MATH 504 ANALYSIS IN EUCLIDEAN SPACES, SPRING TERM 2009, PROBLEMS 8

Return by Monday 23rd March

1. (Exercise 2.6.3) Let $L^1(\mathbb{R}^+, x^{-1}dx)$ denote the set of functions f defined on the positive real numbers $\mathbb{R}^+$ such that

$$\int_0^\infty |f(x)|x^{-1}dx < \infty.$$

(i) Show that the convolution product

$$(f \circ g)(x) = \int_0^\infty f(x/y)g(y)y^{-1}dy$$

induces an algebra on $L^1(\mathbb{R}^+, x^{-1}dx)$.
(ii) Define the "Mellin transform" formally by

$$f^\sharp(t) = \int_0^\infty f(x)x^{-2\pi it}x^{-1}dx.$$

Prove that the map

$$T : f(\cdot) \to f(e^{\cdot})$$

establishes a $1:1$, linear, product–preserving transformation from $L^1(\mathbb{R}^+, x^{-1}dx)$ onto the convolution algebra on $L^1(\mathbb{R})$ and that

$$f^\sharp = \widehat{(Tf)}.$$

2. (Exercise 2.6.8) Let $f \in L^1(\mathbb{R})$ and $x \in \mathbb{R}$.
(i) Prove that

$$\int_a^b \hat{f}(t)e(xt)dt = (f \circ D)(x)$$

where

$$D(u) = \frac{e(bx) - e(ax)}{2\pi ix}.$$

(ii) Prove that if

$$\int_{-1}^1 |f(x+y) - f(y)|\frac{dy}{|y|} < \infty,$$

then

$$f(x) = \lim_{a\to-\infty} \int_a^0 \hat{f}(t)e(xt)dt + \lim_{b\to\infty} \int_0^b \hat{f}(t)e(xt)dt.$$