MATH 504 ANALYSIS IN EUCLIDEAN SPACES, SPRING TERM 2009, PROBLEMS 6

Return by Monday 23rd February

1. Suppose that $\{u_n\}$ is uniformly distributed (mod 1), and let c be a real number. Put $v_n = u_n + c$. Show that $\{v_n\}$ is uniformly distributed.

2. Let $\alpha_n = \log n - \lfloor \log n \rfloor$
(a) Show that

$$\limsup_{N\to\infty} \frac{1}{N}\mathrm{card}\{n : 1 \le n \le N, \alpha_n \in [0, 1/2]\} = \frac{e - e^{1/2}}{e-1}.$$

(b) Show that

$$\liminf_{N\to\infty} \frac{1}{N}\mathrm{card}\{n : 1 \le n \le N, \alpha_n \in [0, 1/2]\} = \frac{e^{1/2} - 1}{e-1}.$$

(c) Show that

$$\frac{1}{N}\sum_{n=1}^{N} e(k \log n) = \frac{N^{2\pi i k}}{2\pi i k + 1} + O\Big(|k|\frac{1+\log N}{N}\Big).$$

Hint: Try comparing the sum on the left with the corresponding integral.
(d) Show that the sequence $\{\alpha_n\}$ is not uniformly distributed (mod 1).

3. Suppose that the sequence α_n satisfies $\lim_{n\to\infty}(\alpha_{n+1} - \alpha_n) = \beta$. Prove that if $\beta \in \mathbb{R}\backslash\mathbb{Q}$ then α_n is uniformly distributed modulo 1. Hint: Consider $\sum_{m=1}^{n} e(h\alpha_{m+1})$.