Chapter 10

Analytic properties of the zeta function and L-functions

1. Functional equations and analytic continuation

In §1.3 we saw that the zeta function can be analytically continued to the half-plane $\sigma > 0$. We now derive an important formula for the Riemann zeta function, one that serves to define the zeta function throughout the complex plane. From this formula we see that the zeta function is analytic at all points except for $s = 1$, and we find that $\zeta(s)$ is related to $\zeta(1-s)$. In preparation for this we first use the Poisson summation formula to establish a corresponding functional equation for theta functions.

Theorem 1. *For arbitrary real α, and complex numbers z with $\Re z > 0$,*

$$\sum_{n=-\infty}^{\infty} e^{-\pi(n+\alpha)^2 z} = z^{-1/2} \sum_{k=-\infty}^{\infty} e(k\alpha) e^{-\pi k^2/z}, \tag{1}$$

and

$$\sum_{n=-\infty}^{\infty} (n+\alpha) e^{-\pi(n+\alpha)^2 z} = -iz^{-3/2} \sum_{k=-\infty}^{\infty} k e(k\alpha) e^{-\pi k^2/z} \tag{2}$$

where the branch of $z^{1/2}$ is determined by $1^{1/2} = 1$.

Proof. We can obtain (2) from (1) by differentiating with respect to α, since the differentiated series are uniformly convergent for α in a compact set. As for (1), we note that if $g(u) = f(u+\alpha)$ then $\widehat{g}(t) = \widehat{f}(t)e(t\alpha)$. (Conventions governing the definition of the Fourier transform $\widehat{f}$ are established in Appendix D.) We apply the Poisson summation formula (Theorem D.3) to $g(u)$, where $f(u) = e^{-\pi u^2 z}$, and it remains only to demonstrate that $\widehat{f}(t) = z^{-1/2} e^{-\pi t^2/z}$. Writing

$$-\pi x^2 z - 2\pi i t x = -\pi(x + it/z)^2 z - \pi t^2/z,$$

we see that

$$\widehat{f}(t) = e^{-\pi t^2/z} \int_{-\infty}^{+\infty} e^{-\pi(x+it/z)^2 z}\, dx.$$

We consider this integral to be a contour integral in the complex plane. We note that the integrand tends to 0 very rapidly as $|\Re x|$ tends to infinity with $|\Im x|$ bounded. Hence by Cauchy's theorem we may translate the path of integration to the line $x - it/z$, $-\infty < x < +\infty$, and we find that the above integral is $\int_{-\infty}^{+\infty} e^{-\pi x^2 z}\,dx$. We now turn the path of integration through an angle $-\frac{1}{2}\arg z$ and again apply Cauchy's theorem. After reparameterizing, we see that our integral is $z^{-1/2}\int_{-\infty}^{+\infty} e^{-\pi x^2}\,dx = z^{-1/2}$. This completes the proof.

Theorem 2. *For any complex number s, except $s = 0$, $s = 1$, and any non-zero complex number z with $\Re z \ge 0$,*

$$\begin{aligned}\zeta(s)\Gamma(s/2)\pi^{-s/2} &= \pi^{-s/2}\sum_{n=1}^{\infty} n^{-s}\Gamma\big(s/2, \pi n^2 z\big)\\ &\quad + \pi^{(s-1)/2}\sum_{n=1}^{\infty} n^{s-1}\Gamma\big((1-s)/2, \pi n^2/z\big)\\ &\quad + \frac{z^{(s-1)/2}}{s-1} - \frac{z^{s/2}}{s}.\end{aligned} \tag{3}$$

Here $\Gamma(s,a)$ is the incomplete gamma function,

$$\Gamma(s,a) = \int_a^{\infty} e^{-w} w^{s-1}\,dw, \tag{4}$$

and we may take the path of integration to be the ray $w = a + u$, $0 \le u < \infty$, so that

$$\Gamma(s,a) = \int_0^{\infty} e^{-u-a}(u+a)^{s-1}\,du.$$

Now $(u+a)^{s-1} \ll |a|^{\sigma-1}$ uniformly for $\Re a \ge 0$, $|a| \ge \varepsilon > 0$, and $|\sigma| \le C$, so that $n^{-s}\Gamma(s/2, \pi n^2 z) \ll n^{-2}$ uniformly for $\Re z \ge 0$, $|z| \ge \varepsilon$, $|s| \le C$. Thus the two sums on the right are uniformly convergent for s in any compact set, and hence by a theorem of Weierstrass they represent entire functions. The last two terms have simple poles at 1 and 0, respectively. As for the left hand side, we note that $\Gamma(s/2)$ has a pole at $s = 0$, and never vanishes, so it follows that $\zeta(s)$ is analytic for all $s \ne 1$. If we simultaneously replace s by $1 - s$ and z by $1/z$ then the two sums on the right in (3) are exchanged, and the last two terms are also exchanged, so that the value of the right hand side is invariant. These observations may be summarized as follows:

Corollary 3. *The function*

$$\xi(s) = \frac{1}{2}s(s-1)\zeta(s)\Gamma(s/2)\pi^{-s/2} \tag{5}$$

is entire, and $\xi(s) = \xi(1-s)$ for all s.

This is the functional equation of the zeta function, first proved by Riemann in 1860. Since $\zeta(s) \ne 0$ for $\sigma \ge 1$, it follows that $\xi(s) \ne 0$ for $\sigma \ge 1$, and by the functional equation that $\xi(s) \neq 0$ for $\sigma \le 0$. The zeros of $\zeta(s)$ in the *critical strip* $0 < \sigma < 1$ coincide precisely with those of $\xi(s)$. As $\Gamma(s/2)$ has simple poles at $s = 0, -2, -4, -6, \ldots$, the zeta function has simple zeros at $s = -2, -4, -6, \ldots$. These are the *trivial zeros* of the zeta function. The only other zeros of the zeta function are the *nontrivial zeros* in the critical strip. The generic nontrivial zero is denoted $\rho = \beta + i\gamma$. By the Schwarz reflection principle, $\xi(\overline{s}) = \overline{\xi(s)}$; hence in particular $\xi(\frac{1}{2} - it) = \overline{\xi}(\frac{1}{2} + it)$. But the functional equation gives $\xi(\frac{1}{2} - it) = \xi(\frac{1}{2} + it)$, so it follows that $\xi(\frac{1}{2} + it)$ is real for all real t. Similarly, if ρ is a zero of $\xi(s)$ then so also are $\overline{\rho}$, $1 - \rho$, and $1 - \overline{\rho}$. The as yet unproved *Riemann Hypothesis* (RH) asserts that all nontrivial zeros of the zeta function have real part $1/2$; that is, all the zeros of $\xi(s)$ lie on the *critical line* $\sigma = 1/2$. We shall find it instructive to explore a number of consequence of this famous conjecture.

Proof of Theorem 2. By Euler's integral formula (Theorem C.2) for $\Gamma(s/2)$ we see that if $\sigma > 0$ then

$$\Gamma(s/2) = \int_0^\infty e^{-x} x^{s/2-1}\, dx. \tag{6}$$

By the linear change of variables $x = \pi n^2 u$ it follows that

$$n^{-s}\Gamma(s/2)\pi^{-s/2} = \int_0^\infty e^{-\pi n^2 u} u^{s/2-1}\, du.$$

We assume that $\sigma > 1$ and sum over n to find that

$$\begin{aligned} \zeta(s)\Gamma(s/2)\pi^{-s/2} &= \sum_{n=1}^\infty \int_0^\infty e^{-\pi n^2 u} u^{s/2-1}\, du \\ &= \int_0^\infty \Big(\sum_{n=1}^\infty e^{-\pi n^2 u}\Big) u^{s/2-1}\, du. \end{aligned} \tag{7}$$

Here the exchange of integration and summation is permitted by absolute convergence. Suppose, for the present, that $\Re z > 0$. We may consider the integral above to be a contour integral in the complex plane, and by Cauchy's theorem we may replace the path of integration by the ray from 0 that passes through z. We now consider separately the integral from 0 to z, and the integral from z to ∞. We call these integrals $\int_1$, $\int_2$, respectively. By reversing the steps we made in passing from (6) to (7) we see immediately that

$$\int_2 = \pi^{-s/2} \sum_{n=1}^\infty n^{-s}\Gamma\big(s/2, \pi n^2 z\big).$$

To treat $\int_1$ we let

$$\vartheta(u) = \sum_{-\infty}^{+\infty} e^{-\pi n^2 u} \tag{8}$$

for $\Re u > 0$. Then the sum in the integrand in (7) is $(\vartheta(u)-1)/2$. Thus

$$\textstyle\int_1 = \frac{1}{2}\int_0^z \vartheta(u)u^{s/2-1}\,du - \frac{1}{2}\int_0^z u^{s/2-1}\,du.$$

Here the second integral is $\frac{2}{s}z^{s/2}$. By Theorem 1 we know that $\vartheta(u) = u^{-1/2}\vartheta(1/u)$. Hence the first term above is

$$\frac{1}{2}\int_0^z \vartheta(1/u)u^{s/2-3/2}\,du = \int_0^z \Big(\sum_{n=1}^\infty e^{-\pi n^2/u}\Big)u^{s/2-3/2}\,du + \frac{1}{2}\int_0^z u^{s/2-3/2}\,du.$$

Here the second integral is $\frac{2}{s-1}z^{(s-1)/2}$. By the change of variable $v = 1/u$ we see that the first term above is

$$\int_{1/z}^\infty \Big(\sum_{n=1}^\infty e^{-\pi n^2 v}\Big)v^{(1-s)/2-1}\,dv.$$

We exchange the order of summation and integration, and make the linear change of variables $x = \pi n^2 v$, to see that this is

$$\pi^{(s-1)/2}\sum_{n=1}^\infty n^{s-1}\Gamma\big((1-s)/2, \pi n^2/z\big).$$

Hence

$$\textstyle\int_1 = \frac{z^{(s-1)/2}}{s-1} - \frac{z^{s/2}}{s} + \pi^{(s-1)/2}\sum_{n=1}^\infty n^{s-1}\Gamma\big((1-s)/2, \pi n^2/z\big),$$

so we have the desired identity for $\sigma > 1$. But, as already noted, the two sums represent entire functions, so the right hand side of (3) is analytic for all s except for simple poles at $s = 1$ and $s = 0$. Hence by the uniqueness of analytic continuation the identity (3) holds for all s except at the poles.

The functional equation of Corollary 3 can also be expressed asymmetrically:

Corollary 4. *For all $s \ne 1$,*

$$\zeta(s) = \zeta(1-s)2^s\pi^{s-1}\Gamma(1-s)\sin\frac{\pi s}{2}. \tag{9}$$

Proof. By the reflection principle (C.6) and the duplication formula (C.9), we see that

$$\frac{\Gamma\big(\frac{1-s}{2}\big)}{\Gamma\big(\frac{s}{2}\big)} = \frac{1}{\pi}\Gamma\Big(\frac{1-s}{2}\Big)\Gamma\Big(1-\frac{s}{2}\Big)\sin\frac{\pi s}{2} = \pi^{-1/2}2^s\Gamma(1-s)\sin\frac{\pi s}{2}.$$

Thus the stated identity follows from Corollary 3.

By Stirling's formula, we can describe $|\zeta(s)|$ in terms of $|\zeta(1-s)|$.

Corollary 5. *Suppose that $A > 0$ is fixed. Then*

$$|\zeta(s)| \asymp \tau^{1/2-\sigma}|\zeta(1-s)|$$

uniformly for $|\sigma| \le A$ and $|t| \ge 1$. Here $\tau = |t| + 4$, as usual.

Proof. Since the above is invariant when s is replaced by $1 - s$, we may suppose that $-A \le \sigma \le 1/2$. We may also suppose that $t \ge 1$, since $|\zeta(\sigma - it)| = |\zeta(\sigma + it)|$. We consider the factors on the right hand side of (9). By Stirling's formula as formulated in (C.18), we see that

$$|\Gamma(1-s)| \asymp \left|(1-s)^{1/2-s}\right| = |1-s|^{1/2-\sigma}\exp(t\arg(1-s)).$$

But $\arg(1-s) = -\arctan t/(1-\sigma) = -\pi/2 + O(1/t)$ and $|1-s| \sim t$, so $|\Gamma(1-s)| \asymp t^{1/2-\sigma}\exp(-\pi t/2)$. On the other hand, $\sin z = (e^{iz} - e^{-iz})/(2i)$, so $|\sin \pi s/2| \asymp \exp(\pi t/2)$, and we obtain the stated result.

Let σ be fixed, and let $\mu(\sigma)$ denote the infimum of those exponents μ such that $\zeta(\sigma + it) \ll \tau^\mu$. This is the *Lindelöf μ-function*. By Corollary 1.17 we know that $\mu(\sigma) = 0$ for $\sigma \ge 1$ and that $\mu(\sigma) \le 1 - \sigma$ for $0 < \sigma \le 1$. By Corollary 5 we see that $\mu(\sigma) = \mu(1-\sigma) + 1/2 - \sigma$. Hence in particular, $\mu(\sigma) = 1/2 - \sigma$ for $\sigma \le 0$. For $0 < \sigma < 1$ the value of $\mu(\sigma)$ is at present unknown, but the *Lindelöf Hypothesis* (LH) asserts that $\zeta(1/2 + it) \ll_\varepsilon \tau^\varepsilon$, which is to say that $\mu(1/2) = 0$. From this it follows that

$$\mu(\sigma) = \begin{cases} 0 & \text{for } \sigma \ge 1/2, \\ 1/2 - \sigma & \text{for } \sigma \le 1/2. \end{cases} \tag{10}$$

Three different proofs that LH implies the above are found in Exercises 18–20. Also, from Exercises 20 and 21 we see that LH is equivalent to a certain assertion concerning the distribution of the zeros of $\zeta(s)$. Since this assertion is visibly weaker than RH, it is evident that RH implies LH. In Chapter 12 we shall show that RH implies a quantitative form of LH.

Concerning special values of the zeta function, we observe first that since $\zeta(s) \sim 1/(s-1)$ for s near 1, it follows from Corollary 4 that

$$\zeta(0) = -1/2. \tag{11}$$

In addition, we note that Corollary B.3 asserts that

$$\zeta(2k) = \frac{(-1)^{k-1}2^{2k-1}B_{2k}}{(2k)!}\pi^{2k} \tag{12}$$

for each positive integer k. Hence by taking $s = 1 - 2k$ in Corollary 4 we deduce that

$$\zeta(1-2k) = \frac{-B_{2k}}{2k} \tag{13}$$

for positive integers k. An alternative proof of this is found in Appendix B. We may also determine the value of $\zeta'(0)$, as follows. Let $f(s) = (s-1)\zeta(s)$. By Corollary 1.16 we know that $f(s) = 1 + C_0(s-1) + \cdots$ for s near 1. On multiplying both sides of (9) by $s-1$ we see that $f(s) = -\zeta(1-s)2^s\pi^{s-1}\Gamma(2-s)\sin \pi s/2$. On differentiating both sides and setting $s=1$ we discover that $C_0 = 2\zeta'(0) - 2\zeta(0)\log 2\pi + 2\zeta(0)\Gamma'(1)$. But $\zeta(0) = -1/2$ and $\Gamma'(1) = -C_0$, so we find that

$$\zeta'(0) = -\frac{1}{2}\log 2\pi. \tag{14}$$

Our treatment of the zeta function extends readily to L-functions.

Theorem 6. *For z with $\Re z > 0$ let*

$$\vartheta_0(z,\chi) = \sum_{n=-\infty}^{\infty} \chi(n)e^{-\pi n^2 z/q},$$

$$\vartheta_1(z,\chi) = \sum_{n=-\infty}^{\infty} n\chi(n)e^{-\pi n^2 z/q}.$$

If χ is a primitive character modulo q then

$$\vartheta_0(z,\chi) = \frac{\tau(\chi)}{q^{1/2}} z^{-1/2}\vartheta_0(1/z,\overline{\chi}),$$

$$\vartheta_1(z,\chi) = \frac{\tau(\chi)}{iq^{1/2}} z^{-3/2}\vartheta_1(1/z,\overline{\chi})$$

where the branch of $z^{1/2}$ is determined by $1^{1/2} = 1$.

Though both these functions are defined for all χ, we note that if $\chi(-1) = -1$ then $\vartheta_0(z,\chi) = 0$ for all z, while if $\chi(-1) = 1$ then $\vartheta_1(z,\chi) = 0$ identically. Thus $\vartheta_0(z,\chi)$ is of interest when $\chi(-1) = 1$, and $\vartheta_1(z,\chi)$ is useful when $\chi(-1) = -1$.

Proof. Since χ is periodic with period q, it follows that

$$\vartheta_0(z,\chi) = \sum_{a=1}^{q} \chi(a) \sum_{m=-\infty}^{\infty} e^{-\pi(mq+a)^2 z/q}.$$

By (1) with $\alpha = a/q$ and z replaced by qz we see that the above is

$$= (qz)^{-1/2} \sum_{a=1}^{q} \chi(a) \sum_{k=-\infty}^{\infty} e^{-\pi k^2/(qz)} e(ak/q) = (qz)^{-1/2} \sum_{k=-\infty}^{\infty} e^{-\pi k^2/(qz)} \sum_{a=1}^{q} \chi(a)e(ak/q).$$

Since χ is primitive, we know by Theorem 9.7 that the inner sum on the right is $\tau(\chi)\overline{\chi}(k)$ for all k. This gives the identity for ϑ_0. The identity for ϑ_1 is proved similarly, using (2).

In order to unify our formulæ we find it convenient to put

$$\kappa = \kappa(\chi) = \begin{cases} 0 & \text{if } \chi(-1) = 1, \\ 1 & \text{if } \chi(-1) = -1. \end{cases} \tag{15}$$

In this notation, the formulæ of Theorem 6 read

$$\vartheta_\kappa(z,\chi) = \frac{\varepsilon(\chi)}{z^{1/2+\kappa}}\vartheta_\kappa(1/z,\overline{\chi}) \tag{16}$$

where

$$\varepsilon(\chi) = \frac{\tau(\chi)}{i^\kappa\sqrt{q}}. \tag{17}$$

Suppose that χ is primitive. Some of our results concerning Gauss sums can be reformulated in terms of $\varepsilon(\chi)$. Firstly, from Theorem 9.7 we see that $|\varepsilon(\chi)| = 1$. Secondly, by Theorems 9.5 and 9.7 we see that $\varepsilon(\chi)\varepsilon(\overline{\chi}) = 1$. Finally, if χ is not only primitive but also quadratic, then $\varepsilon(\chi) = 1$, by Theorem 9.17.

In the same way that Theorem 2 was derived from (8), the following is an immediate consequence of (16).

Theorem 7. *Let χ be a primitive character modulo q with $q > 1$. Then for any complex numbers s and z with $\Re z \geq 0$,*

$$\begin{aligned} L(s,\chi)&\Gamma((s+\kappa)/2)(q/\pi)^{(s+\kappa)/2} \\ &= (q/\pi)^{(s+\kappa)/2}\sum_{n=1}^\infty \chi(n)n^{-s}\Gamma\big((s+\kappa)/2, \pi n^2 z/q\big) \\ &\qquad + \varepsilon(\chi)(q/\pi)^{(1-s+\kappa)/2}\sum_{n=1}^\infty \overline{\chi}(n)n^{s-1}\Gamma\big((1-s+\kappa)/2, \pi n^2/(qz)\big). \end{aligned} \tag{18}$$

As was the case with the zeta function, the above is first proved for $\sigma > 1$. Since each term of the series is entire, and since the series are locally uniformly convergent, the right hand side is an entire funcyion of s, and this provides an analytic continuation of $L(s,\chi)$ to the entire complex plane. If in the above we replace χ by $\overline{\chi}$, s by $1-s$, and z by $1/z$, and then multiply both sides by $\varepsilon(\chi)$ then the right hand side above is unchanged, and thus we obtain a functional equation for $L(s,\chi)$, as follows.

Corollary 8. *Let χ be a primitive character modulo q with $q > 1$. The function*

$$\xi(s,\chi) = L(s,\chi)\Gamma((s+\kappa)/2)(q/\pi)^{(s+\kappa)/2} \tag{19}$$

is entire, and $\xi(s,\chi) = \varepsilon(\chi)\xi(1-s,\overline{\chi})$ for all s.

Let χ be a primitive character modulo q, $q > 1$. We already know that $L(s,\chi) \neq 0$ for $\sigma > 1$. Since the gamma function has no zeros, it follows that $\xi(s,\chi) \neq 0$ in this halfplane. By the functional equation, $\xi(s,\chi) \neq 0$ also for $\sigma < 0$, and hence $L(s,\chi) \neq 0$ for $\sigma < 0$ except that $L(s,\chi)$ must have simple zeros where the gamma factor has simple poles, which is to say at $-\kappa, -\kappa-2, -\kappa-4, \ldots$. These are the *trivial zeros* of $L(s,\chi)$. Zeros $\rho = \beta + i\gamma$ of $L(s,\chi)$ in the *critical strip* $0 \leq \beta \leq 1$ are called *nontrivial*. The conjecture that these latter zeros all lie on the *critical line* $\sigma = 1/2$ is the *Generalized*

Riemann Hypothesis (GRH). If ρ is a nontrivial zero of $L(s,\chi)$, then by the functional equation $1-\rho$ is a zero of $L(s,\overline{\chi})$. Consequently $1-\overline{\rho}$ is a zero of $L(s,\chi)$, since in general $\overline{L(s,\chi)} = L(\overline{s},\overline{\chi})$. The pair of zeros $\rho, 1-\overline{\rho}$ are symmetrically placed with respect to the critical line. Of course, if $\beta = 1/2$ then $\rho = 1-\overline{\rho}$. For complex characters there is no symmetry about the real axis, but if χ is quadratic then $\overline{\chi} = \chi$, and so if ρ is a zero then so also are $\overline{\rho}$, $1-\rho$, and $1-\overline{\rho}$.

The functional equation of an L-function can also be expressed asymmetrically.

Corollary 9. *Suppose that χ is a primitive character* (mod q) *with $q > 1$. Then for all s,*

$$L(s,\chi) = \varepsilon(\chi)L(1-s,\overline{\chi})2^s\pi^{s-1}q^{1/2-s}\Gamma(1-s)\sin\frac{\pi}{2}(s+\kappa).$$

Proof. When $\kappa = 0$ we proceed as in the proof of Corollary 4. When $\kappa = 1$ we use the reflection formula (C.6) and the duplication formula (C.9) to see that

$$\frac{\Gamma(1-s/2)}{\Gamma((s+1)/2)} = \frac{1}{\pi}\Gamma(1-s/2)\Gamma(1/2-s/2)\sin\pi(s+1)/2 = 2^s\pi^{-1/2}\Gamma(1-s)\sin\frac{\pi}{2}(s+1).$$

This, with the identity $\xi(s,\chi) = \varepsilon(\chi)\xi(1-s,\overline{\chi})$ gives the stated result.

By the same method used to prove Corollary 5 we obtain

Corollary 10. *Let χ be a primitive character* (mod q) *with $q > 1$, and suppose that $A > 0$ is fixed. Then*

$$|L(s,\chi)| \asymp (q\tau)^{1/2-\sigma}|L(1-s,\overline{\chi})|$$

uniformly for $|\sigma| \le A$ and $|t| \ge 1$. If $-A \le \sigma \le 1/2$ and $|t| \le 1$ then

$$L(s,\chi) \ll q^{1/2-\sigma}|L(1-s,\overline{\chi})|.$$

Let χ be a character modulo q. If χ is imprimitive then χ is induced by a primitive character $\chi^\star$ modulo d, for some $d|q$, and

$$L(s,\chi) = L(s,\chi^\star)\prod_{p|q}\Big(1-\frac{\chi^\star(p)}{p^s}\Big). \tag{20}$$

If $p|d$ then $\chi^\star(p) = 0$, and thus in the above product we may confine our attention to those primes $p|q$ such that $p \nmid d$. For such a prime, the factor $1-\chi^\star(p)/p^s$ is an entire function whose zeros form an arithmetic progression on the imaginary axis. Thus $L(s,\chi)$ has all the zeros of $L(s,\chi^\star)$, and if there are primes $p|q$ such that $p \nmid d$ then $L(s,\chi)$ has additional zeros on the imaginary axis. Such zeros constitute a finite union of arithmetic progressions. In the special case $\chi = \chi_0$, we have

$$L(s,\chi_0) = \zeta(s)\prod_{p|q}\Big(1-\frac{1}{p^s}\Big).$$

Thus $L(s,\chi_0)$ has a pole at $s = 1$ with residue $\varphi(q)/q$, it has all the zeros of $\zeta(s)$, and it also has zeros of the form $2\pi ik/\log p$ where k takes integral values and $p|q$.

10.1. Exercises

1. Let $\vartheta(u)$ be defined as in (8). Show that $\vartheta'(1) = -\vartheta(1)/4$.

2. Let f be an even function in $L^1(\mathbb{R})$, let $\beta > 1$, suppose that $f(x) = O(x^{-\beta})$ as $x \to \infty$, and that $\widehat{f}(u) = O(u^{-\beta})$ as $u \to \infty$. Show that

$$2\zeta(s)\int_0^\infty f(x)x^{s-1}\,dx = 2\sum_{n=1}^\infty n^{-s}\int_n^\infty f(x)x^{s-1}\,dx + 2\sum_{n=1}^\infty n^{s-1}\int_n^\infty \widehat{f}(u)u^{-s}\,du$$
$$- f(0)/s + \widehat{f}(0)/(s-1)$$

for $1 - \beta < \sigma < \beta$.

3. (Heilbronn (1938); cf Weil (1967)) (a) Show that for $c > 1$, $x > 0$,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \zeta(s)\Gamma(s/2)(\pi x)^{-s/2}\,ds = 2\sum_{n=1}^\infty e^{-\pi n^2 x}.$$

(b) With $\vartheta(x)$ defined as in (8), use the functional equation of the zeta function to show that $\vartheta(x) = x^{-1/2}\vartheta(1/x)$ for $x > 0$.

4. (Lavrik (1965)) (a) Suppose that $\Re z > 0$, that $\sigma_0 > \max(0, -\sigma)$, and that $s \ne 0$, $s \ne -1$, $s \ne -2$, ... By pulling the contour to the left and summing the residues, show that

$$\frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \Gamma(w+s)z^{-w}\,\frac{dw}{w} = \Gamma(s) - \sum_{k=0}^\infty \frac{(-1)^k z^{s+k}}{k!(s+k)}.$$

(b) Show that if $\sigma > 0$ then the right hand side above is $\Gamma(s, z)$.
(c) Argue that both sides are entire functions of s, and hence that the identity

$$\Gamma(s, z) = \frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \Gamma(w+s)z^{-w}\,\frac{dw}{w}$$

holds for all complex s.
(d) Show that if $\sigma_0 > \max(0, (1-\sigma)/2)$ then

$$\pi^{-s/2}\sum_{n=1}^\infty n^{-s}\Gamma\big(s/2, \pi n^2 z\big) = \frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \zeta(s+2w)\Gamma(w+s/2)\pi^{-w-s/2}z^{-w}\,\frac{dw}{w}\,.$$

(e) Suppose now that $s \ne 0$ and $s \ne 1$. Explain why the integrand has poles at $w = 0$, $w = (1-s)/2$, $w = -s/2$, and nowhere else.
(f) Show that when the contour is pulled to the left, the pole at $w = 0$ contributes $\zeta(s)\Gamma(s/2)\pi^{-s/2}$, the pole at $w = (1-s)/2$ contributes $z^{(s-1)/2}/(s-1)$, and the pole at $-s/2$ contributes $-z^{s/2}/s$.
(g) Suppose the contour is pulled to the left to an abscissa $\sigma_1 < \min(0, -\sigma/2)$. By

means of the identity $\zeta(s)\Gamma(s/2)\pi^{-s/2} = \zeta(1-s)\Gamma((1-s)/2)\pi^{(s-1)/2}$ and the change of variable $w \mapsto -w$, show that the expression is $\pi^{(s-1)/2}\sum_{n=1}^\infty n^{s-1}\Gamma\big((1-s)/2, \pi n^2/z\big)$. Thus demonstrate that Theorem 2 can be derived from Corollary 3.

5. Suppose that α is real, that $\Re z > 0$ and that χ is a primitive character (mod q).
(a) Show that

$$\sum_{n=-\infty}^{\infty} \chi(n)e^{-\pi(n+\alpha)^2 z/q} = \frac{\tau(\chi)}{q^{1/2}} z^{-1/2} \sum_{k=-\infty}^{\infty} \overline{\chi}(k)e(k\alpha/q)e^{-\pi k^2/(qz)}.$$

(b) By differentiating with respect to α, or otherwise, show that

$$\sum_{n=-\infty}^{\infty} \chi(n)(n+\alpha)e^{-\pi(n+\alpha)^2 z/q} = \frac{\tau(\chi)}{iq^{1/2}} z^{-3/2} \sum_{k=-\infty}^{\infty} \overline{\chi}(k)ke(k\alpha/q)e^{-\pi k^2/(qz)}.$$

6. Let α and β be real numbers, and suppose that $\Re z > 0$, and put

$$\vartheta_0(z;\alpha,\beta) = \sum_{n=-\infty}^{\infty} e(n\beta)e^{-\pi(n+\alpha)^2 z}.$$

(a) Show that if $f(x) = e(\beta x)e^{-\pi(x+\alpha)^2 z}$ then $\widehat{f}(t) = e(-\alpha\beta)z^{-1/2}$
(b) Show that $\vartheta_0(z;\alpha,\beta) = e(-\alpha\beta)z^{-1/2}\vartheta(1/z,-\beta,\alpha)$.
(c) Without using the result of (b), show that $\vartheta_0(z;\alpha,\beta) = \vartheta_0(z;-\alpha,-\beta)$.

7. Show that

$$\sum_{n=-\infty}^{\infty} \big(1 - 2\pi n^2 x\big)e^{-\pi n^2 x} > \sum_{n=-\infty}^{\infty} \big(2\pi(n+1/2)^2 x - 1\big)e^{-\pi(n+1/2)^2 x} > 0$$

for all $x > 0$.

8. Use the functional equation of the zeta function in any convenient form to show that

$$\zeta(1-s) = \zeta(s)2^{1-s}\pi^{-s}\Gamma(s)\cos\frac{\pi s}{2}.$$

9. Show that if k is a positive integer then

$$\zeta'(-2k) = \frac{(-1)^k(2k)!\zeta(2k+1)}{2^{2k+1}\pi^{2k}}.$$

10. Let $\vartheta(x)$ be defined as in (8). Show that

$$\zeta(s)\Gamma(s/2)\pi^{-s/2} = \frac{1}{s(s-1)} + \frac{1}{2}\int_1^\infty \big(x^{s/2} + x^{(1-s)/2}\big)\big(\vartheta(x) - 1\big)\,\frac{dx}{x}$$

for all s except $s = 1$ or $s = 0$.

11. (Walfisz (**1931**, p. 454)) Show that

$$\sum_{\substack{a=1 \\ (a,b)=1}}^{\infty} \sum_{b=1}^{\infty} \frac{1}{a^2 b^2} = \frac{5}{2}.$$

12. (Mallik (**1977**)) Let χ be a primitive quadratic character.
(a) Show that $\xi'(1/2, \chi) = 0$.
(b) Show that if $L(1/2, \chi) \neq 0$, then $\operatorname{sgn} L'(1/2, \chi) = -\operatorname{sgn} L(1/2, \chi)$.

13. Let χ be a primitive character modulo q, and let θ be a real number such that $e^{2i\theta} = \varepsilon(\chi)$. Thus $e^{i\theta}$ is one of the square-roots of $\varepsilon(\chi)$. Show that $\xi(1/2 + it, \chi)e^{-i\theta}$ is real for all real t.

14. Let χ be a primitive character modulo q with $q > 1$, and suppose that $\chi(-1) = 1$.
(a) For each positive integer k, show that

$$L(2k, \chi) = \frac{(-1)^{k-1} 2^{2k-1} \pi^{2k} \tau(\chi)}{(2k)!\, q} \sum_{a=1}^{q} \overline{\chi}(a) B_{2k}(a/q).$$

(b) For positive integers k, deduce that

$$L(1 - 2k, \chi) = \frac{-q^{2k-1}}{2k} \sum_{a=1}^{q} \chi(a) B_{2k}(a/q).$$

15. Let χ be a primitive character modulo q with $q > 1$, and suppose that $\chi(-1) = -1$.
(a) For each nonnegative integer k, show that

$$L(2k + 1, \chi) = \frac{i(-1)^k 2^{2k} \pi^{2k+1} \tau(\chi)}{(2k + 1)!\, q} \sum_{a=1}^{q} \overline{\chi}(a) B_{2k+1}(a/q).$$

(b) Show that when $k = 0$, the above is consistent with the formula of Theorem 9.9.
(c) For nonnegative integers k, deduce that

$$L(-2k, \chi) = \frac{-q^{2k}}{2k + 1} \sum_{a=1}^{q} \overline{\chi}(a) B_{2k+1}(a/q).$$

16. (a) Let p_1 and p_2 be distinct primes. Show that $(\log p_1)/(\log p_2)$ is irrational.
(b) Let χ be a character modulo q. Show that all zeros of $L(s, \chi)$ on the imaginary axis

are simple, except possibly for zeros at the point $s = 0$.
(c) Let a positive integer m and a primitive character $\chi^\star$ be given. Show that there is a character χ induced by $\chi^\star$ such that $L(s,\chi)$ has a zero at $s = 0$ of exact multiplicity m.

17. (Landau (**1907**)) (a) Let χ denote the character modulo 5 such that $\chi(2) = i$. Show that $L(1,\chi) = (-1-3i)\pi\tau(\chi)/25$.
(b) With χ as above, show that $L(2,\chi^2) = 4\sqrt{5}\pi^2/125$.
(c) Let χ be as above. By using Exercise 9.2.9, or otherwise, show that $\tau(\chi)^2 = (-1-2i)\sqrt{5}$.
(d) With χ as above, show that

$$\frac{L(1,\chi)^2}{L(2,\chi^2)} = 1 + i/2.$$

(e) Let χ denote a nonprincipal character modulo q. Show that

$$\sum_{n=1}^\infty 2^{\omega(n)}\chi(n)n^{-s} = \frac{L(s,\chi)^2}{L(2s,\chi^2)}$$

for $\sigma > 1/2$.
(f) Let $\varepsilon_n = 1$ if $n \equiv 1 \pmod 5$, $\varepsilon_n = -1$ if $n \equiv -1 \pmod 5$, and $\varepsilon_n = 0$ otherwise. Show that

$$\sum_{n=1}^\infty \frac{\varepsilon_n 2^{\omega(n)}}{n} = 1.$$

18. Suppose throughout that $0 < \delta \le 1/2$. (a) Let $\alpha(s) = \sum_{n=1}^\infty a_n n^{-s}$ be a Dirichlet series with abscissa of convergence σ_c. Show that if $\sigma_0 > \max(\delta, \sigma_c)$ then

$$\sum_{n\le x} a_n\big((x/n)^\delta - (n/x)^\delta\big) = \frac{\delta}{\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \alpha(w)\frac{x^w}{(w-\delta)(w+\delta)}\,dw$$

(b) By taking $\alpha(w) = \zeta(1/2 + it + w)$, and considering the residues arising from poles at $w = 1/2 - it$ and at $w = \delta$, show that

$$\begin{aligned}\zeta(1/2+\delta+it) &= x^{-\delta}\sum_{n\le x} n^{-1/2-it}\big((x/n)^\delta - (n/x)^\delta\big)\\ &\qquad + \frac{\delta x^{-\delta}}{\pi}\int_{-\infty}^{\infty}\zeta(1/2+it+iu)\frac{x^{iu}}{u^2+\delta^2}\,du\\ &\qquad - \frac{2\delta x^{1/2-\delta-it}}{(1/2-it-\delta)(1/2-it+\delta)}\\ &= T_1 + T_2 + T_3,\end{aligned}$$

say.
(c) Show that

$$T_1 \ll \big(1 + x^{1/2-\delta}\big)\min\Big(\frac{1}{|\delta-1/2|}, \log x\Big).$$

(d) Let $M(T) = \max_{0\le t\le T} |\zeta(1/2+it)|$. Show that

$$T_2 \ll x^{-\delta} M(2\tau)$$

uniformly for $0 < \delta \le 1/2$.
(e) Show that $T_3 \ll x^{1/2-\delta}/\tau^2$.
(f) By taking $x = M(2\tau)^2$, show that

$$\zeta(\sigma+it) \ll M(2\tau)^{2-2\sigma} \min\Big(\frac{1}{|\sigma-1|}, \log M(2\tau)\Big)$$

uniformly for $1/2 \le \sigma \le 1$.
(g) Show that if $M(T) \ll_\varepsilon T^\varepsilon$ then $\mu(\sigma) = 0$ for $\sigma \ge 1/2$.
(h) By Corollary 5, deduce that if $M(T) \ll_\varepsilon T^\varepsilon$ then $\mu(\sigma) = 1/2 - \sigma$ when $\sigma \le 1/2$.

19. Let $M(\sigma, T) = \max_{1\le t\le T} |\zeta(\sigma+it)|$. Suppose that $\sigma, \sigma_1, \sigma_2$ are fixed, $0 \le \sigma_1 < \sigma < \sigma_2 \le 1$. Let $\mathcal{C}$ denote the rectangular contour with vertices $\sigma_2 - \sigma - i\tau/2$, $\sigma_2 - \sigma + i\tau/2$, $\sigma_1 - \sigma + i\tau/2$, $\sigma_1 - \sigma - i\tau/2$.
(a) Show that

$$\zeta(\sigma+it) = \frac{1}{2\pi i}\int_{\mathcal{C}} \zeta(s+w)\frac{x^w}{w(w+1)}\,dw.$$

(b) Deduce that

$$\zeta(\sigma+it) \ll M(\sigma_1, 2\tau)x^{\sigma_1-\sigma} + M(\sigma_2, 2\tau)x^{\sigma_2-\sigma}\,.$$

(c) By choosing x suitably, show that

$$M(\sigma, T) \ll M(\sigma_1, 2T)^{(\sigma_2-\sigma)/(\sigma_2-\sigma_1)} M(\sigma_2, 2T)^{(\sigma-\sigma_1)/(\sigma_2-\sigma_1)}.$$

(d) Deduce that

$$\mu(\sigma) \le \frac{\sigma_2-\sigma}{\sigma_2-\sigma_1}\mu(\sigma_1) + \frac{\sigma-\sigma_1}{\sigma_2-\sigma_1}\mu(\sigma_2).$$

(e) Conclude that $\mu(\sigma) \le \frac{1}{2}(1-\sigma)$ for $0 \le \sigma \le 1$.
(f) Show that if $\mu(1/2) = 0$ then (10) holds for all σ.

20. (Backlund (1918)) Assume the Lindelöf Hypothesis (LH) throughout, and suppose that δ is a small fixed positive number and t is not the ordinate γ of a zero ρ of $\zeta(s)$.
(a) Show that the number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ in the rectangle $1/2 + \delta \le \beta \le 1$, $T - 1 \le \gamma \le T + 1$ is $o(\log T)$.
(b) Show that

$$\frac{\zeta'}{\zeta}(s) = \sum_\rho \frac{1}{s-\rho} \; + \; o(\log\tau)$$

uniformly for $1/2+2\delta \le \sigma \le 2$ where the sum is over those zeros ρ for which $1/2+\delta \le \beta \le 1$, $t-1 \le \gamma \le t+1$.
(c) Show that if $\sigma_1 < \sigma_2$ and $t \ne \gamma$, then

$$\int_{\sigma_1}^{\sigma_2} \frac{\sigma-\beta}{(\sigma-\beta)^2+(t-\gamma)^2}\,d\sigma = \frac{1}{2}\log\frac{(\sigma_2-\beta)^2+(t-\gamma)^2}{(\sigma_1-\beta)^2+(t-\gamma)^2}.$$

(d) Show that if $1/2 \le \sigma_1 \le 1$ and $t \ne \gamma$, then

$$\int_{\sigma_1}^2 \frac{\sigma-\beta}{(\sigma-\beta)^2+(t-\gamma)^2}\,d\sigma \ge 0.$$

(e) Show that if t is not the ordinate of a zero, then

$$\int_{\sigma_1}^2 \Re\frac{\zeta'}{\zeta}(\sigma+it)\,d\sigma \ge -\varepsilon\log\tau$$

uniformly for $1/2+2\delta \le \sigma \le 2$.
(f) Show that $\mu(\sigma)=0$ for $1/2<\sigma\le 2$.
(g) Deduce that $\mu(\sigma)=1/2-\sigma$ for $-1\le\sigma<1/2$.
(h) Show that

$$\int_{\sigma_1}^{\sigma_2} \frac{t-\gamma}{(\sigma-\beta)^2+(t-\gamma)^2}\,d\sigma = \arctan\frac{t-\gamma}{\sigma_2-\beta} - \arctan\frac{t-\gamma}{\sigma_1-\beta}.$$

(i) Deduce that

$$\left|\int_{\sigma_1}^{\sigma_2} \frac{t-\gamma}{(\sigma-\beta)^2+(t-\gamma)^2}\,d\sigma\right| \le \pi.$$

(j) Conclude that $\arg\zeta(1/2+2\delta+it)=o(\log\tau)$.

21. (Backlund (1918); cf Littlewood (1924)) Suppose now that the number of zeros ρ of $\zeta(s)$ in a rectangle $1/2+\delta\le\beta\le 1$, $t-1\le\gamma\le t+1$ is $o(\log\tau)$ as $t\to\infty$, and put

$$f(s)=\frac{\zeta'}{\zeta}(s)-\sum_\rho\frac{1}{s-\rho}$$

where the sum is over the $o(\log\tau)$ zeros in such a rectangle.
(a) Explain why $f(s)\ll\log\tau$ in the disk $|s-2-it_0|\le 3/2-2\delta$.
(b) Explain why $f(s)=o(\log\tau)$ in the disk $|s-2-it_0|\le 1/2$.
(c) Use Hadamard's three circles theorem to show that $f(s)=o(\log\tau)$ for $|s-2-it_0|\le 3/2-3\delta$.
(d) Deduce that $\zeta(1/2+3\delta+it)\ll\tau^\varepsilon$.
(e) Suppose that our hypothesis concerning the number of zeros in a rectangle holds for every fixed positive δ. Deduce that $\mu(\sigma)=0$ for $\sigma>1/2$.
(f) By Exercise 19(d), conclude that $\mu(1/2)=0$, i.e., that LH follows.

22. For $0<\alpha\le 1$ and $\sigma>1$ let $\zeta(s,\alpha)=\sum_{n=0}^\infty(n+\alpha)^{-s}$ be the Hurwitz zeta function.
(a) Show that

$$\zeta(s,\alpha)\Gamma(s)=\int_0^\infty\frac{x^{s-1}e^{-\alpha x}}{1-e^{-x}}\,dx$$

for $\sigma > 1$.
(b) Let

$$I(s,\alpha) = \int_{\mathcal{C}(r)} \frac{z^{s-1}e^{-\alpha z}}{1-e^{-z}}\,dz$$

where $\mathcal{C}(r)$ is a contour that runs by a straight line from $ir+\infty$ to ir, by a semicircle from ir through $-r$ to $-ir$, and then by a straight line from $-ir$ to $-ir+\infty$. Note that the value of $I(s,\alpha)$ is independent of r for $0 < r < 2\pi$. By letting $r \to 0$ show that $I(s,\alpha) = \big(e^{2\pi i s} - 1\big)\zeta(s,\alpha)\Gamma(s)$ for $\sigma > 1$.
(c) By means of (C.6), show that

$$\zeta(s,\alpha) = \frac{\Gamma(1-s)e^{-\pi i s}}{2\pi i} I(s,\alpha)$$

for $\sigma > 1$.
(d) Show that $I(s,\alpha)$ is an entire function of s. Deduce by the above that $\zeta(s,\alpha)$ is meromorphic.
(e) Show that $I(k,\alpha) = 0$ for $k = 2,3,\ldots$.
(f) Show that $I(1,\alpha) = 2\pi i$.
(g) Deduce that $\zeta(s,\alpha)$ is analytic everywhere except for a simple pole at $s = 1$ with residue 1.
(h) Show that if k is an integer then

$$I(k,\alpha) = \oint_{|z|=1} z^{k-2}\Big(\frac{ze^{(1-\alpha)z}}{e^z - 1}\Big)\,dz.$$

(i) By Exercise B.3, deduce that if k is a nonnegative integer then

$$I(-k,\alpha) = 2\pi i B_{k+1}(1-\alpha)/(k+1)!\,.$$

(j) By Theorem B.1, deduce that if k is a positive integer then

$$\zeta(1-k,\alpha) = \frac{-B_k(\alpha)}{k}.$$

In particular, $\zeta(0,\alpha) = 1/2 - \alpha$.

23. (Lerch (1894); cf Berndt (1985)) Let α be fixed, $0 < \alpha \le 1$. (a) Show that

$$\zeta(s,\alpha) - \zeta(s) = \alpha^{-s} + \sum_{n=1}^{\infty}\big((n+\alpha)^{-s} - n^{-s}\big)$$

for $\sigma > 0$.
(b) Show that

$$(n+\alpha)^{-s} - n^{-s} + \alpha s n^{-s-1} = (s+1)\int_n^{n+\alpha} (u-n-\alpha)u^{-s-2}\,du\,.$$

(c) Deduce that

$$\zeta(s,\alpha)-\zeta(s)+\alpha s\zeta(s+1)=\alpha^{-s}+\sum_{n=1}^{\infty}\left((n+\alpha)^{-s}-n^{-s}+\alpha sn^{-s-1}\right)$$

for $\sigma>-1$, and that the series is locally uniformly convergent in this halfplane.
(d) Show that

$$\begin{aligned}\zeta'(s,\alpha)\;-\;\zeta'(s)\;+\;\alpha\zeta(s+1)\;+\;\alpha s\zeta'(s+1)\\ =-\,\alpha^{-s}\log\alpha\;+\;\sum_{n=1}^{\infty}\Big(\frac{-\log(n+\alpha)}{(n+\alpha)^s}+\frac{\log n}{n^s}+\frac{\alpha}{n^{s+1}}-\frac{\alpha s\log n}{n^{s+1}}\Big)\end{aligned}$$

for $\sigma>-1$. (Here $\zeta'(s,\alpha)$ is meant to denote $\frac{\partial}{\partial s}\zeta(s,\alpha)$.)
(e) By Corollary 1.16, or otherwise, show that $\lim_{s\to 0}\zeta(s+1)+s\zeta'(s+1)=C_0$.
(f) Deduce that

$$\zeta'(0,\alpha)-\zeta'(0)+\alpha C_0=-\log a+\sum_{n=1}^{\infty}\big(-\log(n+\alpha)+\log n+\alpha/n\big)\,.$$

By (14) and the definition (C.1) of the gamma function, conclude that

$$\zeta'(0,\alpha)=\log\frac{\Gamma(\alpha)}{\sqrt{2\pi}}\,.$$

24. (a) Let χ be a character modulo q. Show that

$$L(s,\chi)=q^{-s}\sum_{a=1}^{q}\chi(a)\zeta(s,a/q)\,.$$

(b) Show that if χ is a nonprincipal character modulo q then

$$L(0,\chi)=\frac{-1}{q}\sum_{a=1}^{q}\chi(a)a\,.$$

(c) Show that if χ is a nonprincipal character modulo q then

$$L'(0,\chi)=L(0,\chi)\log q+\sum_{a=1}^{q}\chi(a)\log\Gamma(a/q)\,.$$

25. Let $Q(x,y)=ax^2+bxy+cy^2$ where a,b,c are real numbers, and put $d=b^2-4ac$. Suppose that Q is positive-definite, which is to say that $a>0$ and $d<0$. For z with $\Re z>0$, put

$$\vartheta_Q(z)=\sum_{m,n\in\mathbb{Z}}e^{-2\pi Q(m,n)z/\sqrt{-d}}.$$

(a) Show that

$$\vartheta_Q(z)=\sum_n e^{-\pi z n^2\sqrt{-d}/(2a)}\sum_m e^{-2\pi a(m+bn/(2a))^2 z/\sqrt{-d}}.$$

(b) Apply Theorem 1 to the inner sum, take the sum over n inside, and apply Theorem 1 a second time to show that $\vartheta_Q(z)=\vartheta_Q(1/z)/z$.
(c) For $\sigma>1$ put

$$\zeta_Q(s)=\sum_{(m,n)\neq(0,0)} Q(m,n)^{-s}.$$

Show that if $\Re z\ge 0$ then

$$\begin{aligned}&\zeta_Q(s)\Gamma(s)(-d)^{s/2}(2\pi)^{-s}\\ &\quad=(-d)^{s/2}(2\pi)^{-s}\sum_{(m,n)\neq(0,0)} Q(m,n)^{-s}\Gamma\big(s,2\pi Q(m,n)z/\sqrt{-d}\big)\\ &\qquad+(-d)^{(1-s)/2}(2\pi)^{s-1}\sum_{(m,n)\neq(0,0)} Q(m,n)^{s-1}\Gamma\big(1-s,2\pi Q(m,n)/(z\sqrt{-d})\big)\\ &\qquad+\frac{z^{s-1}}{2(s-1)}-\frac{z^{-s}}{2s}.\end{aligned}$$

(d) Deduce that $\zeta_Q(s)$ is a meromorphic function whose only singularity is a simple pole at $s=1$ with residue $\pi/\sqrt{-d}$.
(e) Put $\xi_Q(s)=\zeta_Q(s)\Gamma(s)(-d)^{s/2}(2\pi)^{-s}$. Show that $\xi_Q(s)=\xi_Q(1-s)$ for all s except $s=0$, $s=1$.
(f) Show that $\zeta_Q(0)=-1/2$.
(g) Show that $\zeta_Q(-k)=0$ for all positive integers k.

26. Let K be an algebraic number field. The *Dedekind zeta function* of K is defined to be $\zeta_K(s)=\sum_{\mathfrak a} N(\mathfrak a)^{-s}$ for $\sigma>1$, where the sum is over all integral ideals in the ring $\mathcal O_K$ of algebraic integers in K. This is a natural generalization of the Riemann zeta function, and indeed $\zeta_{\mathbb Q}(s)=\zeta(s)$. Since ideals in $\mathcal O_K$ factor uniquely into prime ideals, and since $N(\mathfrak{ab})=N(\mathfrak a)N(\mathfrak b)$ for any pair $\mathfrak a,\mathfrak b$ of ideals, it follows that

$$\zeta_K(s)=\prod_{\mathfrak p}\big(1-N(\mathfrak p)^{-s}\big)^{-1}$$

for $\sigma>1$. Let d denote the discriminant of K. In the case that K is a quadratic field, by analyzing how rational primes split in K it emerges that $\zeta_K(s)=\zeta(s)L(s,\chi_d)$ where $\chi_d(n)=\left(\frac{d}{n}\right)_K$ is the Kronecker symbol. Thus the functional equations of $\zeta(s)$ and of $L(s,\chi_d)$ give a functional equation for $\zeta_K(s)$ in this case. From now on, suppose that K is a complex quadratic field, which is to say that $K=\mathbb Q\big(\sqrt d\big)$ where $d<0$ is a fundamental quadratic discriminant. Let w denote the number of units in $\mathcal O_K$, which is to say that $w=6$ if $d=-3$, $w=4$ if $d=-4$, and $w=2$ if $d<-4$. Let h be the class number of K. Then there are precisely h reduced positive definite binary quadratic forms of discriminant

d, say $Q_1, Q_2, \ldots, Q_h$. As m and n run over integral values, $(m,n) \ne (0,0)$, the values $Q_i(m,n)$ run over the the values $N(\mathfrak{a})$ for ideals $\mathfrak{a}$ in the i^{th} ideal class $\mathcal{C}_i$, each value being taken exactly w times. Thus

$$\zeta_{Q_i}(s) = w \sum_{\mathfrak{a}\in\mathcal{C}_i} N(\mathfrak{a})^{-s}$$

in the notation of the preceding exercise, and

$$\zeta_K(s) = \frac{1}{w}\sum_{i=1}^{h} \zeta_{Q_i}(s).$$

(a) For $\Re z > 0$, let

$$\vartheta_K(z) = \sum_{i=1}^{h} \vartheta_{Q_i}(z) = h + w\sum_{n=1}^{\infty} r(n) e^{-2\pi n z/\sqrt{-d}}$$

where $r(n) = r_K(n) = \sum_{k|n} \chi_d(k)$ is the number of ideals in $\mathcal{O}_K$ with norm n. Show that $\vartheta_K(z) = \vartheta_K(1/z)/z$.
(b) Show that if $\Re z \ge 0$ then

$$\begin{aligned}\zeta_K(s)\Gamma(s)(-d)^{s/2}(2\pi)^{-s} = (-d)^{s/2}(2\pi)^{-s}&\sum_{n=1}^{\infty} r(n)n^{-s}\Gamma\big(s, 2\pi n z/\sqrt{-d}\big)\\ &+ (-d)^{(1-s)/2}(2\pi)^{s-1}\sum_{n=1}^{\infty} r(n)n^{s-1}\Gamma\big(1-s, 2\pi n/(z\sqrt{-d})\big)\\ &+ \frac{hz^{s-1}}{2w(s-1)} - \frac{hz^s}{2ws}\end{aligned}$$

(c) Deduce that $\zeta_K(s)$ is a meromorphic function whose only singularity is a simple pole at $s=1$ with residue $h\pi/(w\sqrt{-d})$.
(d) Put $\xi_K(s) = \zeta_K(s)\Gamma(s)(-d)^{s/2}(2\pi)^{-s}$. Show that $\xi_K(s) = \xi_K(1-s)$ for all s except $s=1$ and $s=0$.
(e) Show that $\zeta_K(0) = -h/(2w)$.
(f) Show that $\zeta_K(-k) = 0$ for all positive integers k.
(g) Show that $r(n^2) \ge 1$ for all positive integers n.
(h) Show that if $L(1/2,\chi) \ge 0$ then $h \gg (-d)^{1/4}\log(-d)$.

27. Let α be an arbitrary complex number and z a complex number with $\Re z > 0$. Let $f(u) = e^{-\pi(u+\alpha)^2 z}$. Show that $\widehat{f}(t) = z^{-1/2}e^{2\pi i t\alpha}e^{-\pi t^2/z}$. Deduce that the identities of Theorem 1 hold for arbitrary complex α.

28. Grössencharaktere for $\mathbb{Q}(\sqrt{-1})$, continued from Exercises 4.2.7 and 4.3.8. (a) By two applications of the preceding exercise, show that if z and w are complex numbers with $\Re z > 0$ then

$$\sum_{a,b\in\mathbb{Z}} e^{-\pi(a^2+b^2)}e^{2\pi i(a+ib)w} = \frac{1}{z}\sum_{c,d\in\mathbb{Z}} e^{-\pi(c^2+d^2)/z}e^{2\pi i(c+id)w/z}.$$

(b) Differentiate both sides of the above m times with respect to w, and then set $w = 0$, to show that

$$\sum_{a,b} e^{-\pi(a^2+b^2)z}(a+ib)^m = \frac{1}{z^{m+1}} \sum_{c,d} e^{-\pi(c^2+d^2)/z}(c+id)^m.$$

(c) Explain why the above reduces to $0 = 0$ if $4 \nmid m$.
(d) Let χ_m and $L(s, \chi_m)$ be defined as before. Show that if m is a positive integer and $\Re z \geq 0$, then

$$\begin{aligned} L(s,\chi_m)\Gamma(s+2m)\pi^{-s} = \frac{\pi^{-s}}{4} \sum_{(a,b)\neq(0,0)} \frac{\chi_m(a+ib)}{(a^2+b^2)^s}\Gamma\big(s+2m, \pi(a^2+b^2)z\big) \\ + \frac{\pi^{s-1}}{4} \sum_{(a,b)\neq(0,0)} \frac{\chi_m(a+ib)}{(a^2+b^2)^{1-s}}\Gamma\big(1-s+2m, \pi(a^2+b^2)/z\big). \end{aligned}$$

(e) Deduce that $L(s, \chi_m)$ is an entire function when m is a nonzero integer.
(f) For each positive integer m, put $\xi(s, \chi_m) = L(s, \chi_m)\Gamma(s + 2m)\pi^{-s}$. Show that $\xi(s, \chi_m) = \xi(1 - s, \chi_m)$ for all s.
(g) Show that if m is a positive integer then $L(s, \chi_m)$ has simple zeros at $-2m, -2m - 1, -2m - 2, \ldots$, but no other zeros in the half-plane $\sigma < 0$.
(h) Show that $\xi(\sigma, \chi_m)$ is real for all real σ, and that $\xi(1/2 + it, \chi_m)$ is real for all real t.

2. Products and sums over zeros

If $P(z)$ is a polynomial then we may express $P(z)$ as a product over its zeros z_i,

$$P(z) = c(z - z_1)(z - z_2)\cdots(z - z_n).$$

The question arises whether a more general entire function may be similarly represented as a product over its zeros, say

$$f(z) = c\prod_n \Big(1 - \frac{z}{z_n}\Big). \tag{21}$$

This is an issue that was addressed by Weierstrass and Hadamard. Rather than derive their extensive theory, we establish only a simple part of it that suffices for our purposes. We do not quite achieve a formula of the type (21) for the zeta function, but we obtain a serviceable substitute.

Lemma 11. *Suppose that $f(z)$ is an entire function with a zero of order K at 0, and that $f(z)$ vanishes at the nonzero numbers $z_1, z_2, z_3, \ldots$. Suppose also that there is a constant θ, $1 < \theta < 2$, such that*

$$\max_{|z|\leq R} |f(z)| \leq \exp(R^\theta)$$

for all sufficiently large R. Then there exist numbers $A = A(f)$ and $B = B(f)$, such that

$$f(z) = z^K e^{A+Bz} \prod_{k=1}^{\infty} \Big(1 - \frac{z}{z_k}\Big) e^{z/z_k}$$

for all z. Here the product is uniformly convergent for z in compact sets.

Proof. We may suppose that $K = 0$, since if $K > 0$ then the function $f(z)/z^K$ does not vanish at the origin. Let $N_f(R)$ denote the number of zeros of $f(z)$ in the disk $|z| \le R$. By Jensen's inequality (Lemma 6.1) we find that $N_f(R) \le 8R^\theta$ for all sufficiently large R. Thus $\sum_{R<|z_k|\le 2R} |z_k|^{-2} \le 8R^{\theta-2}$, so by summing over dyadic blocks we see that $\sum_{k=1}^\infty |z_k|^{-2} < \infty$. (Alternatively, if more precision were desired, we could write this sum as $\int_0^\infty r^{-2}\, dN_f(r)$, and integrate by parts.) But $(1-z)e^z = 1 + O(|z|^2)$ uniformly for $|z| \le 1$, so the product

$$g(z) = \prod_{k=1}^{\infty} \Big(1 - \frac{z}{z_k}\Big) e^{z/z_k}$$

is uniformly convergent in compact regions, and hence represents an entire function. Thus $h(z) = f(z)/(f(0)g(z))$ is a non-vanishing entire function with $h(0) = 1$.

Next we derive an upper bound for $M_h(R)$. To this end we write the product above in three parts,

$$g(z) = \prod_{k\in\mathcal{K}_1} \prod_{k\in\mathcal{K}_2} \prod_{k\in\mathcal{K}_3} = P_1(z)P_2(z)P_3(z),$$

where $|z_k| \le R/2$ for $k \in \mathcal{K}_1$, $R/2 < |z_k| \le 3R$ for $k \in \mathcal{K}_2$, and $|z_k| > 3R$ for $k \in \mathcal{K}_3$. Suppose that $R \le |z| \le 2R$. If $|z_k| \le R/2$ then $|1 - z/z_k| \ge |z/z_k| - 1 \ge 1$, and hence

$$|P_1(z)| \ge \prod_{k\in\mathcal{K}_1} e^{-2R/|z_k|}.$$

Now

$$\sum_{k\in\mathcal{K}_1} \frac{1}{|z_k|} \ll R^{\theta-1}.$$

Thus

$$|P_1(z)| \ge e^{-cR^\theta}$$

for all large R. Since $\operatorname{card} K_2 \le 72R^\theta$, it follows that there is an r, $R \le r \le 2R$, for which $|r - |z_k|| \ge 1/R^2$ for all k. If r is chosen in this way and $|z| = r$ then

$$|1 - z/z_k| \ge \frac{|r - |z_k||}{|z_k|} \ge \frac{1}{27R^3}$$

for all $k \in \mathcal{K}_2$. Hence

$$|P_2(z)| \ge e^{-cR^\theta \log R}$$

when $|z| = r$. Finally,

$$|P_3(z)| \ge \prod_{k\in\mathcal{K}_3} e^{-cR^2/|z_k|^2} \ge e^{-cR^\theta}$$

for $|z| \le 2R$. Hence we see that for each large R there is an r, $R \le r \le 2R$, for which $|g(z)| \ge e^{-cR^\theta \log R}$ when $|z| = r$. Thus $|h(z)| \le e^{cR^\theta \log R}$ for such z, and hence by the maximum modulus principle

$$M_h(R) \le e^{cR^\theta \log R}.$$

Now put $j(z) = \log h(z)$ with $j(0) = 0$. Then $\Re j(z) \le cR^\theta \log R$ for all large R, so that by the Borel–Carathéodory Lemma (Lemma 6.2),

$$j(z) \ll R^\theta \log R$$

for all large R. But $\theta < 2$, so $j(z)$ must be a polynomial of degree at most 1, say $j(z) = A + Bz$, and the proof is complete.

In order to apply our lemma to $\xi(s)$ we need an upper bound for $|\xi(s)|$. From Corollary 1.17 we see that $\zeta(s) \ll |s|^{1/2}$ when $\sigma \ge 1/2$ and $|s| \ge 2$. Thus by Stirling's formula (Theorem C.1) it follows that

$$\xi(s) \ll \exp(|s| \log |s|) \tag{22}$$

when $\sigma \ge 1/2$ and $|s| \ge 2$. In view of the functional equation found in Corollary 3, this same upper bound therefore holds for all s with $|s| \ge 2$. Since

$$\xi(s) = (s-1)\zeta(s)\Gamma(1 + s/2)\pi^{-s/2}, \tag{23}$$

it follows from (11) that $\xi(0) = 1/2$. Thus by Lemma 11 we obtain

Theorem 12. *Let $\xi(s)$ be defined as in Corollary 3. There is a constant B such that*

$$\xi(s) = \frac{1}{2} e^{Bs} \prod_\rho \Big(1 - \frac{s}{\rho}\Big) e^{s/\rho} \tag{24}$$

for all s. Here the product is extended over all zeros ρ of $\xi(s)$.

All known zeros of the zeta function are simple, and it is plausible to conjecture that they all are. In the (unlikely) event that a multiple zero is encountered, the associated factor in the above product is to be repeated as many times as the multiplicity.

Thus far we have remarked upon the zeros of $\xi(s)$ without having proved that they exist. However, from (24) we see that if $\xi(s)$ had at most finitely many zeros then there would be a constant $C > 0$ such that $\xi(s) \ll \exp(C|s|)$ for all large s. On the contrary, by Stirling's formula we find that $\xi(\sigma) = \exp\big(\frac{1}{2}\sigma \log \sigma + O(\sigma)\big)$ as $\sigma \to \infty$, so it is evident that $\xi(s)$ has infinitely many zeros. Concerning the density of the zeros, the following estimate is useful.

Theorem 13. *For $T \ge 0$, let $N(T)$ denote the number of zeros $\rho = \beta + i\gamma$ of $\xi(s)$ in the rectangle $0 < \beta < 1$, $0 < \gamma \le T$. Any zeros with $\gamma = T$ should be counted with weight $1/2$. Then*

$$N(T+1) - N(T) \ll \log(T + 2).$$

Proof. We apply Jensen's inequality (Lemma 6.1) to $\xi(s)$, on a disk with centre $2+i(T+1/2)$ and radius $R=11/6$. By taking $r=7/4$, it follows from the estimates of Corollary 1.17 that the number of zeros ρ in the rectangle $1/2\le\beta\le 1$, $T\le\gamma\le T+1$ is $\ll\log(T+2)$. (Alternatively, we could appeal to Theorem 6.8.) But ρ is a zero if and only if $1-\overline{\rho}$ is a zero, so the rectangle $0\le\beta\le 1/2$, $T\le\gamma\le T+1$ contains the same number of zeros as the former one. Thus we have the result.

By summing the above over integral values of T, we deduce that $N(T)\ll T\log T$. Alternatively, this same upper bound follows from (22) by means of Jensen's inequality. Hence $\sum_\rho|\rho|^{-A}<\infty$ for all $A>1$. With a little more work we could show that $\sum 1/|\rho|=\infty$ (see Exercise 1), and indeed that $N(T)\asymp T\log T$ for all large T (see Exercise 4). A much more precise asymptotic formula for $N(T)$ will be derived in Chapter 12.

We recall that the *logarithmic derivative* of a function $f(z)$ is defined to be $f'(z)/f(z)$. Since $f'(z)/f(z)=\frac{d}{dz}\log f(z)$, it follows that the logarithmic derivative of a product is the sum of the logarithmic derivatives of the factors. Although $\log f(z)$ is multiple-valued, the ambiguity involves only an additive constant, so $f'(z)/f(z)$ is a well-defined single-valued analytic function wherever $f(z)$ is analytic and nonzero. If f has a zero at a of multiplicity m then f'/f has a simple pole at a with residue m. If f has a pole at a of multiplicity m then f'/f has a simple pole at a with residue $-m$. Hence if f is meromorphic then f'/f is meromorphic with only simple poles, which occur at the zeros and poles of f.

By taking logarithmic derivatives in the definition (5) of $\xi(s)$ we find that

$$\frac{\xi'}{\xi}(s)=\frac{1}{s}+\frac{1}{s-1}+\frac{\zeta'}{\zeta}(s)+\frac{1}{2}\frac{\Gamma'}{\Gamma}(s/2)-\frac{1}{2}\log\pi. \tag{25}$$

By taking logarithmic derivatives in the functional equation of Corollary 3 we see that

$$\frac{\xi'}{\xi}(s)=-\frac{\xi'}{\xi}(1-s). \tag{26}$$

By logarithmically differentiating the asymmetric form (9) of the functional equation, we discover that

$$\frac{\zeta'}{\zeta}(s)\;=\;-\frac{\zeta'}{\zeta}(1-s)\;+\;\log 2\pi\;-\;\frac{\Gamma'}{\Gamma}(1-s)\;+\;\frac{\pi}{2}\cot\frac{\pi s}{2}\,. \tag{27}$$

By taking logarithmic derivatives of both sides of the identity (24) we obtain

Corollary 14. *Let B be defined as in Theorem* 12. *Then*

$$\frac{\xi'}{\xi}(s)=B+\sum_\rho\Big(\frac{1}{s-\rho}+\frac{1}{\rho}\Big) \tag{28}$$

and

$$\frac{\zeta'}{\zeta}(s)=B+\frac{1}{2}\log\pi-\frac{1}{s-1}-\frac{1}{2}\frac{\Gamma'}{\Gamma}(s/2+1)+\sum_\rho\Big(\frac{1}{s-\rho}+\frac{1}{\rho}\Big). \tag{29}$$

Moreover,

$$(30) \quad B = -\frac{1}{2}\sum_\rho \Big(\frac{1}{1-\rho} + \frac{1}{\rho}\Big) = -\sum_\rho \Re\frac{1}{\rho} = \frac{-1}{2}C_0 - 1 + \frac{1}{2}\log 4\pi = -0.0230957\ldots.$$

In the above, it is to be understood that if $\xi(s)$ has a multiple zero ρ, then the summand arising from ρ is to be repeated as many times as the multiplicity.

Proof. The second identity follows from the first by means of (25). As for (30), we observe first by taking $s = 0$ in (28) that $B = \frac{\xi'}{\xi}(0)$. Also, by taking $s = 1$ in (28) we find that $\frac{\xi'}{\xi}(1) = B + \sum_\rho(1/(1-\rho) + 1/\rho)$. By (26), this is $-B$, so we obtain the first identity in (30). Since B is real, we may write

$$B = -\frac{1}{2}\sum_\rho \Big(\Re\frac{1}{1-\rho} + \Re\frac{1}{\rho}\Big).$$

However, $\sum_\rho \Re 1/(1-\rho)$ and $\sum_\rho \Re 1/\rho$ are absolutely convergent, so these two sums may be written separately, above. Since $1-\rho$ runs over zeros of the zeta function as ρ does, the two sums are equal, and we obtain the second identity in (30). By logarithmically differentiating the fundamental identity $s\Gamma(s) = \Gamma(s+1)$ we see that $1/s + \frac{\Gamma'}{\Gamma}(s) = \frac{\Gamma'}{\Gamma}(s+1)$. Hence (25) may be rewritten as

$$\frac{\xi'}{\xi}(s) = \frac{1}{s-1} + \frac{\zeta'}{\zeta}(s) + \frac{1}{2}\frac{\Gamma'}{\Gamma}(s/2+1) - \frac{1}{2}\log\pi.$$

We obtain the third identity in (30) by taking $s = 0$ in the above, in view of (11), (14), and (C.12).

In order to extend our theory to include L-functions, we need an upper bound for $|L(s,\chi)|$ that corresponds to the bound for the zeta function provided by Corollary 1.17.

Lemma 15. *Let χ be a nonprincipal character modulo q, and suppose that $\delta > 0$ is fixed. Then*

$$L(s,\chi) \ll \big(1 + (q\tau)^{1-\sigma}\big)\min\Big(\frac{1}{|\sigma-1|}, \log q\tau\Big)$$

uniformly for $\delta \le \sigma \le 2$.

Landau noted that an estimate relating to the zeta function often has a 'q-analogue' in which n^{-it} is replaced by $\chi(n)$ and τ is replaced by q. In the above we have a 'hybrid' of the two, with $\chi(n)n^{-it}$ and $q\tau$ throughout.

Proof. Let $S(u,\chi) = \sum_{0<n\le u}\chi(n)$. Then for $\sigma > 0$,

$$\begin{aligned} L(s,\chi) &= \sum_{n\le x}\chi(n)n^{-s} + \int_x^\infty u^{-s}\,dS(u,\chi) \\ &= \sum_{n\le x}\chi(n)n^{-s} + S(u,\chi)u^{-s}\Big|_x^\infty - \int_x^\infty S(u,\chi)\,du^{-s} \\ &= \sum_{n\le x}\chi(n)n^{-s} - S(x,\chi)x^{-s} + s\int_x^\infty S(u,\chi)u^{-s-1}\,du. \end{aligned}$$

This is analogous to Theorem 1.12. To estimate the sum we use (1.29). For the remaining terms we use the trivial estimate $S(u,\chi) \ll q$. The stated estimate then follows by taking $x = q\tau$. bv

Now suppose that χ is a primitive character modulo q, $q > 1$. By Stirling's formula we see that $\xi(s,\chi) \ll q^{1/2+\sigma}\exp(|s|\log|s|)$ when $\sigma \ge 1/2$ and $|s| \ge 2$. By the functional equation of Corollary 8, it follows that

$$\xi(s,\chi) \ll \exp(|s|\log q|s|) \tag{31}$$

for all s with $|s| \ge 2$. Hence by Lemma 11 we obtain

Theorem 16. *Let χ be a primitive character modulo q, $q > 1$, and let $\xi(s,\chi)$ be defined as in Corollary 8. There is a constant $B(\chi)$ such that*

$$\xi(s,\chi) = \xi(0,\chi)e^{B(\chi)s}\prod_\rho \Big(1-\frac{s}{\rho}\Big)e^{s/\rho} \tag{32}$$

for all s. Here the product is extended over all zeros ρ of $\xi(s,\chi)$.

We expect that the zeros of $\xi(s,\chi)$ are all simple, but if a multiple zero is encountered, then the factor that it contributes to the above product is to be repeated as many times as its multiplicity. In analogy to Theorem 13, we have

Theorem 17. *Let χ be a character modulo q. The number of zeros $\rho = \beta + i\gamma$ of $L(s,\chi)$ in the rectangle $0 \le \beta \le 1$, $T \le \gamma \le T+1$ is $\ll \log q(|T|+2)$.*

Proof. First suppose that χ is primitive. We apply Jensen's inequality (Lemma 6.1) to $L(s,\chi)$, on a disk with centre $2 + i(T+1/2)$ and radius $R = 11/6$. By taking $r = 7/4$, it follows from the estimates of Lemma 15 that the number of zeros ρ in the rectangle $1/2 \le \beta \le 1$, $T \le \gamma \le T+1$ is $\ll \log q(T+2)$. But $L(\rho,\chi) = 0$ if and only if $L(1-\overline{\rho},\chi) = 0$ (except possibly for a trivial zero at $s = 0$ if $\chi(-1) = 1$), so the rectangle $0 \le \beta \le 1/2$, $T \le \gamma \le T+1$ contains the same number of zeros as (or at most one more than) the former one. Thus we have the result when χ is primitive.

Suppose now that χ is induced by a primitive character $\chi^\star$ modulo r, with $r|q$. Then

$$L(s,\chi) = L(s,\chi^\star)\prod_{\substack{p|q\\ p\nmid r}}\Big(1-\frac{\chi^\star(p)}{p^s}\Big).$$

Here each factor in the product has zeros forming an arithmetic progression on the imaginary axis with common difference $2\pi i/\log p$. Thus $L(s,\chi)$ has the $\ll \log r(|T|+2)$ zeros of $L(s,\chi^\star)$, and additionally has $\ll \sum_{p|q}\log p \ll \log q$ zeros on the imaginary axis with imaginary part between T and $T+1$. This completes the argument.

Suppose that χ is a primitive character modulo q. By taking logarithmic derivatives in the definition (18) of $\xi(s,\chi)$, we see that

$$\frac{\xi'}{\xi}(s,\chi) \;=\; \frac{L'}{L}(s,\chi) \;+\; \frac{1}{2}\frac{\Gamma'}{\Gamma}((s+\kappa)/2) \;+\; \frac{1}{2}\log q/\pi. \tag{33}$$

By taking logarithmic derivatives in the functional equation of Corollary 8 we see that

$$\frac{\xi'}{\xi}(s,\chi) \;=\; -\frac{\xi'}{\xi}(1-s,\overline{\chi}). \tag{34}$$

By logarithmically differentiating the asymmetric form of the functional equation found in Corollary 9, we discover that

$$\frac{L'}{L}(s,\chi) \;=\; -\frac{L'}{L}(1-s,\overline{\chi}) \;-\; \log\frac{q}{2\pi} \;-\; \frac{\Gamma'}{\Gamma}(1-s) \;+\; \frac{\pi}{2}\cot\frac{\pi}{2}(s+\kappa) \tag{35}$$

By taking logarithmic derivatives of both sides of the identity (31) we obtain

Corollary 18. *Let χ be a primitive character modulo q, $q>1$, and let $B(\chi)$ be defined as in Theorem* 16*. Then*

$$\frac{\xi'}{\xi}(s,\chi) \;=\; B(\chi) \;+\; \sum_\rho\Big(\frac{1}{s-\rho}+\frac{1}{\rho}\Big) \tag{36}$$

and

$$\frac{L'}{L}(s,\chi) \;=\; B(\chi) - \frac{1}{2}\frac{\Gamma'}{\Gamma}((s+\kappa)/2) \;-\; \frac{1}{2}\log q/\pi \;+\; \sum_\rho\Big(\frac{1}{s-\rho} \;+\; \frac{1}{\rho}\Big). \tag{37}$$

Moreover,

$$\Re B(\chi) \;=\; -\frac{1}{2}\sum_\rho\Big(\frac{1}{1-\rho}+\frac{1}{\rho}\Big) \;=\; -\sum_\rho \Re\frac{1}{\rho} \tag{38}$$

and

$$B(\chi) = \frac{-1}{2}\log\frac{q}{\pi} - \frac{L'}{L}(1,\overline{\chi}) + \frac{1}{2}C_0 + (1-\kappa)\log 2\,. \tag{39}$$

As always, multiple zeros are counted multiply.

Proof. The second identity follows from the first by means of (33). To obtain the first identity in (38), we take $s=1$ in (36), and apply (34) to see that

$$B(\chi) + \sum_\rho\Big(\frac{1}{1-\rho}+\frac{1}{\rho}\Big) = \frac{\xi'}{\xi}(1,\chi) = -\frac{\xi'}{\xi}(0,\overline{\chi}) = -B(\overline{\chi}) = -\overline{B(\chi)}.$$

From Theorem 17 we know that the number of zeros ρ of $\xi(s,\chi)$ with $|\rho|\le R$ is $\ll R\log qR$ for $R\ge 2$. Hence the sums $\sum_\rho \Re 1/(1-\rho)$ and $\sum_\rho \Re 1/\rho$ are absolutely convergent. As the map $\rho\mapsto 1-\overline{\rho}$ merely permutes zeros of $\xi(s,\chi)$, the first of these two sums is unchanged if we replace ρ by $1-\overline{\rho}$. Hence the two sums are equal, and we obtain the second part of (38).

To derive (39) we first take $s=0$ in (36) to see that $B(\chi)=\frac{\xi'}{\xi}(0,\chi)$. By (34) it follows that $B(\chi) = -\frac{\xi'}{\xi}(1,\overline{\chi})$. The stated identity now follows by taking $s=1$ in (33), in view of (C.11) and (C.14).

10.2. Exercises

1. Let f satisfy the hypotheses of Lemma 11, and suppose that

$$\sum_{k=1}^{\infty} \frac{1}{|z_k|} < \infty.$$

(a) Show that there are numbers A and B and a non-negative integer K such that $f(z) = z^K e^{A+Bz} g(z)$ where $g(z) = \prod_{k=1}^{\infty}(1 - z/z_k)$.
(b) Observe that for any complex number w, $|1-w| \le e^{|w|}$ and show that there is a number C such that $|g(z)| \le e^{C|z|}$.
(c) Deduce that $\sum_\rho 1/|\rho| = \infty$ where the sum is over all nontrivial zeros of the zeta function.

2. (a) Let B be the constant given in (30). Show that if $\rho = 1/2 + i\gamma$ is a zero of the zeta function on the critical line, then

$$|\gamma| \ge (-1/B - 1/4)^{1/2} = 6.5611\dots.$$

(b) Let γ be given, and put $f(\beta) = \beta/(\beta^2 + \gamma^2)$. Show that if $0 \le \beta \le 1$ then $f(\beta) \ge \beta/(1+\gamma^2)$. Deduce that if $0 \le \beta \le 1$ then $f(\beta) + f(1-\beta) \ge f(0) + f(1)$.
(c) Show that if $\rho = \beta + i\gamma$ is a nontrivial zero of the zeta function with $\beta \ne 1/2$ then

$$|\gamma| \ge (-2/B - 1)^{1/2} = 9.2518\dots.$$

3. (Landau (1903)) Show that

$$\limsup_{m\to\infty} \Big(\frac{1}{m!}\Big|\sum_{n=1}^{\infty} \frac{\mu(n)(\log n)^m}{n}\Big|\Big)^{1/m} = \frac{1}{3}.$$

4. (a) Show that

$$\sum_\rho \Re\frac{1}{\sigma-\rho} = \frac{1}{2}\log\sigma + O(1)$$

for $\sigma \ge 2$, where the sum is over all nontrivial zeros of the zeta function.
(b) Deduce that

$$\sum_\rho \Big(\Re\frac{1}{\sigma-\rho} - \frac{3}{4}\Re\frac{1}{2\sigma-\rho}\Big) = \frac{1}{8}\log\sigma + O(1)$$

for $\sigma \ge 2$.
(c) Show that each summand above is $\le 1/(\sigma-1)$.
(d) Show that if $|\gamma| \ge 3\sigma$ and σ is large, then the summand arising from ρ in the sum above is ≤ 0.
(e) Conclude that $N(T) \gg T\log T$ when T is large.

5. Put $f(s) = \Re\big(\frac{1}{s+1} - \frac{3/4}{s+2}\big)$.
(a) Show that if $t \ge 2$ then

$$\sum_\rho f(1 + it - \rho) = \frac{1}{8}\log t + O(1)$$

where the sum is over all nontrivial zeros ρ of $\zeta(s)$.
(b) Show that $f(s) \le 1$ when $\sigma \ge 0$.
(c) Show that if $0 \le \sigma < 2$ then $f(s) \le 0$ when

$$t^2 \ge \frac{(\sigma+1)(\sigma+2)(\sigma+5)}{2-\sigma}.$$

(d) Deduce that $f(s) \le 0$ if $0 < \sigma < 1$ and $|t| \ge 6$.
(e) Show that $N(T+6) - N(T-6) \gg \log T$ for all $T > T_0$.

6. (a) Show that for s near 1 the Laurent expansion of $\frac{\zeta'}{\zeta}(s)$ begins

$$\frac{\zeta'}{\zeta}(s) = \frac{-1}{s-1} - C_0 + \cdots .$$

(b) Deduce that

$$\frac{\zeta'}{\zeta}(1-s) = \frac{1}{s} - C_0 + O(|s|)$$

for s near 0.
(c) Show that $\frac{\Gamma'}{\Gamma}(1) = -C_0$.
(d) Show that

$$\frac{\pi}{2}\cot\frac{\pi s}{2} = \frac{1}{s} + O(|s|)$$

for s near 0.
(e) Deduce by (27) that $\frac{\zeta'}{\zeta}(0) = \log 2\pi$.
(f) Use this to give a second proof that $\zeta'(0) = -\frac{1}{2}\log 2\pi$.

7. (Taylor (1945)) (a) Show that if $\sigma > 1/2$, then $|\xi(s+1/2)| > |\xi(s-1/2)|$.
(b) Put $f(s) = \xi(s+1/2) + \xi(s-1/2)$. Show that all zeros of $f(s)$ have real part 1/2.
(c) Assume RH. Show that if c is fixed, $c > 0$, then all zeros of $\xi(s+c) + \xi(s-c)$ have real part 1/2.

8. (Vorhauer (2005)) Let $B(\chi)$ denote the constant in Theorem 16. (a) Show that

$$\frac{1-\beta}{(1-\beta)^2+\gamma^2} + \frac{\beta}{\beta^2+\gamma^2} \ge \frac{1}{1+\gamma^2}$$

uniformly for $0 \le \beta \le 1$.
(b) Deduce that

$$\Re B(\chi) \le -\frac{1}{2}\sum_\gamma \frac{1}{1+\gamma^2} .$$

(c) Show that

$$\frac{\xi'}{\xi}(2,\chi) = \frac{1}{2}\log q + O(1)\,.$$

(d) Show that

$$\Re\frac{\xi'}{\xi}(2,\chi) = \sum_\rho \Re\frac{1}{2-\rho}\,.$$

(e) Show that

$$\Re\frac{\xi'}{\xi}(2,\chi) = \frac{1}{2}\sum_\rho \Re\Big(\frac{1}{2-\rho} + \frac{1}{1+\overline{\rho}}\Big)\,.$$

(f) Show that

$$\frac{2-\beta}{(2-\beta)^2+\gamma^2} + \frac{1+\beta}{(1+\beta)^2+\gamma^2} \le \frac{3}{1+\gamma^2}$$

uniformly for $0 \le \beta \le 1$.
(g) Conclude that

$$\Re B(\chi) \le \frac{-1}{6}\log q + O(1)\,.$$

9. Let $K > 0$ be given, and put $E(z) = (1-z)\exp\big(\sum_{k=1}^K z^k/k\big)$.
(a) Show that

$$E'(z) = -z^K \exp\Big(\sum_{k=1}^K \frac{z^k}{k}\Big).$$

(b) Deduce that the power series coefficients of $E'(z)$ are all ≤ 0.
(c) Write $E(z) = \sum_{m=0}^\infty A_m z^m$. Show that $A_0 = 1$, $A_m = 0$ for $1 \le m \le K$, $A_m < 0$ for $m > K$, and that $\sum_{m>K} A_m = -1$.
(d) Show that if $|z| \le r \le 1$ then $|1 - E(z)| \le 1 - E(r) \le r^{K+1}$.

10. Notes

§1. The case $\alpha = 0$ of (1) was given by Poisson (1823). de la Vallée Poussin observed that the left hand side of (1) has period 1 with respect to α, and then computed the Fourier coefficients of this function to obtain (1). This is rather similar to using the Poisson summation formula, as we have done. Theorem 1 is the basis for a very large class of functional equations and was first exploited systematically by Hecke. For the most general version see Tate's thesis, reproduced in Tate (1967). Riemann gave two proofs of Corollary 3. Riemann's second method involved using Theorem 1 to establish the formula of Exercise 10. This is the case $z = 1$ of Theorem 2, with the order of summation and integration reversed. Theorem 2 is due to Lavrik (1965), who derived it from Corollary 3 in the manner outlined in Exercise 4. For further proofs of the functional equation, see Titchmarsh (1986, Chapter 2).

The proof of Theorem 1 can be arranged so that one does not depend on the fact that $\int e^{-\pi x^2}\,dx = 1$. To see this, let c denote the value of this integral. Then the proof given establishes (1) with the factor c on the right hand side. But if $z = 1$ and $\alpha = 0$ the two sides of (1) are visibly equal and positive, so it follows that $c = 1$

The functional equation for $\zeta(s)$ was established by Riemann (1860), and that for $L(s, \chi)$ by de la Vallée Poussin (1896) although it was known in some special cases earlier. See the commentary of Landau (1909, p. 899).

§2 The product formula of Theorem 12 was established by Hadamard (1893). The constant $B(\chi)$ in Theorem 16 was long considered to be mysterious; the simple formula (39) for it is due to Vorhauer (2005).

10. Literature

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