MATH 504 ANALYSIS IN EUCLIDEAN SPACES, SPRING TERM 2009, PROBLEMS 12

Return by Wednesday 27th April

1. Prove the following identities.

(i)
$$\Big(\frac{\sin \pi x}{\pi x}\Big)^2 = \int_{-1}^{1} (1-|t|)e(tx)\,dt;$$

(ii)
$$\frac{(\sin \pi x)^2}{x} = \int_0^1 \frac{\pi \operatorname{sgn}(x)}{2i} e(tx)\,dt;$$

(iii)
$$\sum_{n=-N}^{N} \operatorname{sgn}(n)e(-nt) = -i\cot \pi t + i\frac{\cos \pi(2N+1)t}{\sin \pi t};$$

(iv)
$$\operatorname{sgn}(x) = \frac{2}{\pi}\int_0^\infty \frac{1}{t}\sin 2\pi tx\,dt.$$

2. Let Vaaler's function $V(z)$ be defined by

$$V(z) = \Big(\frac{\sin \pi z}{\pi}\Big)^2\Big(\frac{2}{z} + \sum_{n=-\infty}^{\infty} \frac{\operatorname{sty}(n)}{(z-n)^2}\Big), \quad \text{and put} \quad V_N(z) = \Big(\frac{\sin \pi z}{\pi}\Big)^2\Big(\frac{2}{z} + \sum_{-N}^{N} \frac{\operatorname{sty}(n)}{(z-n)^2}\Big).$$

(i) Using the identities in Exercise 1, or otherwise, show that

$$V_N(x) = 2\int_0^1 \Big((1-t)\cot \pi t + \frac{1}{\pi}\Big)\sin 2\pi tx\,dt - 2\int_0^1 \frac{\cos \pi(2N+1)t}{\sin \pi t}(1-t)\sin 2\pi tx\,dt.$$

(ii) By using the Riemann–Lebesgue lemma, show that

$$V(x) = 2\int_0^1 \big((1-t)\cot \pi t + \tfrac{1}{\pi}\big)\sin 2\pi tx\,dt.$$

(iii) Let

$$\varphi(t) = \begin{cases} 1 & \text{if } t = 0, \\ \pi(1-|t|)t\cot \pi t + |t| & \text{if } 0 < |t| \le 1, \\ 0 & \text{if } |t| > 1. \end{cases}$$

Show that

$$V'(x) = 2\int_{-1}^{1} \varphi(t)e(xt)\,dt.$$

(iv) Show that $\varphi(t)$ is non-negative, continuously differentiable on $\mathbb{R}$ and that is is strictly decreasing on $[0,1]$.
(v) Show that $V(z)$ is an odd entire function, and that

$$V(z) = 1 - 6\Big(\frac{\sin \pi z}{\pi}\Big)^2\int_0^\infty \frac{\{u\}(1-\{u\})}{(z+u)^4}\,du$$

provided that $z \notin (-\infty, 0]$.
(vi) Show that $V(n) = \operatorname{sgn}(n)$ for all integers n, that $V'(n) = 0$ for all integers $n \ne 0$, that $V'(0) = 2$, and that $0 \le V(x) \le 1$ for $x > 0$.
(vii) Show that if $x > 0$ then

$$V(x) - 1 \ll \min(1, x^{-3}), \quad \text{and} \quad V'(x) \ll \min(1, x^{-3}).$$

(viii) Show that all zeros of $V'(x)$ lie on the real axis.
(ix) Show that

$$V(x) - \operatorname{sgn}(x) = \int_{-\infty}^{\infty} \frac{\varphi(t)-1}{\pi i t} e(tx)\,dt.$$