Chapter 6

The Prime Number Theorem

1. A zero-free region

The Prime Number Theorem (PNT) asserts that

$$\pi(x) \sim \frac{x}{\log x}$$

as x tends to infinity. We shall prove this by using Perron's formula, but in the course of our arguments it will be important to know that $\zeta(s) \neq 0$ for $\sigma \geq 1$. In Chapter 1 we saw that $\zeta(s) \neq 0$ for $\sigma > 1$, but it remains to show that $\zeta(1+it) \neq 0$. To obtain a quantitative form of the prime number theorem we take some care to show that $\zeta(s) \neq 0$ for $\sigma \geq 1 - \delta(t)$ where $\delta(t)$ is some function of t. We would like the width $\delta(t)$ of the zero-free region to be as large as possible, as the rate at which $\delta(t)$ tends to 0 determines the size of the estimate we can derive for the error term in the prime number theorem.

We begin by reviewing some basic facts concerning functions of a complex variable. If $P(z)$ is a polynomial then the rate of growth of $|P(z)|$ as $|z| \to \infty$ reflects the number of zeros of $P(z)$. This is generalized to other analytic functions by Jensen's formula. For our purposes we are content to establish the following simple consequence of Jensen's formula.

Lemma 1. (Jensen's inequality) *If $f(z)$ is analytic in a domain containing the disk $|z| \leq R$, if $|f(z)| \leq M$ in this disk, and if $f(0) \neq 0$, then for $r < R$ the number of zeros of f in the disk $|z| \leq r$ does not exceed*

$$\frac{\log M/|f(0)|}{\log R/r}.$$

.

Proof. Let $z_1, z_2, \ldots, z_K$ denote the zeros of f in the disk $|z| \leq R$, and put

$$g(z) = f(z) \prod_{k=1}^{K} \frac{R^2 - z\overline{z}_k}{R(z - z_k)}.$$

The k^{th} factor of the product has been constructed so that it has a pole at z_k, and so that it has modulus 1 on the circle $|z| = R$. Hence g is an analytic function in the disk $|z| \leq R$, and if $|z| = R$ then $|g(z)| = |f(z)| \le M$. Hence by the maximum modulus principle, $|g(0)| \le M$. But

$$|g(0)| = |f(0)| \prod_{k=1}^{K} \frac{R}{|z_k|}.$$

Each factor in the product is ≥ 1, and if $|z_k| \le r$ then the factor is $\ge R/r$. If there are L such zeros then the above is $\ge |f(0)|(R/r)^L$, which gives the stated upper bound for L.

We now show that a bound for the modulus of an analytic function can be derived from a one-sided bound for its real part in a slightly larger region.

Lemma 2. (The Borel-Carathéodory Lemma) *Suppose that $h(z)$ is analytic in a domain containing the disk $|z| \le R$, that $h(0) = 0$, and that $\Re h(z) \le M$ for $|z| \le R$. If $|z| \le r < R$, then*

$$|h(z)| \le \frac{2Mr}{R-r}$$

and

$$|h'(z)| \le \frac{2MR}{(R-r)^2}.$$

Proof. It suffices to show that

$$\left|\frac{h^{(k)}(0)}{k!}\right| \le \frac{2M}{R^k} \tag{1}$$

for all $k \ge 1$, for then

$$|h(z)| \le \sum_{k=1}^{\infty} \left|\frac{h^{(k)}(0)}{k!}\right| r^k \le 2M \sum_{k=1}^{\infty} \left(\frac{r}{R}\right)^k = \frac{2Mr}{R-r},$$

and

$$|h'(z)| \le \sum_{k=1}^{\infty} \frac{|h^{(k)}(0)| k r^{k-1}}{k!} \le \frac{2M}{R} \sum_{k=1}^{\infty} k \left(\frac{r}{R}\right)^{k-1} = \frac{2MR}{(R-r)^2}.$$

To prove (1) we first note that

$$\int_0^1 h(Re(\theta))\, d\theta = \frac{1}{2\pi i} \oint_{|z|=R} h(z) \frac{dz}{z} = h(0) = 0.$$

Moreover, if $k > 0$ then

$$\int_0^1 h(Re(\theta)) e(k\theta)\, d\theta = \frac{R^{-k}}{2\pi i} \oint_{|z|=R} h(z) z^{k-1}\, dz = 0,$$

and

$$\int_0^1 h(Re(\theta)) e(-k\theta)\, d\theta = \frac{R^k}{2\pi i} \oint_{|z|=R} h(z) z^{-k-1}\, dz = \frac{R^k h^{(k)}(0)}{k!}.$$

By forming a linear combination of these identities we see that if $k > 0$ then

$$\int_0^1 h(Re(\theta))(1 + \cos 2\pi(k\theta + \phi))\, d\theta = \frac{R^k e(-\phi) h^{(k)}(0)}{2 \cdot k!}.$$

By taking real parts it follows that

$$\Re\Big(\frac{1}{2} R^k e(-\phi) h^{(k)}(0)/k!\Big) \le M \int_0^1 (1 + \cos 2\pi(k\theta + \phi))\, d\theta = M$$

for $k > 0$. Since this holds for any real ϕ, we are free to choose ϕ so that $e(-\phi)h^{(k)}(0) = |h^{(k)}(0)|$. Then the above inequality gives (1), and the proof is complete.

If $P(z) = c \prod_{k=1}^K (z - z_k)$ then

$$\frac{P'}{P}(z) = \sum_{k=1}^K \frac{1}{z - z_k}.$$

We now generalize this to analytic functions $f(z)$, to the extent that f'/f can be approximated by a sum over its nearby zeros.

Lemma 3. *Suppose that $f(z)$ is analytic in a domain containing the disk $|z| \le 1$, that $|f(z)| \le M$ in this disk, and that $f(0) \ne 0$. Let r and R be fixed, $0 < r < R < 1$. Then for $|z| \le r$ we have*

$$\frac{f'}{f}(z) = \sum_{k=1}^K \frac{1}{z - z_k} + O\Big(\log \frac{M}{|f(0)|}\Big)$$

where the sum is extended over all zeros z_k of f for which $|z_k| \le R$. (*The implicit constant depends on r and R, but is otherwise absolute.*)

Proof. If $f(z)$ has zeros on the circle $|z| = R$, then we replace R by a very slightly larger value. Thus we may assume that $f(z) \ne 0$ for $|z| = R$. Set

$$g(z) = f(z) \prod_{k=1}^K \frac{R^2 - z\overline{z_k}}{R(z - z_k)}.$$

By Lemma 1 we know that

$$K \le \frac{\log M/|f(0)|}{\log 1/R} \ll \log \frac{M}{|f(0)|}. \tag{2}$$

If $|z| = R$ then each factor in the product has modulus 1. Consequently $|g(z)| \le M$ when $|z| = R$, and by the maximum modulus principle $|g(z)| \le M$ for $|z| \le R$. We also note that

$$|g(0)| = |f(0)| \prod_{k=1}^K \frac{R}{|z_k|} \ge |f(0)|.$$

Since $g(z)$ has no zeros in the disk $|z| \le R$, we may put $h(z) = \log(g(z)/g(0))$. Then $h(0) = 0$, and
$$\Re h(z) = \log|g(z)| - \log|g(0)| \le \log M - \log|f(0)|$$
for $|z| \le R$. Hence by the Borel-Carathéodory Lemma we see that

$$h'(z) \ll \log \frac{M}{|f(0)|} \tag{3}$$

for $|z| \le r$. But

$$h'(z) = \frac{g'}{g}(z) = \frac{f'}{f}(z) - \sum_{k=1}^{K} \frac{1}{z - z_k} + \sum_{k=1}^{K} \frac{1}{z - R^2/\overline{z_k}}. \tag{4}$$

Now $|R^2/\overline{z_k}| \ge R$, so that if $|z| \le r$ then $|z - R^2/\overline{z_k}| \ge R - r$. Hence for $|z| \le r$ the last sum above has modulus
$$\le \frac{K}{R-r} \ll \log \frac{M}{|f(0)|}$$
by (2). To obtain the stated result it suffices to combine this estimate and (3) in (4).

We now apply these general principles to the zeta function.

Lemma 4. *If $|t| \ge 7/8$ and $5/6 \le \sigma \le 2$, then*

$$\frac{\zeta'}{\zeta}(s) = \sum_{\rho} \frac{1}{s-\rho} + O(\log \tau)$$

where $\tau = |t| + 4$ and the sum is extended over all zeros ρ of $\zeta(s)$ for which $|\rho - (3/2 + it)| \le 5/6$.

Proof. We apply Lemma 3 to the function $f(z) = \zeta(z + (3/2 + it))$, with $R = 5/6$ and $r = 2/3$. To complete the proof it suffices to note that $|f(0)| \gg 1$ by the (absolutely convergent) Euler product formula (1.17), and that $f(z) \ll \tau$ for $|z| \le 1$ by Corollary 1.17.

If the zeta function were to have a zero of multiplicity m at $1 + i\gamma$ then we would have

$$\frac{\zeta'}{\zeta}(1 + \delta + i\gamma) \sim \frac{m}{\delta}$$

as $\delta \to 0^+$. But
$$\Re \frac{\zeta'}{\zeta}(1 + \delta + i\gamma) = -\sum_{n=1}^{\infty} \Lambda(n) n^{-1-\delta} \cos(\gamma \log n),$$
and in the very worst case this could be no larger than

$$\sum_{n=1}^{\infty} \Lambda(n) n^{-1-\delta} = -\frac{\zeta'}{\zeta}(1+\delta) \sim \frac{1}{\delta}.$$

Thus m is at most 1, and even in this case ζ'/ζ would be essentially as large as it could possibly be. Roughly speaking, this would imply that $p^{i\gamma}$ is near -1 for most primes. But then it would follow that $p^{2i\gamma}$ is near 1 for most primes, so that

$$\frac{\zeta'}{\zeta}(1+\delta+2i\gamma) \sim -\frac{1}{\delta}$$

as $\delta \to 0^+$. Then $\zeta(s)$ would have a pole at $1+2i\gamma$, contrary to Corollary 1.13. The essence of this informal argument is captured very effectively by the following elementary inequality.

Lemma 5. *If $\sigma > 1$, then*

$$\Re\Big(-3\frac{\zeta'}{\zeta}(\sigma) - 4\frac{\zeta'}{\zeta}(\sigma+it) - \frac{\zeta'}{\zeta}(\sigma+2it)\Big) \ge 0.$$

Proof. From Corollary 1.11 we see that the left hand side above is

$$\sum_{n=1}^{\infty} \Lambda(n) n^{-1-\delta}\big(3 + 4\cos(t\log n) + \cos(2t\log n)\big).$$

It now suffices to note that $3 + 4\cos\theta + \cos 2\theta = 2(1+\cos\theta)^2 \ge 0$ for all θ.

We now use Lemmas 4 and 5 to establish the existence of a zero-free region for the zeta function.

Theorem 6. *There is an absolute constant $c > 0$ such that $\zeta(s) \ne 0$ for $\sigma \ge 1 - c/\log\tau$.*

This is the classical zero-free region for the zeta function.

Proof. Since $\zeta(s)$ is given by the absolutely convergent product (1.17) for $\sigma > 1$, it suffices to consider $\sigma \le 1$. From (1.24) we see that

$$\Big|\zeta(s) - \frac{s}{s-1}\Big| \le |s|\int_1^\infty u^{-\sigma-1}\,du = \frac{|s|}{\sigma} \tag{5}$$

for $\sigma > 0$. From this we see that $\zeta(s) \ne 0$ when $\sigma > |s-1|$, i.e., in the parabolic region $\sigma > (1+t^2)/2$. In particular, $\zeta(s) \ne 0$ in the rectangle $8/9 \le \sigma \le 1$, $|t| \le 7/8$. Now suppose that $\rho_0 = \beta_0 + i\gamma_0$ is a zero of the zeta function with $5/6 \le \beta_0 \le 1$, $|\gamma_0| \ge 7/8$. Since $\Re\rho \le 1$ for all zeros ρ of $\zeta(s)$, it follows that $\Re 1/(s-\rho) > 0$ whenever $\sigma > 1$. Hence by Lemma 4 with $s = 1+\delta+i\gamma_0$ we see that

$$-\Re\frac{\zeta'}{\zeta}(1+\delta+i\gamma_0) \le -\frac{1}{1+\delta-\beta_0} + c_1\log(|\gamma_0|+4).$$

Similarly, by Lemma 4 with $s = 1+\delta+2i\gamma_0$ we find that

$$\Re -\frac{\zeta'}{\zeta}(1+\delta+2i\gamma_0) \le c_1\log(|2\gamma_0|+4).$$

From Corollary 1.13 we see that

$$-\frac{\zeta'}{\zeta}(1+\delta) = \frac{1}{\delta} + O(1).$$

On combining these estimates in Lemma 5 we conclude that

$$\frac{3}{\delta} - \frac{4}{1+\delta-\beta_0} + c_2 \log(|\gamma_0|+4) \;\ge\; 0.$$

We take $\delta = 1/(2c_2 \log(|\gamma_0|+4))$. Thus the above gives

$$7c_2 \log(|\gamma_0|+4) \;\ge\; \frac{4}{1+\delta-\beta_0}\,,$$

which is to say that

$$1 + \frac{1}{2c_2 \log(|\gamma_0|+4)} - \beta_0 \;\ge\; \frac{4}{7c_2 \log(|\gamma_0|+4)}\,.$$

Hence

$$1-\beta_0 \;\ge\; \frac{1}{14c_2 \log(|\gamma_0|+4)}\,,$$

so the proof is complete.

In the above argument it is essential that the coefficient of $\zeta(s)$ is larger than the coefficient of $\zeta(\sigma)$. Among nonnegative cosine polynomials $T(\theta) = a_0 + a_1 \cos 2\pi\theta + \cdots + a_N \cos 2\pi N\theta$, the ratio a_1/a_0 can be arbitrarily close to 2, as we see in the Fejér kernel

$$\Delta_N(\theta) = 1 + 2\sum_{n=1}^{N-1}\Big(1-\frac{n}{N}\Big)\cos 2n\pi\theta = \frac{1}{N}\Big(\frac{\sin \pi N\theta}{\sin \pi\theta}\Big)^2 \ge 0,$$

but it must be strictly less than 2 since

$$a_0 - \frac{1}{2}a_1 = \int_0^1 T(\theta)(1-\cos 2\pi\theta)\,d\theta > 0.$$

It is useful to have bounds for the zeta function and its logarithmic derivative in the zero-free region.

Theorem 7. *Let c be the constant in* Theorem 6. *If $\sigma > 1 - c/(2\log\tau)$ and $|t| \ge 7/8$, then*

$$\frac{\zeta'}{\zeta}(s) \;\ll\; \log\tau\,, \tag{6}$$

$$|\log\zeta(s)| \;\le\; \log\log\tau + O(1)\,, \tag{7}$$

and

$$\frac{1}{\zeta(s)} \;\ll\; \log\tau\,. \tag{8}$$

On the other hand, if $1-c/(2\log\tau) < \sigma \le 2$ *and* $|t| \le 7/8$*, then* $\frac{\zeta'}{\zeta}(s) = -1/(s-1) + O(1)$*,* $\log\big(\zeta(s)(s-1)\big) \ll 1$*, and* $1/\zeta(s) \ll |s-1|$*.*

Proof. If $\sigma > 1$ then by Corollary 1.11 and the triangle inequality we see that

$$|\frac{\zeta'}{\zeta}(s)| \le \sum_{n=1}^{\infty} \Lambda(n) n^{-\sigma} = -\frac{\zeta'}{\zeta}(\sigma) \ll \frac{1}{\sigma-1}.$$

Hence (6) is obvious if $\sigma \ge 1 + 1/\log\tau$. Let $s_1 = 1 + 1/\log\tau + it$. In particular we have

$$\frac{\zeta'}{\zeta}(s_1) \ll \log\tau. \tag{9}$$

From this estimate and Lemma 4 we deduce that

$$\sum_{\rho} \Re\frac{1}{s_1-\rho} \ll \log\tau \tag{10}$$

where the sum is over those zeros ρ for which $|\rho - (3/2+it)| \le 5/6$. Suppose that $1 - c/(2\log\tau) \le \sigma \le 1 + 1/\log\tau$. Then by Lemma 4 we see that

$$\frac{\zeta'}{\zeta}(s) - \frac{\zeta'}{\zeta}(s_1) = \sum_{\rho}\Big(\frac{1}{s-\rho} - \frac{1}{s_1-\rho}\Big) + O(\log\tau). \tag{11}$$

Since $|s-\rho| \asymp |s_1-\rho|$ for all zeros ρ in the sum, it follows that

$$\frac{1}{s-\rho} - \frac{1}{s_1-\rho} \ll \frac{1}{|s_1-\rho|^2\log\tau} \ll \Re\frac{1}{s_1-\rho}.$$

Now (6) follows on combining this with (9) and (10) in (11).

To derive (7) we begin as in our proof of (6). From Corollary 1.11 and the triangle inequality we see that if $\sigma > 1$ then

$$|\log\zeta(s)| \le \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} n^{-\sigma} = \log\zeta(\sigma).$$

But by Theorem 1.14 we know that $\zeta(\sigma) < 1 + 1/(\sigma-1)$, so that (7) holds when $\sigma \ge 1 + 1/\log\tau$. In particular (7) holds at the point $s_1 = 1 + 1/\log\tau + it$, so that to treat the remaining s it suffices to bound the difference

$$\log\zeta(s) - \log\zeta(s_1) = \int_{s_1}^{s} \frac{\zeta'}{\zeta}(w)\,dw.$$

We take the path of integration to be the line segment joining the endpoints. Then the length of this interval multiplied by the bound (6) gives the error term O(1) in (7).

The estimate (8) follows directly from (7), since $\log 1/|\zeta| = -\Re\log\zeta$. The remaining estimates follow trivially from (5).

The ideas we have used enable us not only to derive a zero-free region but also to place a bound on the number of zeros ρ that might lie near the point $1+it$.

Theorem 8. *Let $n(r;t)$ denote the number of zeros ρ of $\zeta(s)$ in the disk $|\rho-(1+it)|\le r$. Then $n(r;t)\ll r\log\tau$, uniformly for $r\le 3/4$.*

Proof. If c_1 is a small positive constant and $r<c_1/\log\tau$, then $n(r;t)=0$ by Theorem 6. Suppose that $c_1/\log\tau\le r\le 1/6$, $|t|\ge 7/8$. As in the proof of Theorem 7, the estimate (10) holds when we take $s_1=1+r+it$. In the sum over ρ, each term is nonnegative, and those zeros ρ counted in $n(r;t)$ contribute at least $1/(2r)$ apiece. Hence their number is $\ll r\log\tau$. If $1/6<r\le 3/4$ and $|t|\ge 3$, then the desired bound follows at once by applying Jensen's inequality (Lemma 1 above) to the function $f(z)=\zeta(z+2+it)$, with $R=11/6$, in view of the bounds provided by Corollary 1.17. Note that $|f(0)|\gg 1$ because of the absolute convergence of the Euler product. If $1/6<r\le 3/4$ and $|t|\le 3$ then we apply Jensen's inequality to the function $f(z)=(z+1+it)\zeta(z+2+it)$.

6.1. Exercises

1. (a) Show that if $|z|<R$, $|w|\le R$, and $z\ne w$, then

$$\Big|\frac{z\overline{w}-R^2}{(z-w)R}\Big|\ge 1.$$

(b) Show that if $|w|\le\rho<R$, $|z|=r<R$, and $z\ne w$, then

$$\Big|\frac{z\overline{w}-R^2}{(z-w)R}\Big|\ge\frac{r\rho+R^2}{(r+\rho)R}.$$

(c) Suppose that f is analytic in the disk $|z|\le R$. For $r\le R$ put $M(r)=\max_{|z|\le r}|f(z)|$. Show that if $0<r<R$ and $0<\rho<R$ then the number of zeros of f in the disk $|z|\le\rho$ does not exceed

$$\frac{\log\frac{M(R)}{M(r)}}{\log\frac{r\rho+R^2}{(r+\rho)R}}.$$

2. Suppose that R, M, and ε are positive real numbers, and set $h(z)=2Mz/(z+R+\varepsilon)$.
(a) Show that $h(0)=0$, that $h(z)$ is analytic for $|z|<R+\varepsilon$, and that $\Re h(z)\le M$ for $|z|\le R+\varepsilon$.
(b) Show that if $0<r<R$, then

$$\max_{|z|\le r}|h(z)|=-h(-r)=\frac{2Mr}{R+\varepsilon-r}\,.$$

(c) Show that if $0<r<R$, then

$$\max_{|z|\le r}|h'(z)|=h'(-r)=\frac{2M(R+\varepsilon)}{(R+\varepsilon-r)^2}\,.$$

3. Show that in the situation of the Borel-Carathéodory Lemma (Lemma 2) that if $|z|\le r<R$ then

$$|h''(z)|\le\frac{4MR}{(R-r)^3}\,.$$

4. (Mertens (1898)) Use the Dirichlet series expansion of $\log \zeta(s)$ to show that if $\sigma > 1$, then
$$|\zeta(\sigma)^3\zeta(\sigma+it)^4\zeta(\sigma+2it)| \ge 1.$$

The method used to establish a zero-free region for the zeta function can be applied to any particular Dirichlet L-function, though the constants involved may depend on the function. We shall pursue this systematically in Chapter 11, but in the exercise below we treat one interesting example.

5. Let χ_0 denote the principal character (mod 4), and χ_1 the non-principal character (mod 4).
(a) Show that $L(1,\chi_1) = \pi/4$, and hence that there is a neighborhood of 1 in which $L(s,\chi_1) \ne 0$.
(b) Show that if $\sigma > 1$ then
$$\Re\Big(-3\frac{L'}{L}(\sigma,\chi_0) - 4\frac{L'}{L}(\sigma+it,\chi_1) - \frac{L'}{L}(\sigma+2it,\chi_0)\Big) \ge 0.$$
(c) Show that there is a constant $c > 0$ such that $L(s,\chi_1) \ne 0$ for $\sigma > 1 - c/\log\tau$.
(d) Show that there is a constant $c > 0$ such that if $\sigma > 1 - c/\log\tau$ then
$$\frac{L'}{L}(s,\chi_1) \ll \log\tau,$$
$$|\log L(s,\chi_1)| \le \log\log\tau + O(1),$$
$$\frac{1}{L(s,\chi_1)} \ll \log\tau.$$

6. (a) Show that if $1 < \sigma_1 \le \sigma_2$, then
$$\frac{\zeta(\sigma_2)}{\zeta(\sigma_1)} \le \Big|\frac{\zeta(\sigma_2+it)}{\zeta(\sigma_1+it)}\Big| \le \frac{\zeta(\sigma_1)}{\zeta(\sigma_2)}$$
for all real t.
(b) Show that if $1 < \sigma_1 \le \sigma_2 \le 2$, then
$$\frac{\sigma_1-1}{\sigma_2-1} \ll \Big|\frac{\zeta(\sigma_2+it)}{\zeta(\sigma_1+it)}\Big| \ll \frac{\sigma_2-1}{\sigma_1-1}$$
uniformly in t.

7. (Montgomery & Vaughan (2001)) (a) Show that if $\sigma > 1$, then
$$\Big|\frac{\zeta(\sigma+i(t+1))}{\zeta(\sigma+it)}\Big| \le \exp\Big(2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^\sigma\log n}|\sin(\tfrac12\log n)|\Big)$$

uniformly for all real t.
(b) Put $f(\theta) = |\sin \pi\theta|$, and for integers k set $\widehat{f}(k) = \int_0^1 f(\theta)e(-k\theta)\,d\theta$ where $e(\theta) = e^{2\pi i\theta}$. Show that $\widehat{f}(k) = -2/(\pi(4k^2 - 1))$.
(c) By Corollary D.3, or otherwise, show that

$$|\sin \pi\theta| = \sum_{k=-\infty}^{\infty} \widehat{f}(k)e(k\theta)\,.$$

(d) Show that if $1 < \sigma \le 2$, then

$$\Big|\frac{\zeta(\sigma + i(t+1))}{\zeta(\sigma + it)}\Big| \le \prod_{k=-\infty}^{\infty} |\zeta(\sigma + ik)|^{2\widehat{f}(k)}$$

uniformly for all real t.
(e) Show that if $\sigma > 1$, then

$$(\sigma - 1)^{4/\pi} \ll \Big|\frac{\zeta(\sigma + i(t+1))}{\zeta(\sigma + it)}\Big| \ll (\sigma - 1)^{-4/\pi}$$

uniformly in t.
(f) Show that

$$(\log t)^{-4/\pi} \ll \Big|\frac{\zeta(1 + i(t+1))}{\zeta(1 + it)}\Big| \ll (\log t)^{4/\pi}$$

uniformly for $t \ge 2$.

8. Suppose that a and b are fixed, $0 < a < b < 1$. Suppose that f is analytic in a domain containing the disc $|z| \le R$, that $f(0) \ne 0$, and that $|f(z)| \le M$ for $|z| \le R$. Show that

$$\frac{f'}{f}(z) = \sum_{k=1}^{K} \frac{1}{z - z_k} + O\Big(\frac{1}{R}\log\frac{M}{|f(0)|}\Big)$$

for $|z| \le aR$ where the sum is over those zeros z_k of $f(z)$ for which $|z_k| \le bR$.

9. (Landau (1924)) Suppose that $\theta(t)$ and $\phi(t)$ are functions with the following properties: $\phi(t) > 0$, $\phi(t) \nearrow$, $e^{-\phi(t)} \le \theta(t) \le 1/2$, $\theta(t) \searrow$. Suppose also that

$$\zeta(s) \ll e^{\phi(t)}$$

for $\sigma \ge 1 - \theta(t)$, $t \ge 2$.
(a) Show that

$$\frac{\zeta'}{\zeta}(s) = \sum_{\rho} \frac{1}{s - \rho} + O\Big(\frac{\phi(t+1)}{\theta(t+1)}\Big)$$

for $\sigma \ge 1 - \theta(t+1)/3$ where the sum is over zeros ρ for which $|\rho - (1 + \theta(t+1) + it)| \le 5\theta(t+1)/3$.
(b) Show that there is an absolute constant $c > 0$ such that $\zeta(s) \ne 0$ for

$$\sigma \ge 1 - c\frac{\theta(2t+1)}{\phi(2t+1)}\,.$$

(c) Show that the zero-free region (26) follows from the estimate (25).
(d) By mimicking the proof of Theorem 7, but with $s_1 = 1 + \theta(2t+1)/\phi(2t+1) + it$, show that

$$\frac{\zeta'}{\zeta}(s) \ll \frac{\phi(2t+2)}{\theta(2t+2)},$$
$$|\log\zeta(s)| \le \log\frac{\phi(2t+2)}{\theta(2t+2)} + O(1),$$
$$\frac{1}{\zeta(s)} \ll \frac{\phi(2t+2)}{\theta(2t+2)}$$

for $\sigma \ge 1 - \frac{1}{2}c\theta(2t+2)/\phi(2t+2)$.

10. Suppose that $\zeta(s) \ne 0$ for $\sigma \ge \eta(t)$, $t \ge 2$, where $\eta(t) \searrow$, $\eta(t) \gg 1/\log t$. Show that

$$\frac{\zeta'}{\zeta}(s) \ll \log t$$

for $\sigma \ge 1 - \frac{1}{2}\eta(t+1)$, $t \ge 2$.

2. The Prime Number Theorem

We are now in a position to prove the Prime Number Theorem in a quantitative form. We apply Perron's formula to $\frac{\zeta'}{\zeta}(s)$ to obtain an asymptotic estimate for

$$\psi(x) = \sum_{n\le x}\Lambda(n),$$

and then use partial summation to derive an estimate for $\pi(x)$. It would be more direct to apply Perron's formula to $\log\zeta(s)$, but our approach is technically simpler since $\log\zeta(s)$ has a logarithmic singularity at $s = 1$ while $\frac{\zeta'}{\zeta}(s)$ has only a simple pole there.

Theorem 9. *There is a constant $c > 0$ such that*

$$\psi(x) = x + O\Big(\frac{x}{\exp(c\sqrt{\log x})}\Big), \tag{12}$$

$$\vartheta(x) = x + O\Big(\frac{x}{\exp(c\sqrt{\log x})}\Big), \tag{13}$$

and

$$\pi(x) = \mathrm{li}(x) + O\Big(\frac{x}{\exp(c\sqrt{\log x})}\Big) \tag{14}$$

uniformly for $x \ge 2$.

Here $\mathrm{li}(x)$ is the *logarithmic integral*,

$$\mathrm{li}(x) = \int_2^x \frac{1}{\log u}\,du.$$

By integrating this integral by parts K times we see that

$$\mathrm{li}(x) = x\sum_{k=1}^{K-1}\frac{(k-1)!}{(\log x)^k} + O_K\Big(\frac{x}{(\log x)^K}\Big). \tag{15}$$

On combining this with (14) we see that

$$\pi(x) = \frac{x}{\log x} + O\Big(\frac{x}{(\log x)^2}\Big).$$

This is a quantitative form of the Prime Number Theorem. When this main term is used, the error term is genuinely of the indicated size, since by (14) and (15) again we see that

$$\pi(x) = \frac{x}{\log x} + \frac{x}{(\log x)^2} + O\Big(\frac{x}{(\log x)^3}\Big).$$

Thus we see that in order to obtain a precise estimate fo $\pi(x)$, it is essential to use the logarithmic integral (or some similar function) to express the main term.

Proof. From Corollary 1.11 and Theorem 5.2 we see that

$$\psi(x) = \frac{-1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\,ds + R \tag{16}$$

for $\sigma_0 > 1$, where by Corollary 5.3 we see that

$$R \ll \sum_{x/2<n<2x}\Lambda(n)\min\Big(1, \frac{x}{T|x-n|}\Big) + \frac{(4x)^{\sigma_0}}{T}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{\sigma_0}}.$$

Here the second sum is $-\frac{\zeta'}{\zeta}(\sigma_0)$, which is $\asymp 1/(\sigma_0 - 1)$ for $1 < \sigma_0 \le 2$. To estimate the first sum we note that $\Lambda(n) \le \log n \ll \log x$. For the n that is nearest to x we replace the minimum by its first member, and for all other values of n we replace it by its second member. Thus the second sum is

$$(\log x)\Big(1 + \frac{x}{T}\sum_{1\le k\le x}\frac{1}{k}\Big) \ll \log x + \frac{x}{T}(\log x)^2.$$

Suppose that $2 \le T \le x$ and that $\sigma_0 = 1 + 1/\log x$. Then

$$R \ll \frac{x}{T}(\log x)^2.$$

Put $\sigma_1 = 1 - c/\log T$ where c is a small positive constant, and let $\mathcal{C}$ denote the closed contour that consists of line segments joining the points $\sigma_0 - iT$, $\sigma_0 + iT$, $\sigma_1 + iT$, $\sigma_1 - iT$. From Theorem 6 we know that $\frac{\zeta'}{\zeta}(s)$ has a simple pole with residue -1 at $s = 1$, but that it is otherwise analytic within $\mathcal{C}$. Hence by the calculus of residues,

$$\frac{-1}{2\pi i}\int_{\mathcal{C}} \frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\,ds = x.$$

If c is small then the estimate (6) of Theorem 7 applies on this contour. Hence

$$-\int_{\sigma_0+iT}^{\sigma_1+iT} \frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\,ds \ll \frac{\log T}{T}x^{\sigma_0}(\sigma_0-\sigma_1) \ll \frac{x}{T},$$

and similarly for the integral from $\sigma_1 - iT$ to $\sigma_0 - iT$. Using (6) again, we also see that

$$\begin{aligned}-\int_{\sigma_1+iT}^{\sigma_1-iT} \frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\,ds &\ll x^{\sigma_1}(\log T)\int_{-T}^{T}\frac{dt}{1+|t|} + x^{\sigma_1}\int_{-1}^{1}\frac{dt}{|\sigma_1+it-1|}\\ &\ll x^{\sigma_1}(\log T)^2 + \frac{x^{\sigma_1}}{1-\sigma_1} \ll x^{\sigma_1}(\log T)^2.\end{aligned}$$

On combining these estimates we conclude that

$$\psi(x) = x + O\Big(x(\log x)^2\big(\frac{1}{T} + x^{-c/\log T}\big)\Big).$$

We choose T so that the two terms in the last factor of the error term are equal, i.e., $T = \exp\big(\sqrt{c\log x}\big)$. With this choice of T, the error term above is

$$\ll x(\log x)^2\exp\big(-\sqrt{c\log x}\big) \ll x\exp\big(-c\sqrt{\log x}\big)$$

since we may suppose that $0 < c < 1$. Thus the proof of (12) is complete.

To derive (13) it suffices to combine (12) with the first estimate of Corollary 2.5. As for (14), we note that

$$\pi(x) = \int_{2^-}^{x}\frac{1}{\log u}\,d\vartheta(u) = \mathrm{li}(x) + \int_{2^-}^{x}\frac{1}{\log u}\,d(\vartheta(u)-u).$$

By integrating by parts we see that this last integral is

$$\frac{\vartheta(u)-u}{\log u}\Big|_{2^-}^{x} + \int_2^x \frac{\vartheta(u)-u}{u(\log u)^2}\,du,$$

and by (13) it follows that this is $\ll x \exp\big(-c\sqrt{\log x}\big)$. Thus we have (14), and the proof is complete.

The method we used to derive Theorem 9 is very flexible, and can be applied to many other situations. For example, the summatory function

$$M(x) = \sum_{n \le x} \mu(n)$$

can be estimated by applying the above method with ζ'/ζ replaced by $1/\zeta$. Thus it may be shown that

$$M(x) \ll x \exp\big(-c\sqrt{\log x}\big) \tag{17}$$

for $x \ge 2$. If instead we were to apply the method to the function $1/\zeta(s+1)$, we would find that

$$\sum_{n \le x} \frac{\mu(n)}{n} \ll \exp\big(-c\sqrt{\log x}\big), \tag{18}$$

since $1/(s\zeta(s+1))$ is analytic at $s = 0$. Hence in particular,

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0. \tag{19}$$

6.2. Exercises

1. (Landau (1901); cf Rosser and Schoenfeld (1962)) Use Theorem 9 to show that

$$\pi(2x) - 2\pi(x) = -2(\log 2)x(\log x)^{-2} + O\big(x(\log x)^{-3}\big).$$

Deduce that for all large x, the interval $(x, 2x]$ contains fewer prime numbers than the interval $(0, x]$.

2. Use Theorem 9 to show that if n is of the form $n = \prod_{p \le y} p$ where y is sufficiently large, then $d(n) > n^{(\log 2)/\log\log n}$.

3. (a) Use Theorem 9 to show that

$$\sum_{x < p \le y} \frac{1}{p} = \log \frac{\log y}{\log x} + O\big(\exp(-c\sqrt{\log x})\big).$$

(b) Use the above and Theorem 2.7 to show that

$$\sum_{p \le x} \frac{1}{p} = \log\log x + b + O\big(\exp(-c\sqrt{\log x})\big)$$

where $b = C_0 - \sum_p \sum_{k=2}^{\infty} 1/(kp^k)$.

4. Show that for $x \ge 2$,

$$\sum_{n \le x} \frac{\Lambda(n)}{n} = \log x - C_0 + O\big(\exp\big(-c\sqrt{\log x}\big)\big).$$

5. (cf Cipolla (1902), Rosser (1939)) Let $p_1 < p_2 < \dots$ denote the prime numbers. Show that

$$p_n = n\Big(\log n + \log\log n - 1 + \frac{\log\log n}{\log n} - \frac{2}{\log n} + O\Big(\frac{(\log\log n)^2}{(\log n)^2}\Big).$$

6. (Landau (1900)) Let $\pi_k(x)$ denote the number of integers not exceeding x that are composed of exactly k distinct primes.
(a) Show that

$$\pi_2(x) = \sum_{p \le \sqrt{x}} \pi(x/p) + O\big(x(\log x)^{-2}\big).$$

(b) Show that the sum above is

$$\sum_{p \le \sqrt{x}} \frac{x}{p\log x/p} + O\big(x(\log\log x)(\log x)^{-2}\big).$$

(c) Using Theorem 9 and integration by parts, show that the sum above is

$$x\int_x^{\sqrt{x}} \frac{du}{u(\log x/u)\log u} + O(x/\log x).$$

(d) Conclude that $\pi_2(x) = x(\log\log x)/\log x + O(x/\log x)$.

7. (D. E. Knutson) Let d_n denote the least common multiple of the numbers $1, 2, \dots, n$.
(a) Show that $d_n = \exp(\psi(n))$.
(b) Let $E(z) = \sum_{n=1}^{\infty} z^n/d_n$. Show that this power series has radius of convergence e.
(c) Show that $E(1)$ is irrational.

8. (Landau (1905)) Let $Q(x)$ denote the number of squarefree integers not exceeding x, and define $R(x)$ by the relation $Q(x) = (6/\pi^2)x + R(x)$.
(a) Show that

$$R(x) = M(y)\{x/y^2\} - \sum_{d \le y} \mu(d)\{x/d^2\} + \sum_{m \le x/y^2} M(\sqrt{x/m}) - 2x\int_y^{\infty} M(u)u^{-3}\,du.$$

(b) Taking $y = x^{1/2}\exp\big(-c\sqrt{\log x}\big)$ where c is sufficiently small, show that $R(x) \ll x^{1/2}\exp\big(-c\sqrt{\log x}\big)$.

9. Let $N = N(Q) = 1 + \sum_{q \le Q} \varphi(q)$ be the number of Farey points of order Q, and for $0 \le \alpha \le 1$ write

$$\operatorname{card}\{(a,q) : q \le Q,\ (a,q) = 1,\ a/q \le \alpha\} = N\alpha + R$$

where $R = R(Q,\alpha)$.
(a) Show that if $\alpha = (1/Q)^-$ then $R = -N/Q \asymp -Q$.
(b) Show that if $\alpha = 1 - 1/Q$ then $R = N/Q - 1 \asymp Q$.
(c) Show that

$$R = -\sum_{r \le Q} \{r\alpha\} M(Q/r)$$

for $0 \le \alpha \le 1$.
(d) Show that $R \ll Q$ uniformly for $0 \le \alpha \le 1$.

10. (Landau (1903b), Massias, Nicolas & Robin (1988), (1989)) Let $f(n)$ denote the maximal order of any element of the symmetric group S_n.
(a) Show that $f(n) = \max \operatorname{lcm}(n_1, n_2, \ldots, n_k)$ where the maximum is extended over all sets $\{n_1, n_2, \ldots, n_k)$ of natural numbers for which $n_1 + n_2 + \cdots + n_k \le n$.
(b) Choose y as large as possible so that $\sum_{p \le y} p \le n$. Show that

$$\log f(n) \ge \sum_{p \le y} \log p = (1 + o(1))(n \log n)^{1/2}.$$

(c) Show that $f(n) = \max q_1 q_2 \cdots q_k$ where $q_i = p_i^{a(i)}$, $p_i \ne p_j$ for $i \ne j$, and $\sum q_i \le n$.
(d) Use the arithmetic-geometric mean inequality to show that $\prod q_i \le (n/k)^k$.
(e) Show that if k is the number of q_i's in (c) then $k \le (2 + o(1))(n/\log n)^{1/2}$.
(f) Conclude that $\log f(n) \asymp (n \log n)^{1/2}$.

11. Let $\lambda(n) = (-1)^{\Omega(n)}$ be Liouville's lambda-function.
(a) Show that $\sum_{n=1}^{\infty} \lambda(n) n^{-s} = \zeta(2s)/\zeta(s)$ for $\sigma > 1$.
(b) Using the method of proof of Theorem 9, show that

$$\sum_{n \le x} \lambda(n) \ll x \exp(-c\sqrt{\log x}).$$

(c) Use (17) and the fact that $\lambda(n) = \sum_{d^2 | n} \mu(n/d^2)$ to give a second proof of the above estimate.

12. (Landau (1907, §14)) Let $c_n = 1$ if n is a prime or a prime-power, $c_n = 0$ otherwise.
(a) Show that $\mu(n)\omega(n) = -\sum_{d|n} c_d \mu(n/d)$.
(b) Use (18) and the method of the hyperbola to show that

$$\sum_{n=1}^{\infty} \frac{\mu(n)\omega(n)}{n} = 0.$$

13. Use the method of proof of Theorem 9 to show that

$$\sum_{n\le x}\Lambda(n)n^{-it} = \frac{x^{1-it}}{1-it} + O(x\exp(-c\sqrt{\log x}) + O\Big(x(\log x)^2\exp\Big(-c\frac{\log x}{\log\tau}\Big)\Big)$$

uniformly for $|t|\le x$.

14. Use the method of proof of Theorem 9 to show that for any fixed real t,

$$\sum_{n=1}^{\infty}\mu(n)n^{-1-it} = \frac{1}{\zeta(1+it)}.$$

15. (a) Use the method of proof of Theorem 9 to show that for any fixed $t\ne 0$,

$$\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\log n}n^{-1-it} = \log\zeta(1+it).$$

(b) Deduce that for any $t\ne 0$,

$$\prod_p\Big(1-p^{-1-it}\Big)^{-1} = \zeta(1+it).$$

16. (Landau (1899b), (1901a), (1903c)) Use the method of proof of Theorem 9 to show that

(a) $$\sum_{n=1}^{\infty}\frac{\mu(n)\log n}{n} = -1;$$

(b) $$\sum_{n=1}^{\infty}\frac{\mu(n)(\log n)^2}{n} = -2C_0;$$

(c) $$\sum_{n=1}^{\infty}\frac{\lambda(n)\log n}{n} = -\zeta(2).$$

17. Taking (18) and a quantitative form of the first part of the preceding exercise for granted, use elementary reasoning to show that if $q\le x$ then

(a) $$\sum_{\substack{n\le x\\(n,q)=1}}\frac{\mu(n)}{n} \ll \exp\big(-c\sqrt{\log x}\big),$$

(b) $$\sum_{\substack{n\le x\\(n,q)=1}}\frac{\mu(n)\log n}{n} = -\frac{q}{\varphi(q)} + O\big(\exp\big(-c\sqrt{\log x}\big)\big).$$

18. (Hardy (1921)) Use the method of proof of Theorem 9 to show that

(a)
$$\sum_{n=1}^{\infty}\frac{\mu(n)}{\varphi(n)}=0;$$

(b)
$$\sum_{n=1}^{\infty}\frac{\mu(n)\log n}{\varphi(n)}=0;$$

(c)
$$\sum_{n=1}^{\infty}\frac{\mu(n)(\log n)^2}{\varphi(n)}=4A\log 2$$

where $A=\prod_{p>2}\Big(1-\frac{1}{(p-1)^2}\Big)$.

19. Let $Q(x)$ denote the number of squarefree integers not exceeding x, and recall Theorem 2.2.
(a) Show that

$$Q(x)=\frac{6}{\pi^2}x-x\sum_{n>\sqrt{x}}\frac{\mu(n)}{n^2}-\sum_{n\le\sqrt{x}}\mu(n)\{x/n^2\}$$

where $\{\theta\}=x-[x]$ is the fractional part of θ.
(b) Show that $\sum_{n>y}\mu(n)/n^2\ll y^{-1}\exp\big(-c\sqrt{\log y}\big)$ for $y\ge 2$.
(c) Note that if k is a positive integer then $\{x/n^2\}$ is monotonic for n in the interval $\sqrt{x/(k+1)}<n\le\sqrt{x/k}$. Deduce that if $x\ge 2k^2$ then

$$\sum_{\sqrt{x/(k+1)}<n\le\sqrt{x/k}}\mu(n)\{x/n^2\}\ll\sqrt{x/k}\exp\big(-c\sqrt{\log x}\big).$$

(d) By using the above for $1\le k\le K=\exp\big(-b\sqrt{\log x}\big)$ where b is suitably chosen in terms of c, show that

$$Q(x)=\frac{6}{\pi^2}x+O\Big(x^{1/2}\exp\Big(-\frac{c}{2}\sqrt{\log x}\Big)\Big).$$

20. (Ingham (1945)) Let $F(n)=\sum_{d|n}f(d)$ for all n. From our remarks at the beginning of Chapter 2 we see that it is natural to expect a connection between
(i) $S(x):=\sum_{n\le x}F(n)=cx+o(x)$;
(ii) $\sum_{n=1}^{\infty}f(n)/n=c$.
Neither of these implies the other, but we show now that (i) implies that the series (ii) is (C,1) summable to c.
(a) Show that $S(x)=\sum_{n\le x}f(n)[x/n]$.
(b) Show that

$$\sum_{n\le x}\frac{f(n)}{n}\Big(1-\frac{n}{x}\Big)=\int_1^x S(v)\Big(\sum_{d\le x/v}\mu(d)/d\Big)\frac{dv}{v^2}.$$

(c) Show that

$$\int_1^x \sum_{d\le x/v} \frac{\mu(d)}{d}\,\frac{dv}{v} \to 1$$

as $x \to \infty$.
(d) Use the estimate $\sum_{d\le y} \mu(d)/d \ll (\log 2y)^{-2}$ to show that

$$\int_1^x \Big|\sum_{d\le x/v} \frac{\mu(d)}{d}\Big|\,\frac{dv}{v} \ll 1.$$

(e) Mimic the proof of Theorem 5.5, or use Exercise 5.2.6 to show that if (i) holds then

$$\lim_{x\to\infty} \sum_{n\le x} \frac{f(n)}{n}\Big(1 - \frac{n}{x}\Big) = c.$$

(f) Use Theorem 5.6 to show that if (i) holds and $f(n) = O(1)$, then (ii) follows.
(g) Take $f(n) = \mu(n)$ to deduce that $\sum_{n=1}^\infty \mu(n)/n = 0$. (Of course we used much more above in (d). For a result in the converse direction, see Exercise 8.1.5.)

21. (Landau (1908b)) Let $\mathcal{R}$ be the set of positive integers that can be expressed as a sum of two squares, let $R(x)$ denote the number of such integers not exceeding x, and let χ_1 denote the non-principal character (mod 4), as in Exercise 6.1.5.
(a) Show that

$$\sum_{n\in\mathcal{R}} n^{-s} = (1-2^{-s})^{-1} \prod_{p\equiv 1\,(4)} (1-p^{-s})^{-1} \prod_{p\equiv 3\,(4)} (1-p^{-2s})^{-1}$$

for $\sigma > 1$.
(b) Show that the Dirichlet series above is $f(s)\sqrt{\zeta(s)L(s,\chi_1)}$ where

$$f(s) = (1-2^{-s})^{-1/2} \prod_{p\equiv 3\,(4)} (1-p^{-2s})^{-1/2}$$

is a Dirichlet series with abscissa of convergence $\sigma_c = 1/2$.
(c) Deduce that the Dirichlet series generating function for $\mathcal{R}$ has a quadratic singularity at $s = 1$.
(d) Show that

$$R(x) = \frac{1}{2\pi i}\int_{\mathcal{C}} f(s)\sqrt{\zeta(s)L(s,\chi_1)}\,\frac{x^s}{s}\,ds + O\big(x\exp\big(-c\sqrt{\log x}\big)\big)$$

where $\mathcal{C}$ is the contour running from $1 - c - i\delta$ along a straight line to $1 - i\delta$, then along the semicircle $1 + \delta e^{i\theta}$, $-\pi/2 \le \theta \le \pi/2$, and finally along a straight line to $1 - c + i\delta$. Here c should be sufficiently small and $\delta = 1/\log x$.
(e) Show that the integral above is

$$= \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{g(s)x^s}{\sqrt{s-1}}\,ds$$

where

$$g(s) = \frac{f(s)}{s}\sqrt{(s-1)\zeta(s)L(s,\chi_1)}$$

is analytic in a neighborhood of 1.
(f) Show that

$$g(1) = \sqrt{\frac{\pi}{2}} \prod_{p\equiv 3\,(4)} (1-p^{-2})^{-1/2}.$$

(g) Show that $g(s) = g(1) + O(|s-1|)$ when s is near 1.
(h) By means of Theorem C.3 with $s = 1/2$, or otherwise, show that

$$\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{x^s}{\sqrt{s-1}}\,ds = \frac{x}{\sqrt{\pi \log x}} + O(x^{1-c}).$$

(i) Show that if $\delta = 1/\log x$ then

$$\int_{\mathcal{C}} |s-1|^{1/2} x^{\sigma}\,|ds| \ll \frac{x}{(\log x)^{3/2}}.$$

(j) Show that

$$R(x) = \frac{bx}{\sqrt{\log x}} + O(x(\log x)^{-3/2})$$

where

$$b = 2^{-1/2} \prod_{p\equiv 3\,(4)} (1-p^{-2})^{-1/2}.$$

22. Let $\mathcal{A}$ denote the set of those positive integers that are composed entirely of the prime 2 and primes $\equiv 1 \pmod 4$, and let $\mathcal{B}$ be the the set of those positive integers that are composed entirely of primes $\equiv 3 \pmod 4$.
(a) Explain why any positive integer n has a unique representation in the form $n = a(n)b(n)$ where $a(n) \in \mathcal{A}$ and $b(n) \in \mathcal{B}$.
(b) Let $A(x)$ denote the number of $a \in \mathcal{A}$, $a \le x$. Show that

$$A(x) = \frac{\alpha x}{\sqrt{\log x}} + O\Big(\frac{x}{(\log x)^{3/2}}\Big)$$

where $\alpha = 1/\sqrt{2}$.
(c) Let $B(x)$ denote the number of $b \in \mathcal{B}$, $b \le x$. Show that

$$B(x) = \frac{\beta x}{\sqrt{\log x}} + O\Big(\frac{x}{(\log x)^{3/2}}\Big)$$

where $\beta = \sqrt{2}/\pi$.
(d) For $0 \le \kappa \le 1$ let $N_\kappa(x)$ denote the number of $n \le x$ such that $a(n) \le n^\kappa$. Show that

$$N_\kappa(x) = \sum_{\substack{a \le x^\kappa \\ a\in\mathcal{A}}} \sum_{\substack{a^{1/\kappa - 1} \le b \le x/a \\ b \in \mathcal{B}}} 1.$$

(e) Show that if κ is fixed, $0 \le \kappa \le 1$, then

$$N_k(x) = c(\kappa)x + O\Big(\frac{x}{\sqrt{\log x}}\Big)$$

where

$$c(\kappa) = \frac{1}{\pi}\int_0^\kappa \frac{du}{\sqrt{u(1-u)}}.$$

23. The definition of $\mathrm{li}(x)$ is somewhat arbitrary because of the casual choice of the left hand endpoint of integration. A more intrinsic logarithmic integral is $\mathrm{Li}(x)$, which is defined to be

$$\mathrm{Li}(x) = \lim_{\varepsilon\to 0^+}\Big(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^x\Big)\frac{dt}{\log t} \tag{20}$$

for $x > 1$. (Note that $\mathrm{li}(x) = \mathrm{Li}(x) - \mathrm{Li}(2)$.)
(a) Show that

$$\int_0^{1-\varepsilon}\frac{dt}{\log t} = -\int_{-\log(1-\varepsilon)}^\infty e^{-v}\,\frac{dv}{v}.$$

(b) Show that

$$\int_0^{1-\varepsilon}\frac{dt}{\log t} = \log\varepsilon - \int_0^\infty (\log v)e^{-v}\,dv + O(\varepsilon\log 1/\varepsilon),$$

and explain why the integral on the right is $\Gamma'(1) = -C_0$.
(c) Show that if $x > 1$, then

$$\int_{1+\varepsilon}^x\frac{dt}{\log t} = \int_{\log(1+\varepsilon)}^{\log x} e^v\,\frac{dv}{v}.$$

(d) Show that if $x > 1$, then

$$\int_{1+\varepsilon}^x\frac{dt}{\log t} = \log\log x - \log\varepsilon + \int_1^{\log x}\frac{e^v-1}{v}\,dv + O(\varepsilon).$$

(e) Show that if $x > 1$, then

$$\mathrm{Li}(x) = \log\log x + C_0 + \int_0^{\log x}\frac{e^v-1}{v}\,dv.$$

(f) Expand e^v as a power series, and integrate term-by-term, to show that if $x > 1$, then

$$\mathrm{Li}(x) = \log\log x + C_0 + \sum_{n=1}^\infty\frac{(\log x)^n}{n!n}. \tag{21}$$

24. For $0 < x < 1$ let
$$\mathrm{Li}(x) = \int_0^x \frac{dt}{\log t}.$$

(a) Show that if $0 < x < 1$, then
$$\mathrm{Li}(x) = x \log\log 1/x - \int_{-\log x}^{\infty} e^{-v} \log v \, dv.$$

(b) Show that if $0 < x < 1$, then
$$\mathrm{Li}(x) = x \log\log 1/x + C_0 + \int_0^{-\log x} e^{-v} \log v \, dv.$$

(c) Show that if $0 < x < 1$, then
$$\mathrm{Li}(x) = \log\log 1/x + C_0 - \int_0^{-\log x} \frac{1 - e^{-v}}{v} \, dv.$$

(d) Show that if $0 < x < 1$, then
$$\mathrm{Li}(x) = \log\log 1/x + C_0 + \sum_{n=1}^{\infty} \frac{(\log x)^n}{n!n}.$$

(e) (Pólya & Szegö (1972, p. 8)) Show that
$$\sum_{n=1}^{\infty} \frac{z^n}{n!n} = -e^z \sum_{n=1}^{\infty} \Big(\sum_{k=1}^{n} \frac{1}{k}\Big) \frac{(-z)^n}{n!}.$$

(f) Show that if $0 < x < 1$, then

$$\mathrm{Li}(x) = \log\log 1/x + C_0 - x \sum_{n=1}^{\infty} \Big(\sum_{k=1}^{n} \frac{1}{k}\Big) \frac{(\log 1/x)^n}{n!}. \tag{22}$$

25. By repeated integration by parts we know that
$$\mathrm{Li}(x) = x \sum_{k=1}^{K} \frac{(k-1)!}{(\log x)^k} + O_K\Big(\frac{x}{(\log x)^{K+1}}\Big).$$

Our object is to determine how closely one can approximate to $\mathrm{Li}(x)$ by partial sums of the formal asymptotic expansion
$$\mathrm{Li}(x) \sim x \sum_{k=1}^{\infty} \frac{(k-1)!}{(\log x)^k}.$$

(a) Show that the least term in the sum above occurs when $k = [\log x] + 1$.
(b) Show that if $x \ge e^K$, then

$$\operatorname{Li}(x) = x\sum_{k=1}^{K} \frac{(k-1)!}{(\log x)^k} + \operatorname{Li}(e) + \sum_{k=1}^{K-1}\Big(k!\int_{e^k}^{e^{k+1}} \frac{dt}{(\log t)^{k+1}} - \frac{(k-1)!e^k}{k^k}\Big)$$
$$- \frac{(K-1)!e^K}{K^K} + K!\int_{e^K}^{x} \frac{dt}{(\log t)^{K+1}}.$$

(c) Define $R(x)$ by the relation

$$\operatorname{Li}(x) = x\sum_{k=1}^{[\log x]} \frac{(k-1)!}{(\log x)^k} + R(x).$$

Show that $R(x)$ is increasing, continuous, and convex downward for $x \in [e^K, e^{K+1})$. Let $\alpha_K = R(e^K)$, and let β_K be the limit of $R(x)$ as x tends to e^{K+1} from below.
(d) Show that

$$\int_{e^K}^{e^{K+1}} \frac{dt}{(\log t)^{K+1}} = \frac{e^K}{K^K}\int_0^{1/K} \frac{e^{Kw}}{(1+w)^{K+1}}\,dw.$$

(e) Show that the integrand on the right above is ≤ 1 in the range of integration.
(f) Show that the minimum of $e^{Kw}/(1+w)^{K+1}$ for $w > 0$ occurs when $w = 1/K$.
(g) Show that

$$\frac{e^{K+1}}{(K+1)^{K+1}} < \int_{e^K}^{e^{K+1}} \frac{dt}{(\log t)^{K+1}} < \frac{e^K}{K^{K+1}}.$$

(h) Show that $\alpha_K \nearrow$ and that $\beta_K \searrow$.
(i) Show that $\beta_K - \alpha_K \ll K^{-1/2}$.
(j) Show that $R(x) = c + O\big((\log x)^{-1/2}\big)$ where

$$c = \operatorname{Li}(e) + \sum_{k=1}^{\infty}\Big(k!\int_{e^k}^{e^{k+1}} \frac{dt}{(\log t)^{k+1}} - \frac{(k-1)!e^k}{k^k}\Big).$$

(k) Show that if $x \ge e$, then

$$\alpha_1 \le \operatorname{Li}(x) - x\sum_{k=1}^{[\log x]} \frac{(k-1)!}{(\log x)^k} \le \beta_1 \tag{23}$$

where $\alpha_1 = -0.82316\ldots$ and $\beta_1 = 1.259706\ldots$.

26. (Ingham (1932, pp. 60–63)) Suppose that $\eta(t)$ is defined for $t \ge 2$, that $\eta'(t)$ is continuous, $\eta'(t) \to 0$ as $t \to \infty$, that $\eta(t) \searrow$, that $1/\log t \ll \eta(t) \le 1/2$, and that $\zeta(s) \ne 0$ for $\sigma \ge 1 - \eta(t)$, $t \ge 2$. For $x \ge 2$, put

$$\omega(x) = \min_{2\le t<\infty} \eta(t)\log x + \log t\,.$$

(a) Show that there is an absolute constant $c > 0$ such that

$$\pi(x) = \mathrm{li}(x) + O(x \exp(-c\omega(x)))\,.$$

(b) Show that if $a > 0$ is fixed and (24) holds, then (27) holds with $b = 1/(1+a)$.
(c) Show that (28) follows from (26).

6. Notes

§1. Jensen (1899) proved that if f satisfies the hypotheses of Lemma 1, then

$$|f(0)| \prod_{k=1}^{n} \frac{R}{|z_k|} \; = \; \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(Re^{i\theta}\right)\right| d\theta\right)$$

where $z_1, \ldots, z_n$ are the zeros of f in the disc $|z| \le R$. Here the right hand side may be regarded as being the geometric mean of $|f(z)|$ for z on the circle $|z| = R$. Each factor of the product above is ≥ 1, and if $|z_k| \le r$, then $R/|z_k| \ge R/r$. Thus Lemma 1 follows easily from the above. The products used in the proofs of Lemmas 1 and 3 are known as Blaschke products. Their use (usually with infinitely many factors) is an important tool of complex analysis. Lemma 2 is due to Borel (1897); it refines an earlier estimate of Hadamard. Carathéodory's contributions on this subject are recounted by Landau (1906; §4).

Lemma 4 is implicit in Landau (1909, p. 372), and may have been known earlier. It can also be easily derived from the identity (10.29) that arises by applying Hadamard's theory of entire functions to the zeta function.

The Prime Number Theorem was first proved, in the qualitative form $\pi(x) \sim x/\log x$, independently by Hadamard (1896) and de la Vallée Poussin (1896). In these papers, it was shown that $\zeta(1+it) \ne 0$, but no specific zero-free region was established. The first proof that $\zeta(1+it) \ne 0$ given by de la Vallée Poussin was rather complicated, but later in his long paper he gave a second proof depending on the inequality $1 - \cos 2\theta \le 4(1 + cos\theta)$. This is equivalent to the nonnegativity of the cosine polynomial $3 + 4\cos\theta + \cos 2\theta$, which Mertens (1898) used to obtain the result of Exercise 4. Our Lemma 5 is derived by the same method. The classical zero-free region of Theorem 6 was established first by de la Vallée Poussin (1899). The estimates (6) and (8) of Theorem 7 were first proved by Gronwall (1913).

It has been improved since, by using exponential sum estimates to obtain better upper bounds for $|\zeta(s)|$ when σ is near 1 . The first such improvement was derived by Hardy & Littlewood. Their paper on this was never published, but accounts of their approach have been given by Landau (1924b) and Titchmarsh (1986, Chapter 5). Littlewood (1922) announced that from these estimates he had deduced that $\zeta(s) \ne 0$ for $\sigma \ge 1 - c(\log\log\tau)/\log\tau$. As explained by Ingham (1932, p. 66), Littlewood never published his complicated proof, because the simpler method of Landau (1924a) had become available.

In 1935, Vinogradov introduced a new method for estimating Weyl sums. A *Weyl sum* is a sum of the form $\sum_{n=1}^{N} e(f(n))$ where $f \in \mathbb{R}[x]$. The quality of Vinogradov's estimate depends on rational approximations to the coefficients of f, and on the degree of f. The function $f(x) = t \log x$ is not a polynomial, but by approximating to it by polynomials on can make Vinogradov's method apply. This was first done by Chudakov (1936), who derived estimates for $\zeta(s)$ for σ near 1 that allowed him to deduce that $\zeta(s) \neq 0$ for

$$\sigma > 1 - c(\log \tau)^{-a} \tag{24}$$

for $a > 10/11$. Vinogradov (1936) gave stronger exponential sum estimates, which Titchmarsh (1938) used obtain a zero-free region of the above form for $a > 4/5$. Hua (1949) introduced a further refinement of Vinogradov's method, from which Titchmarsh (1951, Chapter 6) and Tatuzawa (1952) derived the zero-free region

$$\sigma > 1 - c(\log \tau)^{-3/4}(\log \log \tau)^{-3/4}.$$

By refining the passage from Weyl sums to the zeta function, Korobov (1958) obtained (24) for $a > 5/7$, and then Korobov (1958) and Vinogradov (1958) obtained $a > 2/3$. In fact, Vinogradov claimed that one can take $a = 2/3$, but this seems to be still out of reach. Richert's polished exposition of Vinogradov's method is reproduced in Walfisz (1963). Other expositions have since been given by Karatsuba & Voronin (1992, Chapter 4), Montgomery (1994, Chapter 4), and Vaughan (1997). Richert (1967) used Vinogradov's method to show that

$$\zeta(s) \ll t^{100(1-\sigma)^{3/2}}(\log t)^{2/3} \tag{25}$$

for $\sigma \leq 1$, $t \geq 2$. From this it follows that $\zeta(s) \neq 0$ for

$$\sigma \geq 1 - c(\log \tau)^{-2/3}(\log \log \tau)^{-1/3} \tag{26}$$

The methods of Hadamard and de la Vallée Poussin depended on the analytic continuation of $\zeta(s)$, on bounds for the size of $\zeta(s)$ in the complex plane, and on Hadamard's theory of entire functions. The first two of these are achieved most easily by Riemann's functional equation (see Corollaries 10.3–10.5). An abbreviated account of the third is found in Lemma 10.11. Landau (1903a) showed that one can obtain a zero-free region using only the local analytic properties of the zeta function. This enabled Landau to prove the Prime Ideal Theorem, which is the natural extension of the Prime Number Theorem to algebraic number fields: If K is an algebraic number field, then the number of prime ideals $\mathfrak{p}$ in K with $N(\mathfrak{p}) \leq x$ is asymptotic to $x/\log x$ as $x \to \infty$. This could not have been done at that time by the methods of Hadamard and de la Vallée Poussin, since the analytic continuation and functional equation of the Dedekind zeta function $\zeta_K(s)$ was established only later, by Hecke (1917). Landau did not achieve Theorem 6 at the first attempt, but he refined his approach in a series of papers culminating in the polished exposition of Landau (1924a).

§2. Ingham (1932, pp. 60–65; cf Titchmarsh (1986, pp. 56–60)) developed a general system by which any given zero-free region of the zeta function can be used to derive an associated bound for the error term in the Prime Number Theorem. In particular, he showed that if $\zeta(s) \neq 0$ for s in then region (24), then

$$\psi(x) = x + O\big(x \exp\big(-c(\log x)^b\big)\big) \tag{27}$$

where $b = 1/(1+a)$. Similarly, from the zero-free region (26) it follows that

$$\pi(x) = \mathrm{li}(x) + O\big(x \exp\big(-c(\log x)^{3/5}(\log\log x)^{-1/5}\big)\big)\,. \tag{28}$$

Turán (1950) used his method of power sums to show conversely that (27) implies (24). More general converse theorems have since been established by Stás (1961) and Pintz (1980), (1983), (1984). A similar converse theorem, in which an upper bound for $M(x) = \sum_{n\le x}\mu(n)$ is used to produce a zero-free region has been given by Allison (1970).

That $M(x) = o(x)$ was first proved by von Mangoldt (1897). The quantitative estimate (17) is due to Landau (1908a). The relation (19), asserted by Euler(1748; Chapter 15, no. 277), was first proved by von Mangoldt (1897). Landau (1899a) and de la Vallée Poussin (1899) shortly gave simpler proofs.

6. Literature

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