximately $(1-y^{-\sigma})^{-1}$, and hence we expect the product to be approximately $(1-y^{-\sigma})^{-y/\log y}$. If σ is larger (but still < 1) then the general factor is approximately $\exp(p^{-\sigma})$, and hence the product is approximately the exponential of

$$\sum_{p\le y} p^{-\sigma} \sim \int_2^y \frac{dt}{t^{\sigma}\log t} \sim \frac{y^{1-\sigma}}{(1-\sigma)\log y}.$$

We begin by making these relations precise.

Lemma 3. *If* $0 \le \sigma \le 1$, *then*

$$\sum_{p\le y} p^{-\sigma} = \int_2^y \frac{du}{u^\sigma \log u} + O\big(y^{1-\sigma}\exp(-c\sqrt{\log y})\big) + O(1). \tag{18}$$

Proof. We write the left hand side as

$$\int_{2-}^y u^{-\sigma}\, d\pi(u) = \int_{2-}^y u^{-\sigma}\, d\,\mathrm{li}(u) + \int_{2-}^y u^{-\sigma}\, dr(u)$$

where $r(u) = \pi(u) - \mathrm{li}(u)$. The first integral on the right is $\int_2^y u^{-\sigma}(\log u)^{-1}\, du$. By integrating by parts we find that the second integral is

$$y^{-\sigma} r(y) - 2^{-\sigma} r(2^-) + \sigma \int_2^y r(u) u^{-\sigma-1}\, du.$$

Suppose that b is a positive constant chosen so that $r(u) \ll u\exp(-b\sqrt{\log u})$. Then the first two terms above can be absorbed into the error terms in (18) if $c < b$. To complete the proof it suffices to show that

$$\int_2^y u^{-\sigma}\exp(-b\sqrt{\log u})\, du \ll 1 + y^{1-\sigma}\exp(-\tfrac{b}{3}\sqrt{\log y}), \tag{19}$$

for then we have (18) with $c = b/3$.

To prove (19) we note that if $\sigma \ge 1 - b/(2\sqrt{\log y})$ then

$$\begin{aligned} u^{1-\sigma}\exp(-\tfrac{b}{2}\sqrt{\log u}) &= \exp\big((1-\sigma)\log u - \tfrac{b}{2}\sqrt{\log u}\big) \\ &\le \exp\big(\tfrac{b}{2}(\log u)/\sqrt{\log y} - \tfrac{b}{2}\sqrt{\log u}\big) \\ &\le 1 \end{aligned}$$

for $2 \le u \le y$. Hence for σ in this range the integral in (19) is

$$\le \int_2^y \frac{du}{u\exp(\frac{b}{2}\sqrt{\log u})} < \int_2^\infty \frac{du}{u\exp(\frac{b}{2}\sqrt{\log u})} \ll 1.$$

Now suppose that

$$\sigma \le 1 - \frac{b}{2\sqrt{\log y}}. \tag{20}$$

We write the integral in (19) as $\int_2^{y^{1/4}} + \int_{y^{1/4}}^y = I_1 + I_2$, say. Then

$$I_1 \le \int_2^{y^{1/4}} u^{-\sigma}\, du < \frac{y^{(1-\sigma)/4}}{1-\sigma},$$

which by (20) is

$$\ll y^{1-\sigma}\sqrt{\log y}\exp\big(-\tfrac{3}{4}(1-\sigma)\log y\big) \ll y^{1-\sigma}\exp\big(-\tfrac{b}{3}\sqrt{\log y}\big).$$

As for I_2, we note that if $u \ge y^{1/4}$ then $\log u \ge \frac{1}{4}\log y$. Hence

$$\begin{aligned} I_2 &\le \exp\big(-\tfrac{b}{2}\sqrt{\log y}\big)\int_2^y u^{-\sigma}\, du \le \exp\big(-\tfrac{b}{2}\sqrt{\log y}\big)\frac{y^{1-\sigma}}{1-\sigma} \\ &\ll \exp\big(-\tfrac{b}{2}\sqrt{\log y}\big) y^{1-\sigma}\sqrt{\log y} \ll y^{1-\sigma}\exp\big(-\tfrac{b}{3}\sqrt{\log y}\big). \end{aligned}$$

These estimates combine to give (19), so the proof is complete.

Lemma 4. *If $y \ge 2$ and $1 - 4/\log y \le \sigma \le 1$ then*

$$\sum_{p\le y} p^{-\sigma} = \log\log y + O(1). \tag{21}$$

If $y \ge 2$ and $0 \le \sigma \le 1 - 4/\log y$ then

$$\sum_{p\le y} p^{-\sigma} = \frac{y^{1-\sigma}}{(1-\sigma)\log y} + \log\frac{1}{1-\sigma} + O\Big(\frac{y^{1-\sigma}}{(1-\sigma)^2(\log y)^2}\Big). \tag{22}$$

Proof. Suppose that $1 - 4/\log y \le \sigma \le 1$. If $u \le y$ then

$$\begin{aligned} u^{-\sigma} &= u^{-1}u^{1-\sigma} = u^{-1}\exp\big((1-\sigma)\log u\big) = u^{-1}\big(1 + O((1-\sigma)\log u)\big) \\ &= u^{-1} + O\big(u^{-1}(1-\sigma)\log u\big). \end{aligned}$$

Hence

$$\int_2^y \frac{du}{u^\sigma \log u} = \int_2^y \frac{du}{u\log u} + O\Big((1-\sigma)\int_2^y \frac{du}{u}\Big) = \log\log y + O(1).$$

Thus (21) follows from Lemma 3.

To prove (22) we let $v = \exp(4/(1-\sigma))$, and observe that $v \le y$. We write the integral in Lemma 3 as $\int_2^v + \int_v^y = I_1 + I_2$, say. By the above we see that $I_1 = \log\log v + O(1) = \log 1/(1-\sigma) + O(1)$. By integration by parts we see that

$$I_2 = \frac{y^{1-\sigma}}{(1-\sigma)\log y} - \frac{v^{1-\sigma}}{(1-\sigma)\log v} + \frac{1}{1-\sigma}\int_v^y \frac{du}{u^\sigma(\log u)^2}.$$

Here the first term on the right is one of the main terms in (22), and the second term is $O(1)$. Let J denote the integral on the right. To complete the proof it suffices to show that

$$J \ll \frac{y^{1-\sigma}}{(1-\sigma)(\log y)^2}. \tag{23}$$

To this end we integrate by parts again:

$$J = \frac{y^{1-\sigma}}{(1-\sigma)(\log y)^2} - \frac{v^{1-\sigma}}{(1-\sigma)(\log v)^2} + \frac{2}{1-\sigma}\int_v^y \frac{dw}{w^\sigma(\log w)^3}.$$

Here the second term on the right hand side is $e^4 2^{-4}(1-\sigma) \ll 1-\sigma$, while the first term on the right hand side is larger. As for the integral on the right, we observe that if $w \ge v$ then $(\log w)^3 \ge 4(\log w)^2/(1-\sigma)$. Hence the last term on the right above has absolute value not exceeding $J/2$. Thus we have (23), and the proof is complete.

Lemma 5. *Suppose that* $y \ge 2$. *If* $\max\big(2/\log y, 1 - 4/\log y\big) \le \sigma \le 1$, *then*

$$\prod_{p \le y} \big(1 - p^{-\sigma}\big)^{-1} \asymp \log y. \tag{24}$$

If $2/\log y \le \sigma \le 1 - 4/\log y$ *then*

$$\prod_{p \le y} \big(1 - p^{-\sigma}\big)^{-1} = \frac{1}{1-\sigma} \exp\Big(\frac{y^{1-\sigma}}{(1-\sigma)\log y}\Big(1 + O\Big(\frac{1}{(1-\sigma)\log y}\Big) + O\big(y^{-\sigma}\big)\Big)\Big). \tag{25}$$

Proof. The bound (24) is trivial when $\sigma \le 2/3$ since then $y \le e^{12}$. The estimate $(1 - \delta)^{-1} = \exp\big(\delta + O(\delta^2)\big)$ holds uniformly for $|\delta| \le 1/2$. We take $\delta = p^{-\sigma}$ for $p > v = e^{1/\sigma}$ to deduce that

$$\prod_{v < p \le y} \big(1 - p^{-\sigma}\big)^{-1} = \exp\Big(\sum_{v < p \le y} p^{-\sigma} + O\Big(\sum_{v < p \le y} p^{-2\sigma}\Big)\Big).$$

Now (24) follows at once from Lemma 4 when $\sigma \ge 2/3$. Thus it remains to establish (25). The sum in the error term above is $\ll 1$ for $\sigma > 5/8$. If $3/8 \le \sigma \le 5/8$, then by Lemma 4 it is $\ll y^{1/4}/\log y$. If $2/\log y \le \sigma \le 3/8$, then by Lemma 4 the sum is $\ll y^{1-2\sigma}/\log y$. Thus in any case this error term is majorized by the error terms on the right hand side of (25). By Lemma 4, the main term is

$$\sum_{v < p \le y} p^{-\sigma} = \frac{y^{1-\sigma}}{(1-\sigma)\log y} + \log \frac{1}{1-\sigma} + O\Big(\frac{y^{1-\sigma}}{(1-\sigma)^2(\log y)^2}\Big) + O\Big(\frac{v}{\log v}\Big).$$

Since $2/\log y \le \sigma \le 1 - 4/\log y$, y satisfies $y \ge e^6$, and $\sigma(1-\sigma)\log y \ge 2(1 - 2/\log y) \ge 4/3$. Hence $\big(y^{1-\sigma}\big)^{3/4} \ge v$ and the second error term above is dominated by the first.

It remains to consider the contribution of the primes $p \le v$. If $\sigma > 1/3$ then the contribution of these primes is $\ll 1$, so we may suppose that $2/\log y \le \sigma \le 1/3$. In this range

$$1 - p^{-\sigma} \asymp \sigma \log p = \frac{\log p}{\log v}.$$

Since

$$\sum_{p \le v} \log\Big(C \frac{\log v}{\log p}\Big) \ll v,$$

it follows that

$$\prod_{p \le v} \big(1 - p^{-\sigma}\big)^{-1} < \exp(Cv) = \exp\big(Ce^{1/\sigma}\big) \le \exp\big(Cy^{1/2}\big),$$

which suffices. Thus the proof is complete.

We now bound $\psi(x, y)$ by combining Lemma 5 with the inequalities (17).

Theorem 6. *If* $y = x^{1/u}$ *and* $\log x \le y \le x^{1/9}$ *then*

$$\psi(x,y) < x(\log y)\exp\Big(-u\log u - u\log\log u + u - \frac{u\log\log u}{\log u} + O\Big(\frac{u}{\log u}\Big) + O\Big(\frac{u^2\log u}{y}\Big)\Big).$$

Here the first error term is larger than the second if $y \ge (\log x)\log\log x$, while if y is smaller then the second error term dominates.

Proof. We first note that we may suppose that $y \ge 9\log x$, since the bound for smaller y follows by taking $y = 9\log x$. To motivate the choice of σ in (17) we note that the expression to be minimized is approximately

$$x^\sigma \exp\Big(\int_2^y \frac{u^{-\sigma}}{\log u}\,du\Big).$$

On taking logarithmic derivatives, this suggests that we should take σ to be the root of the equation

$$\log x = \frac{y^{1-\sigma}}{1-\sigma}. \tag{26}$$

In actual fact we take

$$\sigma = 1 - \frac{\log u + \log\log u}{\log y}. \tag{27}$$

It is easy to see that for this σ the right hand side of (26) is

$$\log x \frac{\log u}{\log u + \log\log u},$$

so it is reasonable to expect that the simple choice (27) is close enough to the root of (26) for our present purposes.

From the inequalities $9\log x \le y \le x^{1/9}$ it follows that the σ given by (27) satisfies $2/\log y \le \sigma \le 1 - 1/\log y$. Hence the stated upper bound follows by combining (17) with the estimates of Lemma 5.

To obtain companion lower bounds we observe that if k is chosen so that $y^k \le x$, then $\psi(x,y)$ certainly counts all integers n composed of primes $p \le y$ such that $\Omega(n) \le k$. Put $r = \pi(y)$, and suppose that $p_1, p_2, \dots, p_r$ are the primes not exceeding y. Then n is of the form $n = p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}$, and $\psi(x,y)$ is at least as large as the number of solutions of the inequality $a_1 + a_2 + \cdots + a_r \le k$ in non-negative integers a_i. For this quantity we have an exact formula, as follows.

Lemma 7. *Let* $A(r,k)$ *denote the number of solutions of the inequality* $a_1+a_2+\cdots+a_r \le k$ *in non-negative integers* a_i*. Then* $A(r,k) = \binom{r+k}{k}$.

Analytic Proof. Let $a_{r+1} = k - \sum_{i=1}^r a_i$. Then $A(r,k)$ is the number of ways of writing $k = a_1 + a_2 + \cdots + a_{r+1}$, which is the coefficient of x^k in the power series

$$\Big(\sum_{a=0}^\infty x^a\Big)^{r+1} = (1-x)^{-r-1} = \sum_{k=0}^\infty \binom{r+k}{k} x^k$$

by the 'negative' binomial theorem.

Combinatorial Proof. Suppose that we have k circles $\circ$ and r bars $|$ arranged in a line. Let a_1 be the number of circles to the left of the first bar, let a_2 be the number of circles between the first and second bar, and so on, so that a_r is the number of circles between the last two bars. (The number of circles to the right of the last bar is $k - \sum a_i$.) Thus a configuration of circles and bars determines a choice of non-negative a_i with $a_1 + a_2 + \cdots + a_r \le k$. But conversely, a choice of such a_i determines a configuration of circles and bars. The number of ways of choosing the positions of the k circles in the $r+k$ available places is $\binom{r+k}{k}$.

Theorem 8. *If* $\log x \le y \le x$ *then*

$$\psi(x,y) \gg \frac{x}{y}\exp(-u\log\log x + u/2).$$

Proof. Let $r = \pi(y)$ and let k be the largest integer such that $y^k \le x$. That is, $k = [u]$. Then by Lemma 7 and Stirling's formula we see that

$$\psi(x,y) \ge \binom{r+k}{k} \asymp \Big(\frac{r+k}{k}\Big)^k \Big(\frac{r+k}{r}\Big)^r \frac{1}{\sqrt{k}}. \tag{28}$$

The identity

$$k\log(1+r/k) + r\log(1+k/r) = \int_0^r \log(1+k/t)\,dt$$

shows that the left hand side is an increasing function of r. It can be supposed that x is sufficiently large. Let $z = y/(k\log y)$. Then the expression (28) is

$$\gg \Big(1 + \frac{y}{k\log y}\Big)^k \Big(1 + \frac{k\log y}{y}\Big)^{y/\log y} \frac{1}{\sqrt{k}} \ge \big(z(1+1/z)^z\big)^k\,,$$

Moreover $u - 1 < k \le u \le y/\log y$ and $z(1+1/z)^z$ is increasing for $z \ge 1$. Thus the above is $\ge \Big(z'(1+1/z')^{z'}\Big)^k \ge \Big(z'(1+1/z')^{z'}\Big)^{u-1}$ where $z' = y/(u\log y)$. As $z' \le y/\sqrt{k}$ this is

$$\ge \frac{1}{y}\Big(\frac{y}{u\log y}\Big)^u \Big(1 + \frac{u\log y}{y}\Big)^{y/\log y} = \frac{x}{y}\exp\left(-u\log\log x + \frac{y}{\log y}\log(1 + (\log x)/y)\right).$$

The stated inequality now follows on noting that $\log(1+\delta) \ge \delta/2$ for $0 \le \delta \le 1$.

When y is of the form $y = (\log x)^a$ with a not too large, the upper bound of Theorem 6 and the lower bound of Theorem 8 are quite close, and we have

Corollary 9. *If* $y = (\log x)^a$ *and* $1 \le a \le (\log x)^{1/2}/(2\log\log x)$ *then*

$$x^{1-1/a}\exp\Big(\frac{\log x}{5a\log\log x}\Big) < \psi(x,y) < x^{1-1/a}\exp\Big(\frac{(\log a + O(1))\log x}{a\log\log x}\Big).$$

Proof. The lower bound follows from Theorem 8 since $\log y \le (\log x)/(4a\log\log x)$ in the range under consideration. As for the upper bound, we note that $\log u \asymp \log\log x$, so that $\log\log u = \log\log\log x + O(1)$. Hence $\log u + \log\log u = \log\log x - \log a + O(1)$, and the result follows from Theorem 6.

For $1 \le u \le 4$ we may use the differential equation (4) and the initial condition (5) to derive formulæ for $\rho(u)$ (see Exercise 6 below), but for larger u we take a different approach.

Theorem 10. *For any real or complex number s we have*

$$\int_0^\infty \rho(u)e^{-us}\,du = \exp\Big(C_0 + \int_0^s \frac{e^{-z}-1}{z}\,dz\Big) \tag{29}$$

where C_0 is Euler's constant. Conversely, for any $u > 0$ and any real σ_0 we have

$$\rho(u) = \frac{e^{C_0}}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \exp\Big(\int_0^s \frac{e^{-z}-1}{z}\,dz\Big)e^{us}\,ds. \tag{30}$$

Proof. Let $F(s)$ denote the integral on the left hand side of (29); this is the Laplace transform of $\rho(u)$. In view of the rapid decay of $\rho(u)$ established in Lemma 1, we see that the integral converges for all s, and hence that $F(s)$ is an entire function. On integrating by parts we see that

$$F(s) = \frac{1}{s} + \frac{1}{s}\int_1^\infty \rho'(u)e^{-us}\,du,$$

and hence that

$$\big(sF(s)\big)' = -\int_1^\infty u\rho'(u)e^{-us}\,du.$$

The differential–delay identity (4) for $\rho(u)$ thus yields a differential equation for $F(s)$,

$$\big(sF(s)\big)' = e^{-s}F(s).$$

By separation of variables it follows that

$$F(s) = F(0)\exp\Big(\int_0^s \frac{e^{-z}-1}{z}\,dz\Big).$$

To determine the value of $F(0)$ we note that

$$1 = \lim_{s\to+\infty} sF(s) = F(0)\exp\Big(\int_0^1 \frac{e^{-z}-1}{z}\,dz + \int_1^\infty \frac{e^{-z}}{z}\,dz\Big).$$

By integration by parts we see that

$$\int_0^1 \frac{e^{-z}-1}{z}\,dz + \int_1^\infty \frac{e^{-z}}{z}\,dz = \int_0^\infty e^{-z}\log z\,dz = \Gamma'(1) = -C_0 \tag{31}$$

by (C.12) and Theorem C.2. Hence $F(0) = e^{C_0}$. An arithmetic proof of this is found in Exercise 7 below. Thus we have the identity (29), and (30) follows by applying the inverse Laplace transform to both sides.

7.1. Exercises

1. (Chowla and Vijayaraghavan (1947)) Show that if $f(x)$ is a function that tends to infinity in such a way that $\log f(x) = o(\log x)$ then almost all integers n have a prime factor larger than $f(n)$. That is

$$\lim_{x\to\infty} \frac{1}{x} \operatorname{card}\{n \le x : P(n) > f(n)\} = 1$$

where $P(n)$ denotes the largest prime factor of n.

2. (de Bruijn (1951b) Let $P(n)$ denote the largest prime factor of n. Show that

$$\sum_{n\le x} \log P(n) \sim Dx \log x$$

where $D = \int_0^\infty \rho(u)(u+1)^{-2}\,du$ is called *Dickman's constant*.

3. (See Alladi & Erdős (1977)) Let $P(n)$ denote the largest prime factor of n.
(a) Show that

$$\sum_{n\le x} P(n) = \sum_{\sqrt{x}<p\le x} p\left[\frac{x}{p}\right] + O\big(x^{3/2}\big).$$

(b) Show that the sum on the right above is

$$= \sum_{1\le k\le\sqrt{x}} k \sum_{x/(k+1)<p\le x/k} p + O\big(x^{3/2}\big).$$

(c) Show that

$$\sum_{p\le y} p = \frac{y^2}{2\log y} + O\Big(\frac{y^2}{(\log y)^2}\Big).$$

(d) Show that

$$\sum_{k=1}^{\infty} k\Big(\frac{1}{k^2} - \frac{1}{(k+1)^2}\Big) = \frac{\pi^2}{6}.$$

(e) Conclude that

$$\sum_{n\le x} P(n) = \frac{\pi^2}{12}\,\frac{x^2}{\log x} + O\Big(\frac{x^2}{(\log x)^2}\Big).$$

4. Show that $\rho^{(k)}(u)$ has a jump discontinuity at $u = k$, and is continuous for $u > k$.

5. (a) Show that $\rho(u)$ is convex upwards for all $u \ge 1$.
(b) Show that if $u \ge 2$ then $u\rho(u) \ge \rho(u-1/2)$.
(c) Show that if $u \ge 2$ then $(2u-1)\rho(u) \le \rho(u-1)$.

6. (a) Show that if $1 \le u \le 2$ then $\rho(u) = 1 - \log u$.
(b) Show that if $2 \le u \le 3$ then

$$\rho(u) = 1 - \log u + \int_2^u \frac{\log(t-1)}{t}\,dt.$$

(c) Show that if $3 \le u \le 4$ then

$$\rho(u) = 1 - \log u + \int_2^u \frac{\log(t-1)}{t}\,dt - \int_3^u \frac{(\log u/t)\log(t-2)}{t-1}\,dt.$$

7. Let $P(\sigma) = \prod_{p\le y}(1-p^{-\sigma})^{-1}$.
(a) Explain why

$$P(1) = \sum_{p|n \Rightarrow p\le y} \frac{1}{n} = e^{C_0}\log y + O(1).$$

(b) Show that if $\sigma \ge 1$ then $\frac{P'}{P}(\sigma) \ll \log y$.
(c) Deduce that

$$-P'(1) = \sum_{\substack{n \\ p|n\Rightarrow p\le y}} \frac{\log n}{n} \ll (\log y)^2.$$

(d) Conclude that

$$\sum_{\substack{n>x \\ p|n\Rightarrow p\le y}} \frac{1}{n} \ll \frac{(\log y)^2}{\log x}.$$

(e) Show that

$$\sum_{\substack{n\le x \\ p|n\Rightarrow p\le y}} \frac{1}{n} = (\log y)\int_0^u \frac{\psi(y^v, y)}{y^v}\,dv + O(1)$$

where $u = (\log x)/\log y$.
(f) Deduce that

$$\int_0^\infty \rho(u)\,du = e^{C_0}.$$

(g) Show that $\sum_{n=1}^\infty n\rho(n) = e^{C_0}$.

8. (Erdős & Nicolas (1981)) Let α be fixed, $0 < \alpha < 1$.
(a) Let k be the least integer $> \alpha(\log x)/\log\log x$, put $y = x^{1/k}$, and set $r = \pi(y)$. Show that there are at least $\binom{r}{k}$ integers $n \le x$ such that $\omega(n) > \alpha(\log x)/\log\log x$.
(b) Show that the number of integers $n \le x$ such that $\omega(n) > \alpha(\log x)/\log\log x$ is at least $x^{1-\alpha+o(1)}$.
(c) Show that if $\sigma > 1$ and $A \ge 1$ then the number of integers $n \le x$ such that $\omega(n) > \alpha(\log x)/\log\log x$ is at most

$$x^\sigma A^{-k}\sum_{n=1}^\infty \frac{A^{\omega(n)}}{n^\sigma}.$$

(d) Show that if $A = \log x$ and $\sigma = 1 + (\log\log\log x)/\log\log x$ then the above is $x^{1-\alpha+o(1)}$.

9. (de Bruijn (1966)) Assume that $0 < \sigma \le 3/\log y$, and note that this interval covers a range is not treated in Lemma 5.
(a) Show that $1 - p^{-\sigma} \asymp \sigma \log p$, and hence deduce that

$$\prod_{p\le y}\Big(1-p^{-\sigma}\Big)^{-1} \le \exp\Big(\sum_{p\le y}\log\frac{C}{\sigma\log p}\Big) \le \exp\Big(\frac{Cy}{\log y}\log\frac{4}{\sigma\log y}\Big) \tag{32}$$

for a suitable constant C.
(b) Write

$$\prod_{p\le y}\Big(1-p^{-\sigma}\Big)^{-1} = \big(1-y^{-\sigma}\big)^{-\pi(y)}\prod_{p\le y}\frac{1-y^{-\sigma}}{1-p^{-\sigma}} = F_1\cdot F_2,$$

say. Show that

$$F_1 \le \big(1-y^{-\sigma}\big)^{-y/\log y}\exp\Big(\frac{Cy}{(\log y)^2}\log\frac{4}{\sigma\log y}\Big).$$

(c) Note that

$$\frac{1-p^{-\sigma}}{1-y^{-\sigma}} = 1 - \frac{(y/p)^\sigma - 1}{y^\sigma - 1}, \tag{33}$$

and hence deduce that the above is $\ge 1 - c\frac{\log y/p}{\log y}$, so that

$$F_2 \le \exp\Big(\frac{C}{\log y}\sum_{p\le y}\log y/p\Big) \le \exp\big(Cy/(\log y)^2\big).$$

(d) Conclude that

$$\prod_{p\le y}\big(1-p^{-\sigma}\big)^{-1} \le \big(1-y^{-\sigma}\big)^{-y/\log y}\exp\Big(\frac{Cy}{(\log y)^2}\log\frac{4}{\sigma\log y}\Big)$$

for $0 < \sigma \le 3/\log y$.

10. (de Bruijn (1966)) Lemma 5 suffers from a loss of precision when $3/\log y \le \sigma \le (\log\log y)/\log y$. To obtain a refined estimate in this range, write

$$\prod_{p\le y}\big(1-p^{-\sigma}\big)^{-1} = F_1\cdot F_2\cdot F_3$$

where the F_i are products over the intervals $p \le \exp(1/\sigma)$, $\exp(1/\sigma) < p \le y/\exp(1/\sigma)$, and $y/\exp(1/\sigma) < p \le y$, respectively.
(a) Use (32) to show that $F_1 \le \exp\big(C\sigma e^{1/\sigma}\big)$.
(b) Use Lemma 5 to show that

$$F_2 \le \exp\Big(\frac{Cy^{1-\sigma}}{e^{1/\sigma}\log y}\Big).$$

(c) Use the identity (33) to show that

$$\frac{1-p^{-\sigma}}{1-y^{-\sigma}} \ge 1 - \frac{c\sigma \log y/p}{y^\sigma},$$

and hence deduce that

$$F_3 \le \bigl(1-y^{-\sigma}\bigr)^{-\pi(y)} \exp\Bigl(C\sigma \sum_{p\le y} \frac{\log y/p}{y^\sigma}\Bigr) \le \bigl(1-y^{-\sigma}\bigr)^{-y/\log y} \exp\Bigl(\frac{y^{1-\sigma}}{(\log y)^2} + \frac{C\sigma y^{1-\sigma}}{\log y}\Bigr).$$

(d) Conclude that

$$\prod_{p\le y} \bigl(1-p^{-\sigma}\bigr)^{-1} \le \bigl(1-y^{-\sigma}\bigr)^{-y/\log y} \exp\Bigl(\frac{C\sigma y^{1-\sigma}}{\log y}\Bigr)$$

when $3/\log y \le \sigma \le (\log\log y)/\log y$.

11. (de Bruijn (1966)) (a) For $\sigma > 0$ let $f(\sigma) = x^\sigma\bigl(1-y^{-\sigma}\bigr)^{-y/\log y}$. Show that $f(\sigma)$ is minimized precisely when

$$\sigma = \frac{\log\bigl(1+y/\log x\bigr)}{\log y}.$$

(b) Show that for the above σ,

$$f(\sigma) = \exp\Bigl(\frac{\log x}{\log y}\log\Bigl(\frac{y+\log x}{\log x}\Bigr) + \frac{y}{\log y}\log\Bigl(\frac{y+\log x}{y}\Bigr)\Bigr).$$

(c) Show that if $y \le \log x$ then

$$\psi(x,y) \le \exp\Bigl(\frac{\log x}{\log y}\log\Bigl(\frac{y+\log x}{\log x}\Bigr) + \frac{y}{\log y}\Bigl(1+O\Bigl(\frac{1}{\log y}\Bigr)\Bigr)\log\Bigl(\frac{y+\log x}{y}\Bigr)\Bigr).$$

(d) Show that if $\log x \le y \le (\log x)^2$ then

$$\psi(x,y) \le \exp\Bigl(\frac{\log x}{\log y}\Bigl(1+O\Bigl(\frac{1}{\log y}\Bigr)\Bigr)\log\Bigl(\frac{y+\log x}{\log x}\Bigr) + \frac{y}{\log y}\log\Bigl(\frac{y+\log x}{y}\Bigr)\Bigr).$$

12. (Erdős (1963)) Show that

$$\psi(x,\log x) = \exp\Bigl((2\log 2 + o(1))\frac{\log x}{\log\log x}\Bigr).$$

13. (de Bruijn (1966)) Show that if a is fixed, $0 < a < 1$, then

$$\psi(x,(\log x)^a) = \exp\bigl((1/a - 1 + o(1))(\log x)^a\bigr).$$

14. Let $\psi_2(x,y)$ denote the number of squarefree integers $n \le x$ composed entirely of primes $p \le y$.
(a) Show that

$$\psi_2(x,y) = \sum_{\substack{d\le x\\ p|d \Rightarrow p\le y}} \mu(d)\psi(x/d^2, y).$$

(b) (Ivić) Let $\delta > 0$ be fixed. Then

$$\psi_2(x,y) \sim \frac{6}{\pi^2}\psi(x,y)$$

uniformly for $x^\delta \le y \le x$.
(c) Show that $\psi_2(x, \log x) = \psi(x, \log x)^{1/2+o(1)}$.
(d) Show that if $a > 1$ and $y \ge (\log x)^a$ then $\psi_2(x,y) = \psi(x,y)^{1+o(1)}$.
(e) Show that if $0 < a < 1$ and $y \le (\log x)^a$ then $\psi_2(x,y) = \psi(x,y)^{o(1)}$.
(f) Show that $\psi_2(x, c\log x) = \psi(x, c\log x)^{\phi(c)+o(1)}$ for any fixed $c > 0$, where

$$\phi(c) = \begin{cases} \frac{c\log 2}{(c+1)\log(c+1) - c\log c} & (0 < c \le 2),\\ \frac{c\log c - (c-1)\log(c-1)}{(c+1)\log(c+1)-c\log c} & (c \ge 2). \end{cases}$$

2. Numbers composed of large primes

Let $\Phi(x,y)$ denote the number of integers $n \le x$ composed entirely of primes $p \ge y$. The number 1 is such a number as it is an empty product. Thus it is clear that if $y > x$ then

$$\Phi(x,y) = 1 \tag{34}$$

Also, if $x^{1/2} \le y \le x$ then

$$\Phi(x,y) = \pi(x) - \pi(y^-) + O(1) = \frac{x}{\log x} - \frac{y}{\log y} + O\Big(\frac{x}{(\log x)^2}\Big) \tag{35}$$

For smaller values of y we show that

$$\Phi(x,y) \sim \frac{w(u)x}{\log y} \tag{36}$$

where $u = (\log x)/\log y$ and $w(u)$ is a function determined by the initial condition

$$w(u) = 1/u \tag{37}$$

for $1 < u \le 2$ and for $u > 2$ by the differential–delay equation

$$\big(uw(u)\big)' = w(u-1). \tag{38}$$

FIGURE 2. Buchstab's function $w(u)$ and its horizontal asymptote e^{-C_0} for $1 \le u \le 4$.

Before proceeding further we first derive some of the simplest properties of the function $w(u)$. By integrating (38) we deduce that $uw(u) = \int_1^{u-1} w(v)\,dv + C$ for $u > 2$, and by letting u tend to 2 we find that $C = 1$ so that

$$uw(u) = \int_1^{u-1} w(v)\,dv + 1 \tag{39}$$

for $u \ge 2$. From this it is evident that if $w(v) \le 1$ for $v \le u-1$ then $w(v) \le 1$ for $v \le u$, and that if $w(v) \ge 1/2$ for $v \le u-1$ then $w(v) \ge 1/2$ for $v \le u$. Thus we conclude that $1/2 \le w(u) \le 1$ for all $u > 1$. From the identity $uw'(u) = w(u-1) - w(u)$ we deduce that $|w'(u)| \le 1/(2u)$ for all $u > 2$. Let $M(u) = \max_{v \ge u} |w'(v)|$. Since $w(u-1) - w(u) = -w'(\xi)$ for some ξ, $u-1 < \xi < u$, we know that

$$M(u) \le M(u-1)/u.$$

Let k be chosen so that $1 < u - k \le 2$. By using the above inequality k times we find that

$$M(u) \le \frac{M(u-k)}{u(u-1)\cdots(u-k+1)} \ll \frac{1}{\Gamma(u+1)}.$$

That is,

$$w'(u) \ll \frac{1}{\Gamma(u+1)} \tag{40}$$

for $u > 2$. Since $w'(u)$ tends to 0 rapidly, it follows that the integral $\int_2^\infty w'(v)\,dv$ converges absolutely, and hence we see that $\lim_{u\to\infty} w(u)$ exists. Since it is to be expected that $\Phi(x,y)$ is approximately $x\prod_{p<y}(1-1/p)$ when y is small, it is not surprising that

$$\lim_{u\to\infty} w(u) = e^{-C_0}. \tag{41}$$

We shall prove this later, as a consequence of Theorem 12. First we establish the basic asymptotic estimate (36).

Theorem 11. (Buchstab) *Let $\Phi(x, y)$ denote the number of positive integers $n \le x$ composed entirely of prime numbers $p \ge y$, and let $w(u)$ be defined as above. Then*

$$\Phi(x, y) = \frac{w(u)x}{\log y} - \frac{y}{\log y} + O\Big(\frac{x}{(\log x)^2}\Big) \tag{42}$$

uniformly for $1 \le u \le U$ and all $y \ge 2$. Here $u = (\log x)/\log y$, which is to say that $y = x^{1/u}$.

The term $-y/\log y$ can be included in the error term when $y \ll x/\log x$ but, in view of (35), has to be present when y is close to x. It might be difficult to prove that the above holds uniformly for all $u \ge 1$ because of the precise form of the error term, but the weaker assertion (36) can be shown to hold for $u \ge 1 + \varepsilon$, since sieve methods can be used when u is large.

Proof. The number of positive integers $n \le x$ whose least prime factor is p is exactly $\Phi(x/p, p)$. Hence by classifying integers according to their least prime factor we see that

$$\Phi(x, y) = 1 + \sum_{y \le p \le x} \Phi(x/p, p). \tag{43}$$

This is an identity of Buchstab; similar 'Buchstab identities' are important in sieve theory. We show by induction on U that

$$\Phi(x, y) = \frac{w(u)x}{\log y} - \frac{y}{\log y} + O\Big(\frac{x}{(\log x)^2}\Big) \tag{44}$$

for $U \le u \le U + 1$. When $U = 1$ this is (35), and it is only in this first range that the second main term is significant. For the inductive step we apply (43) with $y = x^{1/u}$ and with $y = x^{1/U}$ and subtract to see that

$$\Phi(x, x^{1/u}) = \Phi(x, x^{1/U}) + \sum_{x^{1/u} \le p < x^{1/U}} \Phi(x/p, p).$$

Choose u_p so that $p = (x/p)^{1/u_p}$. Then the above is

$$\Phi(x, x^{1/U}) + \sum_{x^{1/u} \le p < x^{1/U}} \Phi\big(x/p, (x/p)^{1/u_p}\big).$$

But $u_p = (\log x)/\log p - 1 \in [U - 1, U]$, so by the inductive hypothesis, when $U \ge 2$, the above is

$$\frac{Uw(U)x}{\log x} + O\left(\frac{x}{(\log x)^2}\right) + \sum_{x^{1/u} \le p < x^{1/U}} \left(\frac{u_p w(u_p)x}{p \log x/p} + O\left(\frac{x}{p(\log x)^2}\right) + O\left(\frac{p}{\log p}\right)\right).$$

The sum over p of the first error term is $\ll x/(\log x)^2$, and the sum over p of the second is $\ll x^{2/U}/(\log x)^2$, which is acceptable since $U \ge 2$. To estimate the contribution of the main term in the sum we write the Prime Number Theorem in the form $\pi(t) = \mathrm{li}(t) + R(t)$, apply Riemann–Stieltjes integration, and integrate the term involving $R(t)$ by parts, to see that the sum of the main term is

$$\int_{x^{1/u}}^{x^{1/U}} \frac{xw(\frac{\log x}{\log t} - 1)}{t(\log t)^2}\,dt + \Big[f(t)R(t)\Big|_{x^{1/u}-}^{x^{1/U}-} - \int_{x^{1/u}}^{x^{1/U}} R(t)\,df(t) \tag{45}$$

where

$$f(t) = \frac{xw(\frac{\log x}{\log t} - 1)}{t\log t}.$$

Since $f'(t) \ll x/(t^2\log t)$ and $R(t) \ll t/(\log t)^A$, the terms involving $R(t)$ contribute an amount $\ll_U x/(\log x)^A$. By the change of variables $v = (\log x)/\log t - 1$ we see that the first integral in (45) is

$$\frac{x}{\log x}\int_{U-1}^{u-1} w(v)\,dv,$$

which by (39) is

$$= \frac{x}{\log x}(uw(u) - Uw(U)).$$

On combining our estimates we obtain (44), so the inductive step is complete.

We now derive formulæ for $w(u)$ similar to those in Theorem 10 involving $\rho(u)$.

Theorem 12. *If $\Re s > 0$ then*

$$s + s\int_1^\infty w(u)e^{-us}\,du = \exp\Big(-C_0 + \int_0^s \frac{1-e^{-z}}{z}\,dz\Big) \tag{46}$$

where C_0 is Euler's constant. If $u > 1$ and $\sigma_0 > 0$, then

$$w(u) = \frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty}\Big(\exp\Big(\int_s^\infty \frac{e^{-z}}{z}\,dz\Big) - 1\Big)e^{us}\,ds. \tag{47}$$

Since the right hand side of (46) is an entire function, we see that the Laplace transform of $w(u)$ is entire apart from a simple pole at $s = 0$ with residue e^{-C_0}.

Proof. Let $G(s)$ denote the left hand side of (46). Then

$$\Big(\frac{G(s)}{s}\Big)' = -\int_1^\infty w(u)ue^{-us}\,du.$$

By integrating by parts we see that this is

$$\Big[\frac{w(u)ue^{-us}}{s}\Big|_1^\infty - \frac{1}{s}\int_2^\infty w(u-1)e^{-us}\,du = \frac{-e^{-s}G(s)}{s^2}$$

by (37) and (38). That is,

$$G'(s) = G(s)\frac{1-e^{-s}}{s},$$

which by the method of separation of variables implies that

$$G(s) = A \exp\Big(\int_0^s \frac{1-e^{-z}}{z}\,dz\Big)$$

where A is a positive constant. To determine the value of A we note that

$$1 = \lim_{s\to\infty} \frac{G(s)}{s} = A \exp\Big(\int_0^1 \frac{1-e^{-z}}{z}\,dz - \int_1^\infty \frac{e^{-z}}{z}\,dz\Big).$$

From (31) we deduce that $A = e^{-C_0}$, and hence we have (46). To obtain (47) it suffices to take the inverse Laplace transform, since

$$\int_0^s \frac{1-e^{-z}}{z}\,dz = \int_s^\infty \frac{e^{-z}}{z}\,dz + \log s + C_0\,.$$

7.2. Exercises

1. By using (31), or otherwise, show that

$$\int_0^s \frac{1-e^{-z}}{z}\,dz = C_0 + \log s + \int_s^\infty \frac{e^{-z}}{z}\,dz$$

when $\Re s > 0$.

2. (a) Show that

$$w(u) = \frac{1+\log(u-1)}{u}$$

for $2 \le u \le 3$.
(b) Show that

$$w(u) = \frac{1}{u}\Big(1 + \log(u-1) + \int_3^u \frac{\log(v-2)}{v-1}\,dv\Big)$$

for $3 \le u \le 4$.
(c) Show that

$$w(u) = \frac{1}{u}\Big(1 + \log(u-1) + \int_3^u \frac{\log(v-2)}{v-1}\,dv + \int_4^u \frac{\log\frac{u-1}{v-1}\log(v-3)}{v-2}\,dv\Big)$$

for $4 \le u \le 5$.

3. (Friedlander (1972)) Let $\mathcal{S}$ be a set of positive integers not exceeding X, and suppose that $(a,b) \le Y$ whenever $a \in \mathcal{S}$, $b \in \mathcal{S}$, $a \ne b$. Let $M(X,Y)$ denote the maximum cardinality of all such sets $\mathcal{S}$.
(a) Let $\mathcal{S}_0$ be the set of those positive integers $n \le X$ such that if $d|n$, $d < n$, then $d \le Y$.

Show that $\operatorname{card} \mathcal{S}_0 = M(X,Y)$.
(b) Show that if $Y \le X^{1/2}$ then

$$M(X,Y) = 1 + \pi(X) - \pi(Y) + \sum_{p \le Y} \Phi(Y,p).$$

(c) Show that if $X^{1/2} < Y \le X$ then

$$M(X,Y) = 1 + \pi(X) - \pi(Y) + \sum_{p < X/Y} \Phi(Y,p) + \sum_{X/Y \le p \le Y} \Phi(X/p,p).$$

3. Primes in short intervals

Let Jacobstal's function $g(q)$ be the length of the longest gap between consecutive reduced residues modulo q. We show that there are long gaps between primes by showing that there exist integers q for which $g(q)$ is large. Since the average gap between consecutive reduced residues (mod q) is $q/\varphi(q)$, it is obvious that

$$g(q) \ge \frac{q}{\varphi(q)}.$$

If $p_1 < p_2 < \ldots < p_k$ are the distinct primes dividing q, then by the Chinese remainder theorem there is an x such that $x \equiv -i \pmod{p_i}$ for $1 \le i \le k$. Then $(x+i,q) > 1$ for $1 \le i \le k$, and hence

$$g(q) \ge \omega(q) + 1.$$

These observations can be combined: It can be shown that

$$g(q) \gg \frac{q\omega(q)}{\varphi(q)}. \tag{48}$$

This is not quite enough to produce long gaps between primes, but for certain q we improve on the above to establish

Lemma 13. *Let $P = P(z) = \prod_{p \le z} p$. Then*

$$\lim_{z\to\infty} \frac{g\big(P(z)\big)}{z} = \infty.$$

This immediately yields

Theorem 14. (Westzynthius) *Let p_n denote the n^{th} prime number in increasing order. Then*

$$\limsup_{n\to\infty} \frac{p_{n+1}-p_n}{\log p_n} = \infty.$$

Proof of Theorem 14. Suppose that $N = g(P) - 1$ and that M is chosen, $P \le M < 2P$, so that $(M+m,P) > 1$ for $1 \le m \le N$. But $M+m > P \ge (M+m,P)$, and hence $M+m$

is composite because it has the proper divisor $(M+m,P)$. If n is chosen so that p_n is the largest prime not exceeding M then $p_{n+1}-p_n \ge g(P)$ and $p_n < 2P$, which is $< e^{2z}$ when z is large. Hence

$$\frac{p_{n+1}-p_n}{\log p_n} \ge \frac{g(P)}{2z}$$

which tends to infinity as $z \to \infty$.

Proof of Lemma 13. Let L be large and fixed, and put $N = [zL/3]$. We show that if $z > z_0(L)$ then there exists an integer M such that $(M+n, P(z)) > 1$ for $1 \le n \le N$. Put

$$P_1 = \prod_{p\le L} p \qquad P_2 = \prod_{L<p\le L^L} p, \qquad P_3 = \prod_{L^L<p\le z/3} p \qquad P_4 = \prod_{z/3<p\le z} p,$$

and let $\mathcal{N}$ be the set of those integers n, $1 \le n \le N$, such that $(n, P_1P_3) = 1$. The members of $\mathcal{N}$ are (i) 1; (ii) integers n composed entirely of prime factors of P_2; (iii) primes p, $z/3 < p \le N$. Thus

$$\operatorname{card}\mathcal{N} \le 1 + \psi(N, L^L) + \pi(N) - \pi(z/3).$$

If z is sufficiently large then $L^L < \log N$, so that $\psi(N, L^L) < N^\varepsilon$ by Corollary 9. Hence

$$\operatorname{card}\mathcal{N} < \pi(N).$$

We choose $M \equiv 0 \pmod{P_1P_3}$, so that $(M+n, P_1P_3) > 1$ if $1 \le n \le N$, $n \notin \mathcal{N}$. To bound the number of $n \in \mathcal{N}$ such that $(M+n, P_2) = 1$ we average as in the proof of Lemma 3.5. Clearly

$$\sum_{m=1}^{q} \sum_{\substack{n\in\mathcal{N}\\(m+n,q)=1}} 1 = \sum_{n\in\mathcal{N}} \sum_{\substack{m=1\\(m+n,q)=1}}^{q} 1 = \sum_{n\in\mathcal{N}} \varphi(q) = \varphi(q)\operatorname{card}\mathcal{N}$$

for any integer q. Hence

$$\min_m \sum_{\substack{n\in\mathcal{N}\\(m+n,q)=1}} 1 \le (\operatorname{card}\mathcal{N}) \prod_{p\mid q}\Big(1-\frac{1}{p}\Big).$$

By taking $q = P_2$ we see that there is an $M \pmod{P_2}$ such that

$$\operatorname{card}\{n\in\mathcal{N} : (M+n, P_2) = 1\} \le (\operatorname{card}\mathcal{N}) \prod_{p\mid P_2}\Big(1-\frac{1}{p}\Big).$$

For such an M,

$$\operatorname{card}\{1\le n\le N : (M+n, P_1P_2P_3) = 1\} \le \pi(N) \prod_{p\mid P_2}\Big(1-\frac{1}{p}\Big).$$

By Mertens' theorem (Theorem 2.7(e)), the product on the right is $\sim 1/L$ as $L \to \infty$. Suppose that L is chosen sufficiently large to ensure that this product is $\le 3/(2L)$. Then the right hand side above is

$$\lesssim \frac{3N}{2L\log N} \sim \frac{z}{2\log z}.$$

The number of primes dividing P_4 is $\pi(z) - \pi(z/3) \sim 2z/(3\log z)$ as $z \to \infty$. Thus if z is large then there are more such primes than there are integers n, $1 \le n \le N$, for which $(M + n, P_1P_2P_3) = 1$. Hence for each such n we may associate a prime p_n, $p_n|P_4$, in a one-to-one manner, and take $M \equiv -n \pmod{p_n}$. Then $(M + n, P_4) > 1$ and we are done.

The success of the argument just completed can be attributed to the fact that the number of n, $1 \le n \le N$ for which $(n, P_1P_3) = 1$ is considerably smaller than $N\prod_{p|P_1P_3}(1-1/p)$. By considering how L may be chosen as a function of z we obtain a quantitative improvement of Lemma 13 and hence also of Theorem 14.

Theorem 15. (Rankin) *Let p_n denote the n^{th} prime number in increasing order. There is a constant $c > 0$ such that*

$$\limsup_{n\to\infty} \frac{p_{n+1} - p_n}{\left(\frac{(\log p_n)(\log\log p_n)(\log\log\log\log p_n)}{(\log\log\log p_n)^2}\right)} \ge c.$$

Proof. We repeat the argument in the proof of Lemma 13, with the sole change that L is allowed to depend on z. If L is chosen so that

$$\psi(N, L^L) < \frac{N}{(\log N)^2} \tag{49}$$

then $L = o(\log N)$, and hence

$$\psi(N, L^L) = o\Big(\frac{z}{\log N}\Big).$$

Since $z/\log N \le z/\log z \ll \pi(z/3)$, it follows that

$$\psi(N, L^L) = o\big(\pi(z/3)\big),$$

and the proof proceeds as before.

By Theorem 6 we see that

$$\psi(N, N^{1/u}) < \frac{N}{(\log N)^2}$$

if $u\log u \ge 3\log\log N$, which is the case if $u \ge 4(\log\log N)/\log\log\log N$. Taking $u = (\log N)/\log L^L$, we deduce that (49) holds if

$$L\log L < \frac{(\log N)(\log\log\log N)}{4\log\log N}.$$

This is satisfied if

$$L < \frac{(\log N)(\log\log\log N)}{4(\log\log N)^2}$$

since then $\log L < \log\log N$. Since $N > z$ when $L \ge 3$, we conclude that we may take

$$L = \frac{(\log z)(\log\log\log z)}{4(\log\log z)^2}.$$

Hence

$$g\big(P(z)\big) > \frac{z(\log z)(\log\log\log z)}{13(\log\log z)^2}$$

for all $z > z_0$, and this gives the stated result.

Concerning the maximum number of primes in a short interval, by the Brun–Titchmarsh inequality (Theorem 3.9) and the Prime Number Theorem we see that

$$\pi(x+y) - \pi(x) < (2+\varepsilon)\pi(y)$$

for $y > y_0(\varepsilon)$. Let

$$\rho(y) = \limsup_{x\to\infty} \big(\pi(x+y) - \pi(x)\big). \tag{50}$$

Thus $\rho(y) < (2+\varepsilon)\pi(y)$. Very little is known about $\rho(y)$. It was once conjectured that

$$\pi(M+N) \le \pi(M) + \pi(N) \tag{51}$$

for $M > 1, N > 1$, but there is now serious doubt as to the validity of this inequality. Indeed, it seems likely that $\rho(y) > \pi(y)$ for all large y. To see why, let

$$\overline{\rho}(N) = \max_M \sum_{\substack{n=M+1 \\ p\mid n \Rightarrow p>N}}^{M+N} 1. \tag{52}$$

Clearly $\rho(N) \le \overline{\rho}(N)$. We expect that

$$\rho(N) = \overline{\rho}(N) \tag{53}$$

for all N, since this would follow from the

Prime k-tuple Conjecture. *Let $a_1, a_2, \ldots a_k$ be given integers. Then there exist infinitely many positive integers n such that $n+a_1, n+a_2, \ldots, n+a_k$ are all prime, provided that for every prime number p there is an integer n such that $(n+a_i, p) = 1$ for $i = 1, 2, \ldots, k$.*

We now show that $\overline{\rho}(N) > \pi(N)$ for all large N, so that (51) and (53) are inconsistent.

Theorem 16. *There is an absolute constant N_0 such that if $N > N_0$ then $\overline{\rho}(N) - \pi(N) \gg N(\log N)^{-2}$.*

Proof. Suppose that N is even and that $N > 2$. Then for every M,

$$\sum_{\substack{n=M+1 \\ p|n \Rightarrow p>N}}^{M+N} 1 = \sum_{\substack{n=M+1 \\ p|n \Rightarrow p \geq N}}^{M+N} 1 \geq \sum_{\substack{n=M+1 \\ p|n \Rightarrow p>N-1}}^{M+N-1} 1.$$

Hence $\overline{\rho}(N) \geq \overline{\rho}(N-1)$ when N is even, $N > 2$, so it suffices to treat the case when N is odd, say $N = 2K+1$. Let $\mathcal{P}(K)$ denote the set of integers n with $K/(2\log K) < |n| \leq K$ and $|n|$ prime. Then

$$\operatorname{card} \mathcal{P}(K) = 2(\pi(K) - \pi(K/(2\log K))),$$

so by Theorem 6.9,

$$\operatorname{card} \mathcal{P}(K) = \pi(2K+1) + (c + o(1))\frac{K}{(\log K)^2}$$

where $c = 2\log 2 - 1 > 0$. We now show that $\mathcal{P}(K)$ can be translated to form a set of integers $\{M + n : n \in \mathcal{P}(K)\}$ with each member coprime to $\prod_{p \leq N} p$. By the Chinese Remainder Theorem it suffices to show that for every prime number $p \leq N$ there is a residue class $r_p \pmod p$ that contains no element of $\mathcal{P}(K)$.

Obviously each element of $\mathcal{P}(K)$ is coprime to each prime $p \leq K/(2\log K)$, so we may take $r_p = 0$ for such primes. It remains to treat the primes p for which $K/(2\log K) < p \leq 2K+1$. This is accomplished by means of a clever application of Lemma 13. Suppose that $K/(2\log K) < p \leq 2K+1$. We show that there is an r_p such that if $|hp + r_p| \leq K$, then $hp + r_p \notin \mathcal{P}(K)$. By Lemma 13 there is an interval $\mathcal{J} = [M_1 - 3\log K, M_1 + 3\log K]$ in which every integer j is divisible by a prime p_j with $p_j \leq \frac{1}{3}\log K$. By the Chinese Remainder Theorem, we can choose r_p so that $r_p \equiv M_1 p \pmod{p_j}$ for each $j \in \mathcal{J}$. This can be done with $0 < r_p \leq \exp\big(\vartheta(\frac{1}{3}\log K)\big) < K^{1/2}$. If $|h| \leq 3\log K$ then $h = j - M_1$ for some $j \in \mathcal{J}$ and so $h \equiv -M_1 \pmod{p_j}$. Hence $hp + r_p \equiv -M_1 p + r_p \equiv 0 \pmod{p_j}$, which implies that $hp + r_p \notin \mathcal{P}(K)$. On the other hand, if $|h| > 3\log K$, then $|hp + r_p| \geq \big(\frac{3}{2} - o(1)\big)K > K$, so that $hp + r_p \notin \mathcal{P}(K)$ in this case also. Since the arithmetic progression $hp + r_p$ has no element in common with $\mathcal{P}(K)$ the proof is complete.

7.3. Exercises

1. Show that the function $\overline{\rho}(N)$ is weakly increasing.

2. (a) Show that in the prime k-tuple conjecture, the hypothesis that for every prime p the numbers a_j do not cover all residue classes (mod p) is satisfied for all $p > k$, so that it is enough to verify the hypothesis for $p \leq k$ (a finite calculation for any given set of a_j).
(b) Prove the converse of the prime k-tuple conjecture: If there exist infinitely many integers n for which $n + a_j$ is prime for all j, $1 \leq j \leq k$, then for every prime p there is a residue class $x \pmod p$ such that $x + a_j \not\equiv 0 \pmod p$ $(1 \leq j \leq k)$.

3. Show that $g(q) \gg q\omega(q)/\varphi(q)$.

4. (cf Erdős (1951)) Show that if $0 < c < 1/2$ then there exist arbitrarily large numbers x such that the interval $(x, x + c(\log x)/\log\log x)$ contains no squarefree number.

5. (cf Erdős (1946), Montgomery (1987)) Suppose that $2 \le h \le x$. Let $\mathcal{P}$ denote the set of all primes $p \le h$, let $\mathcal{D}$ denote the set of positive integers composed entirely of primes in $\mathcal{P}$, and let $f(n) = \prod_{p|n, p\in\mathcal{P}}(1 - 1/p)$.
(a) Show that $f(n) = \sum_{d|n, d\in\mathcal{D}} \mu(d)/d$.
(b) Show that

$$\sum_{x<n\le x+h} f(n) = \frac{6}{\pi^2}h + O(\log h)$$

uniformly in x.
(c) Show that

$$\frac{\varphi(n)}{n} \ge f(n) - \sum_{\substack{p|n \\ p>h}} \frac{1}{p}.$$

(d) Among those primes $p > h$ that divide an integer in the interval $(x, x + h]$, let $\mathcal{Q}$ be those for which $p \le h \log x$, and $\mathcal{R}$ those for which $p > h \log x$. Show that

$$\sum_{p\in\mathcal{Q}} \frac{1}{p} \ll \log\log\log x.$$

(e) Explain why

$$\prod_{\substack{p\in\mathcal{R} \\ U<p\le 2U}} p \,\Bigg|\, \prod_{x<n\le x+h} n,$$

and deduce that

$$\operatorname{card}\{p \in \mathcal{R} : U < p \le 2U\} \ll \frac{h \log x}{\log U}.$$

(f) By summing over $U = 2^k h \log x$, show that

$$\sum_{p\in\mathcal{R}} \frac{1}{p} \ll \frac{1}{\log(h \log x)}.$$

(g) Show that

$$\frac{6}{\pi^2}h + O(\log h) + O(\log\log\log x) \le \sum_{x<n\le x+h} \frac{\varphi(n)}{n} \le \frac{6}{\pi^2}h + O(\log h).$$

6. (cf Pillai & Chowla (1930)) Show that there is an absolute constant $c > 0$ such that there exist arbitrarily large x for which $\varphi(n)/n < 1/4$ when $x < n \le x + c\log\log\log x$. Deduce that

$$\sum_{n\le x} \frac{\varphi(n)}{n} - \frac{6}{\pi^2}x = \Omega(\log\log\log x).$$

7. (Hausman & Shapiro (1973); cf Montgomery & Vaughan (1986)) (a) Show that

$$\sum_{n=1}^{q}\Big(\sum_{\substack{m=1\\(m+n,q)=1}}^{h} 1-\frac{\varphi(q)}{q}h\Big)^2=\frac{\varphi(q)^2}{q}\sum_{\substack{r|q\\r>1}}\mu(r)^2\frac{r^2}{\varphi(r)^2}\{h/r\}(1-\{h/r\})\prod_{\substack{p|q\\p\nmid r}}\frac{p(p-2)}{(p-1)^2}.$$

(b) Use the inequality $\{\alpha\}(1-\{\alpha\})\le\alpha$ to show that

$$\sum_{n=1}^{q}\Big(\sum_{\substack{m=1\\(m+n,q)=1}}^{h} 1-\frac{\varphi(q)}{q}h\Big)^2\le h\varphi(q).$$

8. (Erdős (1951)) Put $P=\prod_{p\le y}p^2$, set $k=\pi(y)$, and let $2=p_1<p_2<\cdots<p_k$ denote the primes $p\le y$.
(a) Explain why there is an x, $1\le x\le P$, such that $p_i^2|(x+i)$ for $1\le i\le k$.
(b) Show that $k\gg(\log x)/\log\log x$.
(c) Show that there exist arbitrarily large integers x such that there is no squarefree integer between x and $x+c(\log x)/\log\log x$. Here c is a suitably small positive constant.

9. (Erdős (1951)) (a) For a positive integer q, let $\mathcal{S}(q)$ denote the set of those residue classes s modulo q^2 such that (s,q) is a perfect square. Show that if q is squarefree, then $\mathcal{S}(q)$ contains exactly $\prod_{p|q}(p^2-p+1)$ elements.
(b) Show that if q is squarefree and $1\le h\le q^2$, then there is an integer a such that the number of members of $\mathcal{S}(q)$ in the interval $(a,a+h]$ is at most

$$h\prod_{p|q}\Big(1-\frac{1}{p}+\frac{1}{p^2}\Big).$$

(c) From now on, suppose that q is the product of those primes $p\le y$ such that $p\equiv 3\pmod 4$. By recalling Corollary 4.12, or otherwise, show that the expression above is $\asymp h/\sqrt{\log y}$.
(d) Show that if an integer n can be expressed as a sum of two squares, then $n\in\mathcal{S}(q)$.
(e) Let $\mathcal{R}$ be the set of those primes p, $y<p\le Cy$, such that $p\equiv 3\pmod 4$. Here C is an absolute constant, taken to be sufficiently large to ensure that $\mathcal{R}$ has at least $y/\log y$ elements. Note that such a constant exists, in view of Exercise 4.3.5(e). Let r denote the product of all members of $\mathcal{R}$. Suppose that the number of members of $\mathcal{S}(q)$ lying in the interval $(a,a+h]$ is $<y/\log y$. For each $s\in\mathcal{S}(q)$ satisfying $a<s\le a+h$, associate a prime $p\in\mathcal{R}$. Suppose that the integer b is chosen modulo p^2 so that $s+bq^2\equiv p\pmod{p^2}$. Show that the interval $(a+bq^2,a+bq^2+h]$ does not contain a sum of two squares.
(f) Show that a and b can be chosen so that $0<a+bq^2<(qr)^2$.
(g) Show that $\log qr\ll y$.
(h) Show that this construction succeeds with $h\asymp y/\sqrt{\log y}\gg(\log qr)/(\log\log qr)^{1/2}$.
(i) Conclude that there exist arbitrarily large x such that there is no sum of two squares

between x and $x + c(\log x)/(\log\log x)^{1/2}$. Here c is a suitably small positive constant. (Note that a stronger result is established in the next exercise.)

10. (Richards (1982)) For every prime $p \le y$, let $\beta(p)$ denote the greatest positive integer such that $p^\beta \le y$, and put

$$q = \prod_{\substack{p\le y \\ p\equiv 3\,(4)}} p^{2\beta(p)}.$$

(a) Show that $q = \exp(2\psi(y;4,3))$.
(b) Show that $\log q \ll y$.
(c) Suppose that $1 \le n \le y$. Show that if $n \equiv 3 \pmod 4$, then there is a prime $p|q$ such that p divides n to an odd power.
(d) Let $x = (q-1)/4$. Show that x is an integer, and that $4x \equiv -1 \pmod q$.
(e) Show that if $1 \le i \le y/4$ and $p|q$, then the power of p that exactly divides $x+i$ is the same as the power of p that exactly divides $4i-1$.
(f) Deduce that no integer in the interval $(x, x+y/4]$ can be expressed as a sum of two squares.
(g) Conclude that there exist arbitrarily large numbers x such that no number between x and $x + c\log x$ is a sum of two squares. Here c is a suitably small positive constant.

4. Numbers composed of a prescribed number of primes

Let $\sigma_k(x)$ denote the number of integers n with $1 \le n \le x$ and $\Omega(n) = k$. Then $\sigma_1(x) = \pi(x) \sim x/\log x$. Consider $\sigma_2(x)$. Clearly

$$\sigma_2(x) = \sum_{\substack{p_1,p_2 \\ p_1\le p_2 \\ p_1p_2\le x}} 1 = \sum_{p\le\sqrt{x}} \big(\pi(x/p) - \pi(p) + O(1)\big)\,.$$

By the Prime Number Theorem this is

$$= \sum_{p\le\sqrt{x}} (1+o(1))\frac{x}{p(\log x/p)} + O\Big(\frac{x}{\log x}\Big)\,.$$

Thus, by partial summation and a further application of the Prime Number Theorem we find that

$$\sigma_2(x) \sim \frac{x\log\log x}{\log x}. \tag{54}$$

By inducting on k in this manner it can be shown that

$$\sigma_k(x) \sim \frac{x(\log\log x)^{k-1}}{(k-1)!\log x} \tag{55}$$

for any fixed k. Since the sum over all $k \ge 1$ of the right hand side is exactly x, it is tempting to think that the above holds quite uniformly in k. However this is not the case, as we shall presently discover. To obtain precise estimates that are uniform in k we apply analytic methods. In §2.4 we determined the asymptotic distribution of the additive function $\Omega(n)-\omega(n)$ by establishing the mean value of the multiplicative function $z^{\Omega(n)-\omega(n)}$. In the same spirit we shall derive information concerning the distribution of $\Omega(n)$ from mean value estimates of $z^{\Omega(n)}$. Since the Euler product of this latter function behaves badly when $|z|$ is large, we start not with $z^{\Omega(n)}$ but with $d_z(n)$ defined by the identities

$$\zeta(s)^z = \prod_p \left(1-p^{-s}\right)^{-z} = \sum_{n=1}^{\infty} d_z(n) n^{-s} \qquad (\sigma > 1). \tag{56}$$

Since $d_z(p) = z = z^{\Omega(p)}$, the functions $d_z(n)$ and $z^{\Omega(n)}$ are 'nearby', and hence the mean value of $z^{\Omega(n)}$ can be derived from that for $d_z(n)$ by elementary reasoning.

Theorem 17. *Let $D_z(x) = \sum_{n\le x} d_z(n)$, and let R be any positive real number. If $x \ge 2$, then*

$$D_z(x) = \frac{x(\log x)^{z-1}}{\Gamma(z)} + O\big(x(\log x)^{\Re z-2}\big)$$

uniformly for $|z| \le R$.

Proof. Let $a = 1 + 1/\log x$. Then by Corollary 5.3,

(57)

$$D_z(x) - \frac{1}{2\pi i}\int_{a-iT}^{a+iT} \zeta(s)^z \frac{x^s}{s}\, ds \ll \sum_{\frac12 x<n<2x} |d_z(n)| \min\Big(1, \frac{x}{T|x-n|}\Big) + \frac{x^a}{T}\sum_n |d_z(n)| n^{-a}.$$

Since $|d_z(n)|$ is erratic, we must exercise some care in estimating the error terms above. Let $\mathcal{A} = \{n : |n-x| \le x/(\log x)^{2R+1}\}$. Without loss of generality we may suppose that R is an integer. We note that $|d_z(n)| \le d_{|z|}(n) \le d_R(n)$. By the method of the hyperbola we see by induction on R that

$$D_R(x) = xP_R(\log x) + O_R\big(x^{1-1/R}\big)$$

where P_R is a polynomial of degree $R-1$. Hence the contribution to the first sum in the error term in (57) of the $n \in \mathcal{A}$ is

$$\ll \sum_{n\in\mathcal{A}} |d_z(n)| \ll x(\log x)^{-R-2}$$

The contribution of the $n \notin \mathcal{A}$ is

$$\ll T^{-1}(\log x)^{2R+1} x(\log x)^{R-1}.$$

We take $T = \exp\big(\sqrt{\log x}\big)$ to see that this is also $\ll x(\log x)^{-R-2}$. The second sum in the error term in (57) is $\ll \zeta(a)^R \ll (\log x)^R$. Thus the total error term is $\ll x(\log x)^{-R-2}$.

If z is a positive integer then $\zeta(s)^z$ has a pole at $s = 1$, and we can extract a main term by the calculus of residues, as in our proof of the Prime Number Theorem (Theorem

6.9). On the other hand, if z is not an integer then $\zeta(s)^z$ has a branch point at $s = 1$, so greater care must be exercised in moving the path of integration. Put $b = 1 - c/\log T$ where c is a small positive constant, and replace the contour from $a - iT$ to $a + iT$ by a path consisting of $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ where $\mathcal{C}_1$ is polygonal with vertices $a - iT$, $b - iT$, $b - i/\log x$, $\mathcal{C}_2$ begins with a line segment from $b - i/\log x$ to $1 - i/\log x$, continues with the semicircle $\{1 + e^{i\theta}/\log x : -\pi/2 \le \theta \le \pi/2\}$, and concludes with the line segment from $1 + i/\log x$ to $b + i/\log x$, and finally $\mathcal{C}_3$ is polygonal with vertices $b + i/\log x$, $b + iT$, $a + iT$. By Theorem 6.7, $\zeta(s)^z \ll (\log x)^R$ on the new path, so the integrals over $\mathcal{C}_1$ and $\mathcal{C}_3$ contribute an amount $\ll x(\log x)^{-R-2}$. On $\mathcal{C}_2$ we have $\zeta(s)^z/s = (s-1)^{-z}\big(1 + O(|s-1|)\big)$. Hence

$$(58) \qquad \frac{1}{2\pi i}\int_{\mathcal{C}_2} \zeta(s)^z \frac{x^s}{s}\,ds = \frac{1}{2\pi i}\int_{\mathcal{C}_2} (s-1)^{-z} x^s\,ds + O\Big(\int_{\mathcal{C}_2} |s-1|^{1-\Re z} x^\sigma\,|ds|\Big).$$

By the change of variables $s = 1 + w/\log x$ we see that the main term above is

$$x(\log x)^{z-1}\frac{1}{2\pi i}\int_{\mathcal{H}_2} w^{-z} e^w\,dw$$

where $\mathcal{H}_2$ starts at $-\beta - i$, loops around 0, and ends at $-\beta + i$ where $\beta = c(\log x)/\log T$. Let $\mathcal{H}_1$ be the contour $\mathcal{H}_1 = \{w = u - i : -\infty < u \le -\beta\}$, and similarly let $\mathcal{H}_3 = \{w = u + i : -\infty < u \le -\beta\}$. If we integrate over the union of the $\mathcal{H}_i$ then we obtain Hankel's formula (see Theorem C.3) for $1/\Gamma(z)$. The integral over $\mathcal{H}_1$ is $\ll_R \int_\beta^\infty e^{-u/2}\,du \ll_R e^{-\beta/2}$, which is small since $T = \exp\big(\sqrt{\log x}\big)$. Thus we see that the main term in (58) is $x(\log x)^{z-1}/\Gamma(z) + O_R(x\exp(-c\sqrt{\log x}))$ for some constant c. On the semicircular part of $\mathcal{C}_2$ the integrand in the error term in (58) is $\ll x(\log x)^{\Re z-1}$, so the contribution is $\ll x(\log x)^{\Re z-2}$. By the change of variables $s = 1 + w/\log x$ we see that the linear portions of $\mathcal{C}_2$ contribute an amount

$$\ll x(\log x)^{\Re z-2}\int_0^\infty (u^2+1)^{(R-1)/2} e^{-u}\,du \ll_R x(\log x)^{\Re z-2}.$$

Thus we have the stated estimate, and the proof is complete.

We now establish a procedure by which we can pass from $d_z(n)$ to other nearby functions.

Theorem 18. *Suppose that $\sum_{m=1}^\infty |b_z(m)|(\log m)^{2R+1}/m$ is uniformly bounded for $|z| \le R$, and for $\sigma \ge 1$ let*

$$F(s,z) = \sum_{m=1}^\infty b_z(m) m^{-s}.$$

Let $a_z(n)$ be defined by the relation

$$\zeta(s)^z F(s,z) = \sum_{n=1}^\infty a_z(n) n^{-s} \qquad (\sigma > 1)$$

and let $A_z(x) = \sum_{n \le x} a_z(n)$. Then for $x \ge 2$,

$$A_z(x) = \frac{F(1,z)}{\Gamma(z)} x(\log x)^{z-1} + O\big(x(\log x)^{\Re z-2}\big).$$

Proof. Since $a_z(n) = \sum_{m|n} b_z(m)d_z(n/m)$, we see by Theorem 17 that

$$\begin{aligned} A_z(x) &= \sum_{m\le x/2} b_z(m)D_z(x/m) + \sum_{x/2<m\le x} b_z(m) \\ &= \frac{x}{\Gamma(z)} \sum_{m\le x/2} \frac{b_z(m)}{m}(\log x/m)^{z-1} + O\Big(x\sum_{m\le x}\frac{|b_z(m)|}{m}(\log 2x/m)^{\Re z-2}\Big). \end{aligned} \tag{59}$$

The error term here is

$$\ll x(\log x)^{\Re z-2}\sum_{m\le\sqrt{x}}\frac{|b_z(m)|}{m} + x(\log x)^{-R-2}\sum_{m>\sqrt{x}}\frac{|b_z(m)|}{m}(\log m)^{2R} \ll x(\log x)^{\Re z-2}.$$

In the main term, when $m \le x^{1/2}$ we write

$$(\log x/m)^{z-1} = (\log x)^{z-1} + O\big((\log m)(\log x)^{\Re z-2}\big).$$

Thus the first sum on the right hand side of (59) is

$$\begin{aligned} &= (\log x)^{z-1}\sum_{m\le x/2}\frac{b_z(m)}{m} + O\Big((\log x)^{\Re z-2}\sum_{m\le\sqrt{x}}\frac{|b_z(m)|}{m}\log m + (\log x)^{R-1}\sum_{m>\sqrt{x}}\frac{|b_z(m)|}{m}\Big) \\ &= (\log x)^{z-1}F(1,z) + O\Big((\log x)^{\Re z-2}\sum_{m}\frac{|b_z(m)|}{m}(\log m)^{2R+1}\Big), \end{aligned}$$

which gives the result.

Suppose that $R < 2$, and let

$$F(s,z) = \prod_p \Big(1-\frac{z}{p^s}\Big)^{-1}\Big(1-\frac{1}{p^s}\Big)^{z} \tag{60}$$

for $\sigma > 1$, $|z| \le R$. Then $a_z(n) = z^{\Omega(n)}$ in the notation of Theorem 18. Hence, with $\sigma_k(x)$ defined as at the beginning of this section we find that

$$A_z(x) = \sum_{n\le x} z^{\Omega(n)} = \sum_{k=0}^{\infty}\sigma_k(x)z^k.$$

Here the power series on the right is actually a polynomial, since $\sigma_k(x) = 0$ for sufficiently large k, when x is fixed. Our asymptotic estimate for $A_z(x)$ enables us to recover an estimate for the power series coefficients $\sigma_k(x)$, since Cauchy's formula asserts that

$$\sigma_k(x) = \frac{1}{2\pi i}\int_{|z|=r}\frac{A_z(x)}{z^{k+1}}\,dz \tag{61}$$

for $r < 2$.

Theorem 19. *Suppose that $R < 2$, that $F(s,z)$ is given by (60), and that $G(z) = F(1,z)/\Gamma(z+1)$. Then*

$$\sigma_k(x) = G\Big(\frac{k-1}{\log\log x}\Big)\frac{x(\log\log x)^{k-1}}{(k-1)!\log x}\bigg(1 + O_R\Big(\frac{k}{(\log\log x)^2}\Big)\bigg) \tag{62}$$

uniformly for $1 \le k \le R\log\log x$.

Since $G(0) = G(1) = 1$, we see that (55) holds when $k = o(\log\log x)$, and also when $k = (1+o(1))\log\log x$, but that (55) does not hold in general. The restriction to $R < 2$ is necessary because of the contribution of the prime $p = 2$ in the Euler product (60) for $F(s,z)$. If $z \ge 2$ then the behaviour is different; see Exercises 5 and 6, below.

Proof. Our quantitative form of the Prime Number Theorem (Theorem 6.9) gives the case $k = 1$, so we may assume that $k > 1$. We substitute the estimate of Theorem 18 in (61) with $r = (k-1)/\log\log x$. The error term contributes an amount

$$\ll x(\log x)^{r-2}r^{-k} = \frac{x}{(\log x)^2}e^{k-1}\frac{(\log\log x)^k}{(k-1)^k} \ll \frac{x(\log\log x)^k}{(k-1)!(\log x)^2} \ll \frac{x(\log\log x)^{k-3}}{(k-1)!\log x}.$$

This is majorized by the error term in (62) since $G((k-1)/\log\log x) \gg 1$. The main term we obtain from (61) is $xI/\log x$ where

$$\begin{aligned} I &= \frac{1}{2\pi i}\int_{|z|=r} G(z)(\log x)^z z^{-k}\,dz \\ &= \frac{G(r)}{2\pi i}\int_{|z|=r}(\log x)^z z^{-k}\,dz + \frac{1}{2\pi i}\int_{|z|=r}(G(z)-G(r))(\log x)^z z^{-k}\,dz. \end{aligned}$$

By integration by parts we find that

$$\frac{r}{2\pi i}\int_{|z|=r}(\log x)^z z^{-k}\,dz = \frac{1}{2\pi i}\int_{|z|=r}(\log x)^z z^{1-k}\,dz.$$

We multiply both sides by $G'(r)$ and combine with the former identity to see that

$$I = \frac{G(r)}{2\pi i}\int_{|z|=r}(\log x)^z z^{-k}\,dx + \frac{1}{2\pi i}\int_{|z|=r}(G(z)-G(r)-G'(r)(z-r))(\log x)^z z^{-k}\,dz. \tag{63}$$

Here the first integral is $(\log\log x)^{k-1}/(k-1)!$ by Cauchy's theorem, which gives the desired main term. On the other hand,

$$G(z) - G(r) - G'(r)(z-r) = \int_r^z (z-w)G''(w)\,dw \ll |z-r|^2,$$

so that if we write $z = re^{2\pi i\theta}$ then the second integral in (63) is

$$\ll r^{3-k}\int_{-1/2}^{1/2}(\sin\pi\theta)^2 e^{(k-1)\cos 2\pi\theta}\,d\theta.$$

But $|\sin x| \le |x|$ and $\cos 2\pi\theta \le 1 - 8\theta^2$ for $-1/2 \le \theta \le 1/2$, so the above is

$$\ll r^{3-k}e^{k-1}\int_0^\infty \theta^2 e^{-8(k-1)\theta^2}\,d\theta \ll r^{3-k}e^{k-1}(k-1)^{-3/2} = \frac{(\log\log x)^{k-3}e^{k-1}}{(k-1)^{k-3/2}}$$
$$\ll k(\log\log x)^{k-3}/(k-1)!.$$

This completes the proof of the theorem.

The decomposition in (63) is motivated by the observation that $|(\log x)^z|$ is largest, for $|z| = r$, when $z = r$. We take the Taylor expansion to the second term because

$$\Big|\int (z-r)^2(\log x)^z z^{-k}\,dz\Big| \asymp \int |z-r|^2|(\log x)^z z^{-k}|\,|dz|,$$

whereas

$$\Big|\int (z-r)(\log x)^z z^{-k}\,dz\Big| = o\Big(\int |z-r||(\log x)^z z^{-k}|\,|dz|\Big).$$

By the calculus of residues we may write

$$\begin{aligned} I &= \frac{1}{(k-1)!}\,\frac{d^{k-1}}{dz^{k-1}}\big(G(z)(\log x)^z\big)\Big|_{z=0} \\ &= \sum_{\nu=0}^{k-1}\frac{G^{(\nu)}(0)}{\nu!}\,\frac{(\log\log x)^{k-1-\nu}}{(k-1-\nu)!}. \end{aligned}$$

This gives a more accurate, but more complicated, main term.

In §2.3 we saw that $\Omega(n)$ rarely differs very much from $\log\log n$. In particular, from Theorem 2.12 we see that if $r < 1$ then the number of $n \le x$ for which $\Omega(n) < r\log\log n$ is $\ll_r x/\log\log x$. We now give a much sharper upper bound for the number of occurrences of such large deviations.

Theorem 20. *Let $A(x,r)$ denote the number of $n \le x$ such that $\Omega(n) \le r\log\log x$, and let $B(x,r)$ denote the number of $n \le x$ for which $\Omega(n) \ge r\log\log x$. If $0 < r \le 1$ and $x \ge 2$ then*

$$A(x,r) \ll x(\log x)^{r-1-r\log r}.$$

If $1 \le r \le R < 2$ and $x \ge 2$ then

$$B(x,r) \ll_R x(\log x)^{r-1-r\log r}.$$

Proof. We argue directly from Theorem 18, using a modified form of Rankin's method. If $0 \le r \le 1$ and $\Omega(n) \le r\log\log x$ then $r^{r\log\log x} \le r^{\Omega(n)}$. Hence

$$A(x,r) \le (\log x)^{-r\log r}\sum_{n\le x} r^{\Omega(n)}.$$

By Theorem 18 this is

$$\sim \frac{F(1,r)}{\Gamma(r)} x(\log x)^{r-1-r\log r}$$

where $F(s,z)$ is taken as in (60). This gives the result since $F(1,r) \ll 1$ and $\Gamma(r) \gg 1$ uniformly for $0 < r \le 1$.

Now suppose that $1 \le r \le R < 2$ and that $\Omega(n) \ge r \log\log x$. Then $r^{\Omega(n)} \ge r^{r\log\log x}$, and hence

$$B(x,r) \le (\log x)^{-r\log r} \sum_{n\le x} r^{\Omega(n)}.$$

Thus we have only to proceed as before to obtain the result.

In discussing Theorem 2.12 we proposed a probabilistic model, which in conjunction with the Central Limit Theorem would predict that the quantity

$$\alpha_n = \frac{\Omega(n) - \log\log n}{\sqrt{\log\log n}} \tag{64}$$

is asymptotically normally distributed. We now confirm this.

Theorem 21. *Let α_n be given by (64) and suppose that $Y > 0$. Then the number of n, $3 \le n \le x$, such that $\alpha_n \le y$ is*

$$\Phi(y)x + O_Y\Big(\frac{x}{\sqrt{\log\log x}}\Big)$$

uniformly for $-Y \le y \le Y$ where

$$\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2}\,dt.$$

Proof. Let

$$\beta_n = \frac{\Omega(n) - \log\log x}{\sqrt{\log\log x}}.$$

Since $\Phi'(y) \ll 1$ and $\alpha_n - \beta_n \ll 1/\sqrt{\log\log x}$ when $x^{1/2} \le n \le x$ and $\Omega(n) \le 2\log\log x$, it suffices to consider β_n in place of α_n. We may of course also suppose that x is large.

Let k be a natural number and let u be defined by writing $k = u + \log\log x$. If $|u| \le \frac{1}{2}\log\log x$ then by Stirling's formula (see (B.26) or the more general Theorem C.1) we see that

$$\frac{(\log\log x)^{k-1}}{(k-1)!} = \frac{e^u \log x}{\sqrt{2\pi \log\log x}} \Big(1 + \frac{u}{\log\log x}\Big)^{\frac{1}{2}-\log\log x - u} \Big(1 + O\Big(\frac{1}{\log\log x}\Big)\Big).$$

The estimate $\log(1+\delta) = \delta - \delta^2/2 + O(|\delta|^3)$ holds uniformly for $|\delta| \le 1/2$. By taking $\delta = u/\log\log x$ we find that

$$\Big(1 + \frac{u}{\log\log x}\Big)^{\frac{1}{2}-\log\log x-u} = \exp\Big(-u + \frac{u-u^2}{2\log\log x} - \frac{u^2}{4(\log\log x)^2} + O\Big(\frac{|u|^3}{(\log\log x)^2}\Big)\Big).$$

Suppose now that $|u| \le (\log\log x)^{2/3}$. By considering separately $|u| \le (\log\log x)^{1/2}$ and $(\log\log x)^{1/2} < |u| \le (\log\log x)^{2/3}$ we see that

$$\frac{u}{\log\log x} \ll \frac{1}{\sqrt{\log\log x}} + \frac{|u|^3}{(\log\log x)^2}.$$

Similarly, by considering $|u| \le 1$ and $|u| > 1$ we see that

$$\frac{u^2}{(\log\log x)^2} \ll \frac{1}{\sqrt{\log\log x}} + \frac{|u|^3}{(\log\log x)^2}.$$

On combining these estimates we deduce that

$$\frac{(\log\log x)^{k-1}}{(k-1)!} = \frac{\log x}{\sqrt{2\pi\log\log x}} \exp\Big(\frac{-u^2}{2\log\log x}\Big)\Big(1 + O\Big(\frac{1}{\sqrt{\log\log x}}\Big) + O\Big(\frac{|u|^3}{(\log\log x)^2}\Big)\Big)$$

uniformly for $|u| \le (\log\log x)^{2/3}$. In Theorem 19 we have $G(1) = 1$ and

$$G\Big(\frac{k-1}{\log\log x}\Big) = G(1) + O\Big(\frac{1+|u|}{\log\log x}\Big).$$

Hence by Theorem 19,

$$\sigma_k(x) = \frac{x\exp\Big(\frac{-(k-\log\log x)^2}{2\log\log x}\Big)}{\sqrt{2\pi\log\log x}}\Big(1 + O\Big(\frac{1}{\sqrt{\log\log x}}\Big) + O\Big(\frac{|k-\log\log x|^3}{(\log\log x)^2}\Big)\Big).$$

By Theorem 20 we know that the contribution of $k \le \log\log x - (\log\log x)^{2/3}$ is negligible. We sum over the range

$$\log\log x - (\log\log x)^{2/3} \le k \le \log\log x + y(\log\log x)^{1/2}.$$

This gives rise to three sums, one for the main term and two for error terms. Each of these sums can be considered to be a Riemann sum for an associated integral, and the stated result follows.

7.4. Exercises

1. Let $p_1, p_2, \dots, p_K$ be distinct primes. Show that the number of $n \le x$ composed entirely of the p_k is

$$\frac{(\log x)^K}{K!\prod_{k=1}^K \log p_k} + O\big((\log x)^{K-1}\big).$$

2. (a) Let $d_z(n)$ be defined as in (56), and suppose that $|z| \le R$. Show that $|d_z(n)| \le d_{|z|}(n) \le d_R(n)$.
(b) Let $F(s,z)$ be defined as in (60). Show that if $0 < r < 1$ and $\sigma > 1/2$ then $0 < F(\sigma, r) < 1$.

(c) Let $F(s,z)$ be defined as in (60). Show that if $1 < r < 2$ then the Dirichlet series coefficients of $F(s,r)$ are all non-negative.

3. (a) Show that if

$$F(s,z) = \prod_p \Big(1 + \frac{z}{p^s - 1}\Big)\Big(1 - \frac{1}{p^s}\Big)^z$$

then $F(s,z)$ converges for $\sigma > 1/2$, uniformly for $|z| \le R$.
(b) Show that if $F(s,z)$ is taken as above, and if $a_z(n)$ is defined as in Theorem 18 then $a_z(n) = z^{\omega(n)}$.
(c) Let $\rho_k(x)$ denote the number of $n \le x$ for which $\omega(n) = k$. Show that if $x \ge 2$ then

$$\rho_k(x) = G\Big(\frac{k-1}{\log\log x}\Big)\frac{x(\log\log x)^{k-1}}{(k-1)!\log x}\Big(1 + O_R\Big(\frac{k}{(\log\log x)^2}\Big)\Big)$$

uniformly for $1 \le k \le R\log\log x$ where $G(z) = F(1,z)/\Gamma(z+1)$.
(d) Show that $G(0) = G(1) = 1$.
(e) Let $A(x,r)$ denote the number of $n \le x$ for which $\omega(n) \le r\log\log x$. Show that

$$A(x,r) \ll x(\log x)^{r-1-r\log r}$$

uniformly for $0 < r \le 1$.
(f) Let $B(x,r)$ denote the number of $n \le x$ for which $\omega(n) \ge r\log\log x$. Show that

$$B(x,r) \ll x(\log x)^{r-1-r\log r}$$

uniformly for $1 \le r \le R$.

4. (a) Show that if

$$F(s,z) = \prod_p \Big(1 + \frac{z}{p^s}\Big)\Big(1 - \frac{1}{p^s}\Big)^z$$

then $F(s,z)$ converges for $\sigma > 1/2$, uniformly for $|z| \le R$.
(b) Show that if $F(s,z)$ is taken as above, and if $a_z(n)$ is defined as in Theorem 18 then $a_z(n) = \mu(n)^2 z^{\omega(n)}$.
(c) Let $\pi_k(x)$ denote the number of squarefree $n \le x$ for which $\omega(n) = k$. Show that if $x \ge 2$ then

$$\pi_k(x) = G\Big(\frac{k-1}{\log\log x}\Big)\frac{x(\log\log x)^{k-1}}{(k-1)!\log x}\Big(1 + O_R\Big(\frac{k}{(\log\log x)^2}\Big)\Big)$$

uniformly for $1 \le k \le R\log\log x$ where $G(z) = F(1,z)/\Gamma(z+1)$.
(d) Show that $G(0) = G(1) = 1$.

5. (a) Show that if $x \ge 2$ then

$$\sum_{n\le x} 2^{\Omega(n)} = cx(\log x)^2 + O(x\log x)$$

where c is a positive constant.
(b) Show that if $x \ge 2$ then

$$\sum_{n\le x} 2^{\omega(n)} = cx\log x + O(x)$$

where c is a positive constant.

6. Show that if $(2+\varepsilon)\log\log x \le k \le R\log\log x$ then $\sigma_k(x) \sim c2^{-k}x\log x$.

7. Show that if $\delta \le r \le 1-\delta$ (or $1+\delta \le r \le 2-\delta$) then $A(x,r)$ (or $B(x,r)$ respectively) is $\asymp x(\log x)^{r-1-r\log r}/\sqrt{\log\log r}$.

8. Show that if x is large then there is a k such that

$$\sigma_k(x) \ge \frac{x}{3\sqrt{\log\log x}}.$$

9. Show that the mean value $\sum_{n\le x} d(n) \sim x\log x$ is due to the numbers $n \le x$ for which $|\omega(n) - 2\log\log x| \ll \sqrt{\log\log x}$.

10. Suppose that $1/2 \le r \le R$. Show that the number of squarefree $n \le x$ that can be written as a sum of two squares and for which $\omega(n) \ge r\log\log x$ is $\ll_R x(\log x)^{r-1-r\log 2r}$.

11. (Addison (1957)) Let $M_{q,k}(x)$ denote the number of $n \le x$ such that $\Omega(n) \equiv k \pmod q$.
(a) Show that if q is fixed then $M_{q,k}(x) \sim x/q$ as $x \to \infty$.
(b) Show that if q is fixed, $q > 2$, then

$$M_{q,k}(x) - \frac{x}{q} = \Omega_\pm\Big(\frac{x}{(\log x)^\kappa}\Big)$$

where $\kappa = 1 - \cos 2\pi/q$.

12. Show that

$$\sum_{1<n\le x} \frac{1}{\omega(n)} \sim \frac{x}{\log\log x}$$

as $x \to \infty$.

13. Show that if $x \ge 2$ then

$$\sum_{1<n\le x} \frac{\Omega(n)}{\omega(n)} = x + O\Big(\frac{x}{\log\log x}\Big).$$

14. Suppose that $0 \le \alpha \le 1$. Show that

$$\sum_{n\le x} \frac{\operatorname{card}\{m : m|n, m \le n^\alpha\}}{d(n)} = \frac{2}{\pi} x \arcsin\sqrt{\alpha} + O\Big(\frac{x}{\sqrt{\log x}}\Big).$$

15. Show that if $x \ge 16$ then

$$\sum_{\substack{n\le x\\ (n,\Omega(n))=1}} 1 = \frac{6}{\pi^2} x + O\Big(\frac{x}{\log\log\log x}\Big).$$

7. Notes

§1. Theorem 2 was first proved by Dickman (1930), and was rediscovered by Chowla and Vijayaraghavan (1947), Ramaswami (1949), and Buchstab (1949). de Bruijn (1951a) gave a more precise estimate for $\psi(x, y)$, over a longer range of y. There is a considerable range of applications of $\Psi(x, y)$, such as those to the distribution of kth power residues, Waring's problem, and the complexity of arithmetical algorithms in computer science. As a reflection of this there have been two significant survey articles, by Norton (1971) and by Hildebrand and Tenenbaum (1993).

Our treatment of $\psi(x, y)$ is fairly elementary, but it would be natural to take a more analytic approach, and use Perron's formula to write

$$\psi(x, y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \prod_{p\le y}\big(1-p^{-s}\big)^{-1}\frac{x^s}{s}\,ds = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \zeta(s)\prod_{p>y}\big(1-p^{-s}\big)\frac{x^s}{s}\,ds.$$

For s not too large an approximation to the product over $p > y$ is provided by the Prime Number Theorem, and this suggests the main term

$$\Lambda(x, y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \zeta(s)\exp\Big(-\int_y^\infty v^{-s}(\log v)^{-1}\,dv\Big)\frac{x^s}{s}\,ds.$$

It can be shown that this is indeed a good approximation to $\psi(x, y)$ over a very long range, but the technical details are rather heavy. By Theorem 10 it is not hard to show that

$$\Lambda(x, y) = x\int_{0-}^{\infty} \rho(u-v)\,d\big([y^v]\,y^{-v}\big)$$

where we use (30) to extend the definition of $\rho(u)$ to $u \le 0$. It follows that

$$\Lambda(x, y) \sim \rho(u)x$$

for a large range of u. For the further development of the theory, especially on the analytic side, see Hildebrand and Tenenbaum (1993).

§2. Theorem 11 is due to Buchstab (1937). The finer details of the behaviour of $\Phi(x, y)$ when u is large are intimately connected with sieve theory, especially that of the linear sieve, i.e., the sieve in which on average one residue class (mod p) is removed. The standard references are Greaves (2001), Halberstam and Richert (1974), Selberg (1991).

§3. Theorem 14 was first proved by Westzynthius (1931). Erdős (1935) showed that

$$\limsup_{n\to\infty} \frac{p_{n+1}-p_n}{(\log p_n)(\log\log p_n)/(\log\log\log p_n)^2} > 0,$$

and then Rankin (1938) obtained Theorem 15 with $c = 1/3$. The value of c has been successively improved by Schönhage (1963), Rankin (1963), Maier and Pomerance (1990), culminating in the value $c = 2e^{C_0}$ of Pintz (1997). Erdős offered a \$10,000 prize for the first proof that Theorem 15 is valid for all $c > 0$.

Early studies of $g(P(z))$ were conducted by Backlund (1929), Brauer & Zeitz (1930), Ricci (1935), and Chang (1938). The size of $g(P(z))$ is not known; possibly it is $\asymp z \log z$. However, it is conceivable that infinitely often $p_{n+1} - p_n$ is as large as $(\log p_n)^\theta$ where $\theta > 1$. In particular, Cramér (1936) conjectured that

$$\limsup_{n\to\infty} \frac{p_{n+1}-p_n}{(\log p_n)^2} = 1.$$

Theorem 16 is due to Hensley & Richards (1973).

§4. The analysis of $\sigma_k(x)$ is based on Selberg's exposition (1954) of Sathe (1953a,b, 1954a,b). Sathe (1954b) also shows that the bound $R \log\log x$ cannot be replaced by $2\log\log x + 1$. Arguments giving rise to versions of Theorem 20 occur in Erdős (1935b). A qualitative version of Theorem 21 is a special case of Erdős and Kac (1940). Quantitative versions with various weaker error terms were obtained by LeVeque (1949) and Kubilius (1956). Theorem 21 had been conjectured by LeVeque and was established by Rényi and Turán (1958). They also showed that the error term is both uniform in x and best possible.

7. Literature

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