MATH 504 ANALYSIS IN EUCLIDEAN SPACES, SPRING TERM 2009, PROBLEMS 5

Return by Monday 16th Febraury

1. Evaluate the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3}.$$

This is $L(3,\chi)$ where χ is the non–trivial Dirichlet character modulo 4. Hint: Problems 3 can be useful.

2. Find a Fourier series proof that if m and n are non–negative integers, then

$$\sum_{k=0}^{n} \binom{n}{k}\binom{k}{m} = \binom{n}{m} 2^{n-m}.$$

3. Suppose that $g : [0,1] \to \mathbb{C}$, $g \in L^2[0,1]$, $c \in \mathbb{C}$, $f(x) = \int_0^x g(y)dy + c$ $(0 \le x \le 1)$, $c_0(f) = \int_0^1 f(x)dx$, $c_0(g) = \int_0^1 g(x)dx$. Then prove that

$$\int_0^1 \left|f(x) - \big(f(1) - f(0)\big)(x - \tfrac{1}{2}) - c_0(f)\right|^2 dx \le \frac{1}{4\pi^2} \int_0^1 |g(x) - c_0(g)|^2 dx$$

with equality if and only if f is of the form $ax + b + c^+e^{2\pi i x} + c^-e^{-2\pi i x}$.